

Remarks on two connected papers about Keller–Segel systems with nonlinear production

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Abstract. These notes aim to provide a deeper insight on the specifics of two articles dealing with chemotaxis models with nonlinear production. More precisely, we are referring to the papers “Boundedness of solutions to a quasilinear parabolic–parabolic chemotaxis model with nonlinear signal production” by X. Tao, S. Zhou and M. Ding [*J. Math. Anal. Appl.* **474**:1 (2019) 733–747] and “Boundedness for a fully parabolic Keller–Segel model with sublinear segregation and superlinear aggregation” by S. Frassu and G. Viglialoro [*Acta Appl. Math.* **171**:1 (2021), 19]. These works, independently published in these last years, present results leaving open room for further improvement. Indeed, in the first a gap in the proof of the main claim appears, whereas the cornerstone assumption in the second is not sharp. In these pages we give a more complete picture to the relative underlying comprehension.

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1. Motivations and main result

In this short document we focus on [2, Theorem 1.1] and [1, Theorem 2.1] where chemotaxis models for two coupled parabolic equations are so formulated:

$$(1.1) \quad \begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + g(u), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, t > 0. \end{cases}$$

Herein, $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with smooth boundary, and $\frac{\partial}{\partial \nu}$ denotes the differentiation with respect to the outward normal of $\partial\Omega$. Additionally, the initial data (u_0, v_0) is assumed to satisfy

$$(1.2) \quad \begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ is nonnegative with } u_0 \not\equiv 0, \\ v_0 \in C^1(\overline{\Omega}) \text{ is nonnegative,} \end{cases}$$

whereas, for all $u \geq 0$ and appropriate real numbers $d_0, d_1, s_1, \alpha, \alpha_1, \beta, g_1, \gamma$, the diffusion and sensitivity laws $D, S \in C^2([0, \infty))$ and the production growth $g \in C^1([0, \infty))$ are such that

$$(1.3) \quad d_0(1+u)^{-\alpha} \leq D(u) \leq d_1(1+u)^{-\alpha_1}, \quad 0 \leq S(u) \leq s_1u(1+u)^{\beta-1},$$

and

$$(1.4) \quad 0 \leq g(u) \leq g_1u^\gamma.$$

The aforementioned results in [2] and [1] are collected as follows.

Theorem 1.1. *Let $n \geq 2$ and (u_0, v_0) satisfy (1.2). Suppose that D, S and g fulfill (1.3) and (1.4). Then problem (1.1) admits a unique nonnegative classical solution (u, v) which is globally bounded provided that:*

I) [2, Theorem 1.1] $0 < \gamma \leq 1$ and

$$(1.5) \quad \alpha + \beta + \gamma < 1 + \frac{2}{n};$$

II) [1, Theorem 2.1] $\alpha = \alpha_1 = 0$, $0 < \gamma < \frac{2}{n}$, $\beta \geq \frac{2}{n}$ and

$$(1.6) \quad \beta + \frac{\gamma}{2} < 1 + \frac{1}{n}.$$

These two theorems have been proved, in an independent way the one from the other, recently. Moreover, when investigating a variant of Keller–Segel systems like those in (1.1), the authors of this report realized that:

- for $0 < \gamma < \frac{1}{n}$, the proof leading to condition (1.5) has a mathematical inconsistency; in this same range, even for the linear diffusion case $\alpha = \alpha_1 = 0$, the condition cannot hold true and has to be replaced by (1.6);

- for $\frac{1}{n} \leq \gamma < \frac{2}{n}$ and $\alpha = \alpha_1 = 0$, assumption (1.6) is less accurate than (1.5).

Since this gap leaves the general theory about models (1.1) somehow incomplete and fragmented, we understand that it is of primary importance giving a revised and unified conclusion. Precisely, the role behind the forthcoming theorem is twofold: correcting [2, Theorem 1.1] and improving [1, Theorem 2.1].

Theorem 1.2. *Let $n \geq 2$ and (u_0, v_0) satisfy (1.2). Suppose that D, S and g fulfill (1.3) and (1.4). If $0 < \gamma \leq 1$ and*

$$(1.7) \quad \begin{cases} \alpha + \beta + \gamma < 1 + \frac{2}{n} & \text{if } \gamma \in [\frac{1}{n}, 1], \\ \alpha + \beta < 1 + \frac{1}{n} & \text{if } \gamma \in (0, \frac{1}{n}), \end{cases}$$

then problem (1.1) admits a unique nonnegative classical solution (u, v) which is globally bounded.

2. Identification of the gap

Once combined with well-known extensibility criteria, global boundedness for local classical solutions to problem (1.1), defined in $\Omega \times (0, T_{\max})$, is achieved by controlling $\|u(\cdot, t)\|_{L^p(\Omega)}$ and $\|\nabla v(\cdot, t)\|_{L^q(\Omega)}$ on $(0, T_{\max})$, and for p, q large enough. In particular, if we refer to [2], such boundedness relies on the ensuing

Proposition 2.1 ([2, Proposition 3.1]). *Let $n \geq 2$ and (u_0, v_0) satisfy (1.2). Suppose that D, S and g fulfill (1.3) and (1.4). If $0 < \gamma \leq 1$, α and β are constrained by assumption (1.5), then for all $p \in [1, \infty)$ and each $q \in [1, \infty)$, there exists $C = C(p, q, \alpha, \alpha_1, \beta, \gamma) > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{and} \quad \|\nabla v(\cdot, t)\|_{L^q(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Unfortunately, the proof of this proposition contains an error in the case $\gamma \in (0, \frac{1}{n})$: specifically, in [2, (3.1) in Section 3] the authors claim that for any $0 < \gamma \leq 1$ it is possible to find $s \in [1, \frac{n}{(n\gamma-1)_+})$ such that

$$(2.1) \quad \gamma - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha - \beta.$$

If from the one hand for $\gamma \in [\frac{1}{n}, 1]$ such a relation and (1.5) fit, from the other hand they do not when $\gamma \in (0, \frac{1}{n})$, and some counterexamples of (2.1) can be encountered. For instance, the triplet $(\alpha, \beta, \gamma) = (1, \frac{1}{n}, \frac{1}{2n})$ is adjusted to (1.5), but oppositely it implies that (2.1) is rewritten as $-\frac{1}{2n} < \frac{1}{s} < 0$, not satisfied for any $s \geq 1$. Since relation (2.1) is crucial in the derivation of Proposition 2.1, the machinery to show [2, Theorem 1.1], of the item (I) above, misses its validity for $\gamma \in (0, \frac{1}{n})$.

3. Correction of Proposition 2.1 and proof of Theorem 1.2: some hints

As specified, we can only confine to the case $\gamma \in (0, \frac{1}{n})$. By putting $\gamma_0 := \frac{1}{n}$, we note from (1.7) that

$$(3.1) \quad \alpha + \beta + \gamma_0 < 1 + \frac{2}{n}.$$

Hence we can fix $s \in [1, \infty)$, rigorously $s \in (\frac{1}{\gamma_0}, \infty)$ (see Remark 3.1 below), such that

$$(3.2) \quad 0 = \gamma_0 - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha - \beta.$$

We next pick $p \geq \bar{p}$ and $q \geq \bar{q}$, where \bar{p} and \bar{q} are defined as in [2, Section 3], and set

$$\phi(z) := \int_0^z \int_0^\rho \frac{(1+\sigma)^{p-\alpha-2}}{D(\sigma)} d\sigma d\rho \quad \text{for } z \geq 0.$$

We can derive (3.9) in [2] unconditionally, that is, we can find $C_1 = C_1(q) > 0$ such that on $(0, T_{\max})$ the local solution of problem (1.1) complies with

$$(3.3) \quad \begin{aligned} & \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} dx + \frac{q-1}{q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 dx \\ & \leq g_1^2 \left(2(q-1) + \frac{n}{2} \right) \int_{\Omega} u^{2\gamma} |\nabla v|^{2(q-1)} dx + (C_1 - 2) \int_{\Omega} |\nabla v|^{2q} dx. \end{aligned}$$

From the condition $\gamma < \gamma_0$ and Young's inequality it follows that for all $t \in (0, T_{\max})$

$$\begin{aligned} \int_{\Omega} u^{2\gamma} |\nabla v|^{2(q-1)} dx & \leq \frac{\gamma}{\gamma_0} \int_{\Omega} u^{2\gamma_0} |\nabla v|^{2(q-1)} dx + \left(1 - \frac{\gamma}{\gamma_0} \right) \int_{\Omega} |\nabla v|^{2(q-1)} dx \\ & \leq \frac{\gamma}{\gamma_0} \int_{\Omega} u^{2\gamma_0} |\nabla v|^{2(q-1)} dx + \left(1 - \frac{\gamma}{\gamma_0} \right) \left[\left(1 - \frac{1}{q} \right) \int_{\Omega} |\nabla v|^{2q} dx + \frac{|\Omega|}{q} \right]. \end{aligned}$$

Therefore, by plugging this inequality into (3.3), we see that there exist $C_2 = C_2(q) > 0$ and $C_3 = C_3(q, |\Omega|) > 0$ providing

$$(3.4) \quad \begin{aligned} & \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} dx + \frac{q-1}{q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 dx \\ & \leq C_2 \int_{\Omega} u^{2\gamma_0} |\nabla v|^{2(q-1)} dx + C_2 \int_{\Omega} |\nabla v|^{2q} dx + C_3 \quad \text{on } (0, T_{\max}). \end{aligned}$$

Since $\gamma_0 = \frac{1}{n} \in [\frac{1}{n}, 1]$ and (3.1) holds, we can estimate the first term on the right-hand side of (3.4) as in the proof of [2], so arriving at [2, (3.19)], with C_{11} involving also the constant C_3 . Finally, thanks to relation (3.2), we complete the proof by similar arguments to those employed in [2, Proposition 3.1]. \square

Remark 3.1 (Comparison between [2, Theorem 1.1] and [1, Theorem 2.1]). The proof of [2, Proposition 3.1] relies, *inter alia*, on the conservation of mass property $\|u(\cdot, t)\|_{L^1(\Omega)} = \int_{\Omega} u_0(x) dx = m$ for all $t \in (0, T_{\max})$, as well as on the bound $\|v(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C$, valid for any $s \in \left(\frac{1}{\gamma}, \frac{n}{(n\gamma-1)_+}\right)$, throughout all $t \in (0, T_{\max})$ and for some $C = C(s, \gamma) > 0$. The first is obtainable by integrating over Ω the equation for u in (1.1). For the second, Neumann semigroup estimates, in conjunction with $\int_{\Omega} g(u)^{\frac{1}{\gamma}} \leq g_1^{\frac{1}{\gamma}} m$, entail for some $C_0 > 0$, $\mu > 0$, and all $t \in (0, T_{\max})$ and $\frac{1}{2} < \rho < 1$

$$\|v(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C_0 \|v_0\|_{W^{1,s}(\Omega)} + C_0 \int_0^t (t-r)^{-\rho-\frac{n}{2}(\gamma-\frac{1}{s})} e^{-\mu(t-r)} \|u^\gamma(\cdot, r)\|_{L^{\frac{1}{\gamma}}(\Omega)} dr.$$

Conversely, in [1, Lemma 3.1] only a uniform bound for $v(\cdot, t)$ in $W^{1,n}(\Omega)$ and for any $0 < \gamma < \frac{2}{n}$ is derived. Subsequently, since $\frac{n}{(n\gamma-1)_+} > n$, one concludes that for s close enough to $\frac{n}{(n\gamma-1)_+}$, the succeeding $W^{1,s}$ -estimates involving v , have to play a sharper role on the final result than the $W^{1,n}$ -estimates do. This is reflected on condition (1.5), milder than (1.6).

References

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