

A GENERAL NOTION OF UNIFORM ELLIPTICITY AND THE REGULARITY OF THE STRESS FIELD FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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ABSTRACT. For solutions of $\operatorname{div}(DF(Du)) = f$ we show that the quasiconformality of $z \mapsto DF(z)$ is the key property leading to the Sobolev regularity of the stress field $DF(Du)$, in relation with the summability of f . This class of nonlinearities encodes in a general way the notion of uniform ellipticity and encompasses all known instances where the stress field is known to be Sobolev regular. We provide examples showing the optimality of this assumption and present two applications: a nonlinear Cordes condition for equations in divergence form and some partial results on the $C^{p'}$ -conjecture.

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1. INTRODUCTION

In this work we are interested in Sobolev regularity results for the often called “stress field” $DF(Du)$ corresponding to solutions of

$$(1.1) \quad \operatorname{div}(DF(Du)) = f,$$

seen as the Euler-Lagrange equation for the energy functional

$$(1.2) \quad J(w, \Omega) = \int_{\Omega} F(Dw) + f w \, dx,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, $f \in L^m(\Omega)$ for some $m > 1$, and $F \in C^1(\mathbb{R}^N)$ is a strictly convex function obeying a suitable local form of uniform ellipticity condition. Questions regarding regularity of the stress field recently gained increasing interest as a basic tool to attack further regularity and locality properties of solutions to divergence form equation, see e. g. [6, 8, 14, 15, 23, 24, 34]. Despite its usefulness, however, most of the results are constrained to special kinds of nonlinearities of either p -Laplacian type or having Uhlenbeck structure. Simple nonlinearities such as, for example,

$$(1.3) \quad F(z) = |z - z_0|^p + |z + z_0|^p, \quad 1 < p < 2, \quad z_0 \neq 0,$$

do not fall within most of the currently available regularity theory. Since we focus on the stress field instead of Du itself, we briefly justify this point of view recalling the general situation for functionals of the Calculus of Variations and their local minimizers.

Let u solve (1.1). If $F \in C^2(\mathbb{R}^N)$ is uniformly elliptic, i. e.

$$\lambda |\xi|^2 \leq (D^2F(z) \xi, \xi) \leq \Lambda |\xi|^2, \quad \forall z, \xi \in \mathbb{R}^N,$$

it is a classical result that $f \in L^2_{\text{loc}}(\Omega) \Leftrightarrow u \in W^{2,2}_{\text{loc}}(\Omega)$, and in this case the regularity of Du and $V = DF(Du)$ coincide, since $Du = DF^{-1}(V)$ and DF is bi-Lipschitz. Regarding $W^{2,m}$ regularity for $m \neq 2$, it is clear that if $u \in W^{2,m}_{\text{loc}}(\Omega)$, then $DF(Du) \in W^{1,m}_{\text{loc}}(\Omega)$, and thus $f \in L^m_{\text{loc}}(\Omega)$. Conversely, suppose $f \in L^m_{\text{loc}}(\Omega)$: differentiating (1.1) gives

$$\operatorname{div}(D^2F(Du) Dv_k) = \partial_k f, \quad v_k = \partial_k u, \quad k = 1, \dots, N.$$

If $f \in L^m_{\text{loc}}(\Omega)$ for $m > N$, then Du is Hölder continuous by De Giorgi-Nash's theorem; so, freezing the (now continuous) coefficients of the matrix $D^2F(Du)$, we can apply the Calderón-Zygmund theory to obtain that $u \in W^{2,m}_{\text{loc}}(\Omega)$, i. e. $DF(Du) \in W^{1,m}_{\text{loc}}(\Omega)$.

Next, consider the p -Poisson equation

$$(1.4) \quad \Delta_p u = f, \quad p > 1,$$

corresponding to the integrand $F(z) = |z|^p/p$. In this case F satisfies

$$\lambda |z|^{p-2} |\xi|^2 \leq (D^2F(z) \xi, \xi) \leq \Lambda |z|^{p-2} |\xi|^2,$$

and the Sobolev regularity of Du is much more involved. We are aware of only one result giving second order Sobolev regularity of u from L^m assumption on f : in the non-degenerate case $p \in (1, 2]$, the regularity $u \in W^{2,p}(\mathbb{R}^N)$ if $f \in L^{p'}(\mathbb{R}^N)$ ($\frac{1}{p} + \frac{1}{p'} = 1$, as usual) has been proved in [44] for global solutions and in [28] for local ones.

Through difference quotients and Caccioppoli inequalities one can usually infer Sobolev regularity of Du from Sobolev regularity of f . In this framework, [17, 39] treat the case when $p > 2$ is near uniform ellipticity proving, respectively, $u \in W^{2,2}_{\text{loc}}(\Omega)$ for $2 \leq p < 3$ and $u \in W^{2,m}_{\text{loc}}(\Omega)$ for any m , as long as $p - 2$ is sufficiently small. The postulated regularity for f is $f \in W^{1,2}_{\text{loc}}(\Omega)$ in [17] and $f \in W^{1,m}_{\text{loc}}(\Omega)$ for $m > N$ in [39].

For $p > 2$ and only assuming L^m regularity of f in (1.4), the best results available prove *fractional* differentiability of Du , see [40, 41, 42, 43, 44]. The main idea of [34, 40, 41], which goes back to Uhlenbeck [46], is to study the regularity properties of the field

$$(1.5) \quad \mathcal{V} = |Du|^{\frac{p-2}{2}} Du$$

and deduce from the latter suitable regularity of Du . This approach is nowadays widespread, but still failed to produce estimates in terms of Lebesgue norm of f paralleling the second-order Calderón-Zygmund theory depicted above in the non-degenerate case.

An alternative route is to consider the regularity of the stress field

$$V = |Du|^{p-2} Du,$$

which arises as an interesting object *per se* in a variety of situations, e. g. in the framework of nonconvex variational problems [16] and in the dual formulation of traffic congestion problems (see [13]). In particular, the applicability of the DiPerna-Lions theory in the latter is tied to the Sobolev regularity of the stress field of the dual problem, which has been the main concern of [13] for a very degenerate functional of the form (1.2). When f is Sobolev regular, variants of Caccioppoli inequalities and difference quotients methods have been used in the cited works to obtain Sobolev regularity of V (see also [27]).

For less regular f the seminal paper is [37], treating the case $f \in L^m$ with $m \geq \max\{2, N/p\}$. In more recent years, the regularity of the stress field has been the object of fruitful investigations, also thanks to the fact that it seems to provide more natural estimates than (1.5). Starting from [31], a first-order nonlinear Calderón-Zygmund theory for the p -Laplacian problem with right-hand side in divergence form

$$\operatorname{div} |Du|^{p-2} Du = \operatorname{div} G$$

is nowadays well developed, showing the principle that the divergence operator can be “canceled out” to get estimates for $|Du|^{p-2} Du$ in terms of G in the same space. We refer to [8, 14, 15] and the literature therein for this line of research, but let us remark that the order of differentiability for V considered in these works is always less than 1.

Indeed, regarding the second-order Calderón-Zygmund theory (i. e. full Sobolev regularity for V), much less is known. A natural conjecture for solutions of (1.4) via the same principle would be

$$(1.6) \quad f \in L_{\text{loc}}^m(\Omega) \quad \Leftrightarrow \quad V \in W_{\text{loc}}^{1,m}(\Omega).$$

The case $m = +\infty$, $p > 2$, corresponds to the well-known $C^{p'}$ conjecture, which will be discussed later, proved in the plane in [4]. The endpoint $m = 1$ of (1.6) is considered by [6] where, e. g., for $f \in L_{\text{loc}}^1(\Omega)$, it is proved that $V \in W^{1-\varepsilon,1}$ for all $\varepsilon > 0$ (actually, f can be a general Radon measure in [6]).

The only positive result of the type (1.6) (beyond the close one [37]) concerns the Hilbertian case $m = 2$, which has been recently proved in [21] for equations having *Uhlenbeck structure*, i. e., of the form

$$(1.7) \quad \operatorname{div}(a(|Du|) Du) = f \quad (\text{or } \operatorname{div} G).$$

The role of this structural assumption is in fact the main motivation of this work: indeed, (with the exception of [40, 6]), the higher order Calderón-Zygmund theory exposed so far is restricted to equations of the form (1.7) and in this case it can be actually extended to systems (see [14, 15, 22] and the recently appeared [7]). As the L^2 -theory seems to be the basic step to deal with the general problem (1.6), it is worth investigating *to what extent the Uhlenbeck structure is necessary to develop such a theory* and whether the general nonlinear problem (1.1) enjoys the same Sobolev regularity for its stress field $V = DF(Du)$.

It turns out that such regularity holds true when the map $z \mapsto DF(z)$ is *quasiconformal*. A quasiconformal map $G : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a homeomorphism belonging to $W_{\text{loc}}^{1,N}(\mathbb{R}^N)$ such that, for some finite K ,

$$(1.8) \quad |\lambda_{\max}(DG(z))|^N \leq K |\det DG(z)|$$

almost everywhere, with $\lambda_{\max}(DG)$ being the maximum singular value of DG ; we refer to [38] for a nice survey on quasiconformal mappings. The main outcome of our results is that quasiconformality of DF in (1.1) is a natural and robust notion of uniform ellipticity, flexible enough to encompass anisotropic examples such as (1.3), still allowing a reasonable regularity theory.

Quasiconformal maps which are gradients of convex functions (or being, more generally, monotone) have been systematically studied by Kovalev and Maldonado in [36, 35]; convex potentials of quasiconformal mappings are called *quasiuniformly convex* functions.

Definition 1.1. *A convex $F \in C^1(\mathbb{R}^N)$ is called K -quasiuniformly convex for some $K \in [1, +\infty)$ if it is not affine, $DF \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$, and*

$$(1.9) \quad \lambda_{\max}(D^2F(z)) \leq K \lambda_{\min}(D^2F(z)) \quad \text{for a. e. } z \in \mathbb{R}^N,$$

where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and maximum eigenvalues of M .

For the most part, this will be the main assumption on F for the study of (1.1), and we will use the acronym ‘q. u. c.’ for ‘quasiuniformly convex’. Clearly, (1.9) and (1.8) for $G = DF$ are equivalent (but (1.9) implies (1.8) with a constant K^{N-1}); in Proposition 2.3 we will show that any q. u. c. function is strictly convex and of (p, q) -growth. In section 3.3 we will discuss concrete examples but, in the meantime, we note that $z \mapsto |z|^p$ is q. u. c. for any $p > 1$, and sum of q. u. c. functions is q. u. c., hence (1.3) is q. u. c..

1.1. Main results. We now present our main results, referring to the appropriate theorems in the following sections for more complete statements. The first one concerns local minimizers $u \in W_{\text{loc}}^{1,p}(\Omega)$ for the functional J in (1.2), i. e. those u obeying $J(u) < +\infty$ and $J(u, B) \leq J(u + w, B)$ for all $B \Subset \Omega$ and $w \in W_0^{1,p}(B)$.

Theorem 1.2 (Theorem 3.3). *Let $\Omega \subseteq \mathbb{R}^N$ be open, $F \in C^1(\mathbb{R}^N)$ be K -q. u. c., and $f \in L^2(\Omega) \cap W^{-1,p'}(\Omega)$ for $p = 1 + 1/K$. Then any local minimizer $u \in W_{\text{loc}}^{1,p}(\Omega)$ of J fulfills*

$$\|DF(Du)\|_{W^{1,2}(B_R)} \leq C \left(1 + \|f\|_{L^2(B_{2R})} + \|F(Du)\|_{L^1(B_{2R})}^K \right)$$

for some $C = C(K, N, R)$ and all $B_{4R} \subseteq \Omega$.

Remark 1.3. Let us briefly discuss the main assumptions. Other comments may be found in Remark 3.4.

- *Assumptions on F .* The Sobolev regularity assumption $DF \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ is necessary, as shown in example 3.5, where a strictly convex, radial $F \in C^1(\mathbb{R}^N)$ obeying (1.9) is constructed in such a way that the stress field of a suitable solution to (1.1) with $f = 0$ is not absolutely continuous.

The q. u. convexity condition fails in simple examples such as the orthotropic p -Laplacian related to the integrand

$$F(z) = \sum_{i=1}^N |z_i|^p.$$

We remark that Giaquinta's example [33] in \mathbb{R}^6 is a local minimizer of an analytic functional whose integrand is not q. u. c. and such that the stress field is in $W_{\text{loc}}^{1,s}(\mathbb{R}^N)$ only for $s < 5/4$. Playing around with examples of similar structure suggests intricate interplays between the maximal Sobolev regularity of the stress field and the possibly non-standard structure of DF ; hence, it is not clear what to expect from functionals with non-quasiconformal gradient mapping.

In example 3.6 we discuss integrands of the form $F(z) = F(|z|)$, while examples 3.7, 3.8 investigate more general anisotropic functionals.

- *Assumptions on f .* The hypothesis $f \in W^{-1,p'}(\Omega)$ has been made for expository reasons: for $p < 2N/(N+2)$, $L^2(\Omega)$ is not embedded in $W^{-1,p'}(\Omega)$, destroying the variational framework we chose to be in. One should then resort to the notion of *approximable solutions* (briefly described in the last section), in order to deal with those cases. We refer to [1] for a comprehensive theory of approximable solutions in the anisotropic framework. Anyway, in terms of summability, $f \in W^{-1,p'}(\Omega)$ is implied by $f \in L^{(p^*)}'(\Omega)$, which is a weaker summability than the one in [37].

We will give two applications of Theorem 1.2. The first one deals with nonlinear Cordes conditions for variational problems. Cordes conditions usually refer to the L^m -theory for elliptic equations in non-divergence form with measurable coefficients, namely, to solutions of

$$(1.10) \quad \sum_{ij=1}^N a_{ij} D_{ij} u = f$$

with measurable coefficients $a_{ij} : \Omega \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ fulfilling

$$\lambda |\xi|^2 \leq \sum_{ij=1}^N a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and a. e. $x \in \Omega$. Under these measurability assumptions alone, there is no hope in general for the Calderón-Zygmund inequality

$$(1.11) \quad \|u\|_{W^{2,m}(\Omega)} \leq C_m \|f\|_{L^m(\Omega)}, \quad \forall m > 1.$$

Some regularity has to be assumed on a_{ij} in order to obtain (1.11) for all $m > 1$ (VMO regularity suffices; see [18]). Roughly stated, a Cordes condition for (1.10) with discontinuous a_{ij} says that (1.11) holds if either m is sufficiently near 2, or Λ/λ is sufficiently near 1. A similar situation takes place for nonlinear equations in divergence form.

Theorem 1.4 (Theorem 4.3). *Let F obey the assumptions of Theorem 1.2 and $f \in L^m(\Omega) \cap W^{-1,p'}(\Omega)$ for some $m > 1$. Then any local minimizer $u \in W^{1,p}(\Omega)$ for J in (1.2) is such that $DF(Du) \in W_{\text{loc}}^{1,m}(\Omega)$ and*

$$\|DF(Du)\|_{W^{1,m}(B_R)} \leq C \left(\|f\|_{L^m(B_{2R})} + \|DF(Du)\|_{L^m(B_{2R})} \right),$$

for $C = C(K, N, m, R)$ and all $B_{4R} \subseteq \Omega$, in either of the following cases:

- (i) $K \leq K_0$, with $K_0 = K_0(N, m) > 1$.
- (ii) $|m - 2| \leq \delta_0$, for $\delta_0 = \delta_0(K, N) > 0$.

As a second application we will derive some partial results pertaining the $C^{p'}$ conjecture, which states that any solution of the p -Poisson equation with bounded right-hand side is $C^{p'}$ regular if $p > 2$ (hereafter we use the notation $C^\gamma = C^{[\gamma], \gamma - [\gamma]}$, being $[\gamma]$ the integer part of γ).

Theorem 1.5 (Corollary 4.4, Theorem 4.6, and Corollary 4.7). *Let $u : \Omega \rightarrow \mathbb{R}$ be an approximable solution of (1.4).*

- (i) *If $f \in L^\infty(\Omega)$, then for all sufficiently small $|p - 2|$ it holds $u \in C^{2-\alpha_p}(\Omega)$, where $\alpha_p = c(N) |p - 2| > 0$.*
- (ii) *For any $m, p > 1$, if u and Ω have cylindrical symmetry, then $|Du|^{p-2} Du \in W_{\text{loc}}^{1,m}(\Omega)$ whenever $f \in L_{\text{loc}}^m(\Omega)$. In particular, any cylindrical approximable solution of (1.4) belongs to $C^{\min\{p', 2\} - \varepsilon}(\Omega)$ for any $\varepsilon > 0$.*

Remark 1.6.

- Item (i) confirms the validity of the $C^{p'}$ conjecture near uniform ellipticity, and is inspired by [39]. Here, however, we obtain an explicit rate of the Hölder exponent and, more substantially, we do not require Sobolev regularity on f . When $f \in L^\infty(\Omega)$ the usual notion of weak solution suffices.
- Point (ii) gives a weak form of the $C^{p'}$ conjecture (namely, $u \in C^{p'-}$) in the class of cylindrical solutions for any $p \geq 2$, with a different approach than the one of [5]. We need the notion of approximable solutions as in general f may fail to belong to $W^{-1,p'}(\Omega)$ for small $m > 1$, destroying the variational setting. For details on such a notion we refer to Section 4.2.

By a cylindrical solution we mean a function of the form $u(x) = v(|x'|)$, $x' \in \mathbb{R}^k$, $k \leq N$. It is worth noticing that the domain Ω may not contain the origin, in which case the approach of [5] cannot be easily applied.

1.2. Outline of the proofs. Consider, as a first step, a smooth compactly supported solution u of

$$\operatorname{div} DF(Du) = f \quad \text{in } \mathbb{R}^N$$

with F and f smooth. Our starting point is the well-known identity

$$(1.12) \quad \|DV\|_{L^2(\mathbb{R}^N)}^2 = \|\operatorname{div} V\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \|\operatorname{curl} V\|_{L^2(\mathbb{R}^N)}^2 \quad \forall V \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N),$$

where $\operatorname{curl} V = DV - DV^t$. Applying (1.12) to the stress field $V = DF(Du)$, we are reduced to estimate $\operatorname{curl} V$.

The main observation is that, in the smooth setting, DV is of special type, namely

$$DV = D^2F(Du) D^2u,$$

where $D^2F(Du)$ is a symmetric positive definite matrix and D^2u is symmetric. An elementary lemma shows that any matrix of the form

$$X = PS, \quad P \text{ symmetric positive definite, } S \text{ symmetric,}$$

fulfills

$$(1.13) \quad |X - X^t|_2^2 \leq 2 \left(1 - \frac{\lambda_{\min}}{\lambda_{\max}}\right)^2 |X|_2^2,$$

where λ_{\min} and λ_{\max} are, respectively, the minimum and maximum eigenvalues of P . Thus the curl term in (1.12) can be reabsorbed to the left if $\lambda_{\max} \leq K \lambda_{\min}$ holds a. e. for the matrix D^2F , giving

$$\|DV\|_{L^2(\mathbb{R}^N)}^2 \leq K^2 \|f\|_{L^2(\mathbb{R}^N)}^2$$

for $V = DF(Du)$ in the smooth, global setting.

In order to prove Theorem 1.2 we have to localize the estimate and to suitably build smooth approximating problems. We regularize the integrand, the source, and the boundary data through convolution but, in order to have strongly elliptic problems, we would like to add a small multiple of a strongly elliptic functional. Since we do not want to alter the q. u. convexity constant, the only viable choice is to add small multiples of $|z|^2$ to the regularized integrands. This is a quite unnatural choice if F is not of standard quadratic growth, and it forces some interplay between the regularization parameters. Here, the explicit *a-priori* Lipschitz estimate for the corresponding solutions taken from [11] plays a key role.

The main step of the proof of Theorem 1.4 is to represent the solutions of

$$\begin{cases} \operatorname{div} V = f \\ \operatorname{curl} V = G \end{cases}$$

in \mathbb{R}^N through Riesz transforms and generalize (1.12) to the L^m -setting as

$$\|DV\|_{L^m(\mathbb{R}^N)} \leq C(m, N) (\|\operatorname{div} V\|_{L^m(\mathbb{R}^N)} + \|\operatorname{curl} V\|_{L^m(\mathbb{R}^N)}).$$

Thus, inequality (1.13) does the trick in case (i) of Theorem 1.4, allowing reabsorption of the curl term for K sufficiently near 1. A standard Riesz interpolation argument, together with a careful choice of the norms involved, allows to prove case (ii).

Finally, point (i) of Theorem 1.5 is an immediate consequence of point (i) of Theorem 1.4, while point (ii) stems from this observation: if u exhibits cylindrical symmetry,

then the stress field $a(|Du|) Du$ is irrotational; by the Helmholtz decomposition, it can be locally represented as the gradient of a solution of $\Delta v = f \in L^m$, so the standard Calderón-Zygmund theory applies.

1.3. Structure of the paper. In section 2 we recall some functional inequalities and properties about quasiuniform convexity. In section 3.1 we develop our basic estimate in the smooth setting; section 3.2 is devoted to the proof of Theorem 1.2, where the main approximation procedure, used also later, is described; section 3.3 contains the relevant examples depicted above. In section 4 we focus on the applications: first we treat the Cordes conditions, and finally we collect the partial results pertaining the $C^{p'}$ conjecture.

Notations:

- The euclidean norm of a vector $v \in \mathbb{R}^N$ is denoted by $|v|$ and (v, w) denotes the scalar product. By Ω we denote a bounded open subset of \mathbb{R}^N , while B_r denotes a ball of radius r not necessarily centered at the origin. Similarly, B denotes a ball with unspecified center and radius; if $B = B_r(x_0)$, then we set $\lambda B = B_{\lambda r}(x_0)$.
- For a $N \times N$ matrix $A = (a_{ij})$, its transpose is denoted by A^t ; on such matrices we consider the Frobenius norm

$$|A|_2 = \left(\sum_{i,j=1}^N |a_{ij}|^2 \right)^{1/2},$$

arising from the scalar product $(A, B)_2 = \text{Tr}(AB^t)$. For $v, w \in \mathbb{R}^N$, $v \otimes w$ is matrix with entries $(v_i w_j)$, while $v \wedge w = v \otimes w - w \otimes v$; I denotes the identity matrix, while Id the identity function of \mathbb{R}^N ; O_N is the orthogonal group of $N \times N$ matrices such that $AA^t = A^t A = I$; if A is symmetric, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ are its maximum and minimum eigenvalues.

- If a domain of integration is missing, it means that we are integrating on \mathbb{R}^N . We also set for brevity $\|f\|_m = \|f\|_{L^m(\mathbb{R}^N)}$. Finally, we sometimes will use the L^m “norm” also for $m \in (0, 1)$, when it is only positively 1-homogeneous.

2. PRELIMINARIES

2.1. Functional inequalities. For $0 < \theta < m$, $m > 1$, starting from the inequality

$$(2.1) \quad \int_{B_1} |v|^m dx \leq \int_{B_1} |Dv|^m dx + C(N, m, \theta) \left(\int_{B_1} |v|^\theta dx \right)^{m/\theta},$$

(obtained by a standard compactness argument) and replicating, with the obvious modifications, the proof of [21, eq. (5.4)], we get the following functional inequality.

Lemma 2.1. *Let $m > 1$, $0 < \theta < m$, and $0 < r < s < R$. There exists $C = C(N, m, \theta)$ such that for any $v \in W^{1,m}(B_s \setminus B_r)$ and any $\delta \in (0, 1)$ it holds*

$$\int_{B_s \setminus B_r} |v|^m dx \leq \delta^m R^m \int_{B_s \setminus B_r} |Dv|^m dx + \frac{C}{(\delta^N (s-r) R^{N-1})^{\frac{m-\theta}{\theta}}} \left(\int_{B_s \setminus B_r} |v|^\theta dx \right)^{m/\theta}.$$

The following lemma is a straightforward generalization of the well-known identity

$$(2.2) \quad \int |DV|_2^2 dx = \frac{1}{2} \int |\operatorname{curl} V|_2^2 dx + \int (\operatorname{div} V)^2 dx,$$

valid for $V \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$.

Lemma 2.2. *If $V \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, then for any $\varphi \in C_c^2(\mathbb{R}^N)$ it holds*

$$(2.3) \quad \begin{aligned} \int \varphi^2 |DV|_2^2 dx &= \frac{1}{2} \int \varphi^2 |\operatorname{curl} V|_2^2 dx + \int \varphi^2 (\operatorname{div} V)^2 dx \\ &+ \int [2(D\varphi^2, V) \operatorname{div} V + (D^2\varphi^2, V \otimes V)_2] dx. \end{aligned}$$

Proof. Write, through parallelogram identity,

$$|DV|_2^2 = \frac{1}{4}|DV - DV^t|_2^2 + \frac{1}{4}|DV + DV^t|_2^2,$$

then multiply by φ^2 and integrate to obtain

$$(2.4) \quad \begin{aligned} \int \varphi^2 |DV|_2^2 dx &= \int \frac{\varphi^2}{4} |\operatorname{curl} V|_2^2 dx + \int \frac{\varphi^2}{4} \sum_{ij} (D_j V^i + D_i V^j)^2 dx \\ &= \int \frac{\varphi^2}{4} |\operatorname{curl} V|_2^2 dx + \int \frac{\varphi^2}{2} |DV|_2^2 dx + \int \frac{\varphi^2}{2} \sum_{ij} D_i V^j D_j V^i dx. \end{aligned}$$

The last term is computed integrating by parts twice: for any $i, j = 1, \dots, N$

$$\begin{aligned} \int \varphi^2 D_i V^j D_j V^i dx &= - \int D_i \varphi^2 V^j D_j V^i dx - \int \varphi^2 V^j D_{ij} V^i dx \\ &= \int D_{ij} \varphi^2 V^j V^i dx + \int D_i \varphi^2 V^i D_j V^j dx \\ &+ \int D_j \varphi^2 V^j D_i V^i dx + \int \varphi^2 D_j V^j D_i V^i dx, \end{aligned}$$

so that summing over i, j gives

$$\int \varphi^2 \sum_{ij} D_i V^j D_j V^i dx = \int [\varphi^2 (\operatorname{div} V)^2 + 2(D\varphi^2, V) \operatorname{div} V + (D^2\varphi^2, V \otimes V)_2] dx.$$

Inserting this formula into (2.4) yields (2.3). □

2.2. Quasiuniform convexity. In this section we show that the q. u. convexity condition provides, in a unified way, many of the properties that the usual integrands of the Calculus of Variations satisfy.

Let us begin by observing that any convex F possesses a distributional second derivative which belongs to $BV_{\text{loc}}(\mathbb{R}^N)$ (see [2]). Accordingly, the second derivative can be decomposed in an absolutely continuous part, a jump part, and a Cantor part. If $F \in C^1(\mathbb{R}^N)$, the jump part vanishes, hence by requiring that $F \in C^1(\mathbb{R}^N) \cap W_{\text{loc}}^{2,1}(\mathbb{R}^N)$ we are actually excluding that D^2F has a Cantor part.

We now discuss more in detail the q. u. convexity condition, yielding the consequences we will use in the rest of the paper.

Proposition 2.3 (Properties of K -quasiuniformly convex functions). *Let F be a K -quasiuniformly convex function. Then*

- (i) DF is K^{N-1} -quasiconformal, hence $C^{1/K}(\mathbb{R}^N)$.
- (ii) F is strictly convex and of (p, q) growth, i. e., there exists $C = C(N, K, F) > 0$ and $1 < p < q < +\infty$ such that

$$(2.5) \quad C^{-1}|z|^p - C \leq F(z) \leq C(|z|^q + 1)$$

$$(2.6) \quad C^{-1}|z|^{p-1} - C \leq |DF(z)| \leq C(|z|^{q-1} + 1)$$

for all $z \in \mathbb{R}^N$. More precisely, one can take $p = 1 + 1/K$ and $q = 1 + K$.

- (iii) If $\varphi \in C_c^\infty(\mathbb{R}^N; [0, +\infty))$, then $F * \varphi$ is K -q. u. c..
- (iv) Its Moreau-Yoshida regularization

$$(2.7) \quad F_\delta(z) = \inf_{y \in \mathbb{R}^N} \left\{ F(y) + \frac{1}{2\delta} |y - z|^2 \right\}$$

is K -q. u. c..

Proof. (i) By the Alexandrov theorem DF is differentiable a. e., and [47, Theorem 32.3] ensures that $DF \in W_{\text{loc}}^{1,N}(\mathbb{R}^N)$. Since (1.8) and (1.9) are equivalent up to changing the constants, u is quasiuniformly convex in the sense of [36]. In particular [36, Theorem 3.1] shows that DF is K^{N-1} -quasiconformal. The regularity statement holds for any quasiconformal mapping (see [38, Theorem 2.14]).

- (ii) The strict convexity of F follows from [36, Lemma 3.2]. If z_0 is the unique minimum point for F we can consider $F(z_0 + \cdot) - F(z_0)$, so there is no loss in generality assuming $DF(0) = 0$, $F(0) = 0$. Let $G = DF$, which then is K^{N-1} -quasiconformal. By [38, Theorem 2.14]

$$|G(z)| \leq C(N, K) \sup_{y \in B_1} |G(y)| |z|^{1/K}, \quad z \in B_1.$$

Since G^{-1} is still K^{N-1} quasiconformal, it obeys a similar estimate, proving the lower bound

$$|G(z)| \geq C(N, K, G) |z|^K, \quad z \in B_1.$$

Finally, the inversion

$$G^*(x) = \frac{G(x/|x|^2)}{|G(x/|x|^2)|^2}$$

is again K^{N-1} quasiconformal on B_1 , so that the previous estimates are transferred to the outside of B_1 as

$$(2.8) \quad C^{-1} |z|^{1/K} \leq |G(z)| \leq C |z|^K \quad |z| \geq 1,$$

where $C = C(N, K, G)$. For $G = DF$, $p = 1 + 1/K$, $q = 1 + K$ we thus obtained (2.6). Moreover, by [35, Lemma 18], DF is δ -convex for some $\delta = \delta(N, K) > 0$, meaning that

$$(2.9) \quad (DF(z) - DF(y), z - y) \geq \delta |DF(z) - DF(y)| |z - y| \quad \forall z, y \in \mathbb{R}^N,$$

Integrating (2.8) on the ray $[0, z]$ provides the upper bound in (2.5), while using (2.9) and (2.8), besides integrating on rays again, we get

$$F(z) = \int_0^1 (DF(tz), z) dt \geq \delta \int_0^1 |DF(tz)| |z| dt \geq \frac{\delta}{pC} |z|^p,$$

producing the lower bound in (2.5).

- (iii) Let $\lambda_{\min}(z) = \lambda_{\min}(D^2F(z))$, where z is a second order differentiability point for F . From the representation

$$\lambda_{\min}(z) := \inf \{ (D^2F(z) \xi, \xi) : \xi \in D \},$$

where D is a fixed countable dense subset of \mathbb{S}^{N-1} , we infer that λ_{\min} is measurable and in $L^1_{\text{loc}}(\mathbb{R}^N)$. Then, for any $z \in \mathbb{R}^N$ and $\xi \in D$,

$$\begin{aligned} (D^2F * \varphi(z) \xi, \xi) &= \int \varphi(z-y) (D^2F(y) \xi, \xi) dy \\ &\geq \int \varphi(z-y) \lambda_{\min}(y) |\xi|^2 dx =: \tilde{\lambda}_{\min}(z) |\xi|^2, \end{aligned}$$

while

$$\int \varphi(z-y) (D^2F(y) \xi, \xi) dy \leq \int \varphi(z-y) K \lambda_{\min}(y) |\xi|^2 dy = K \tilde{\lambda}_{\min}(z) |\xi|^2,$$

implying the claim.

- (iv) Recall that the minimum in (2.7) is attained at a unique point $P_\delta(z)$ fulfilling

$$(2.10) \quad P_\delta(z) + \delta DF(P_\delta(z)) = z, \quad DF_\delta(z) = DF(P_\delta(z)),$$

and the so-defined function $P_\delta = (\text{Id} + \delta DF)^{-1}$ is 1-Lipschitz and a homeomorphism of \mathbb{R}^N , since $F \in C^1(\mathbb{R}^N)$. Let E be the set of points where DF fails to be differentiable. The map $\text{Id} + \delta DF$ is the gradient of a K -quasiuniformly convex function, hence by point (i) is quasiconformal and satisfies the Lusin (N) property, i. e., it sends null-measure sets to null-measure sets (see [47]). Thus, $P_\delta^{-1}(E)$ has zero measure. Moreover, since P_δ is Lipschitz continuous, Rademacher's theorem ensures that the set M where P_δ is not differentiable has zero measure. We will prove (1.9) at any point

$$z \notin M \cup P_\delta^{-1}(E),$$

the latter set having zero measure. Indeed, at any such point z we have that DF is differentiable at $P_\delta(z)$ and P_δ is differentiable at z . The Chain Rule applied to (2.10) then gives

$$(I + \delta D^2F(P_\delta(z))) DP_\delta(z) = I, \quad D^2F_\delta(z) = D^2F(P_\delta(z)) DP_\delta(z),$$

which entails

$$(2.11) \quad D^2F_\delta(z) = D^2F(P_\delta(z)) (I + \delta D^2F(P_\delta(z)))^{-1}.$$

Let the eigenvalues of $D^2F(P_\delta(z))$ be $\lambda_{\min} = \lambda_1 \leq \dots \leq \lambda_N = \lambda_{\max}$. The matrices $D^2F(P_\delta(z))$ and $(I + \delta D^2F(P_\delta(z)))^{-1}$ have the same basis of eigenvectors, with

eigenvalues λ_i and $(1 + \delta \lambda_i)^{-1}$ respectively. Hence (2.11) implies that $D^2 F_\delta(z)$ has eigenvalues $\lambda_i/(1 + \delta \lambda_i)$. As $t \mapsto t/(1 + \delta t)$ is increasing, its minimum and maximum eigenvalues are

$$\lambda_{\delta, \min} := \frac{\lambda_{\min}}{1 + \delta \lambda_{\min}}, \quad \lambda_{\delta, \max} := \frac{\lambda_{\max}}{1 + \delta \lambda_{\max}},$$

which obey $\lambda_{\delta, \max} \leq K \lambda_{\delta, \min}$ as long as $\lambda_{\max} \leq K \lambda_{\min}$. \square

Due to the previous proposition, we will denote henceforth by p and q the power of the lower and upper bound, respectively, for a given K -q. u. c. function F .

2.3. Extensions. We conclude with a couple of tools which will be occasionally used in the following.

Lemma 2.4. *Let $F \in C^1(B_R)$ be a non-negative, strictly convex function, and $\sigma \in (0, 1)$.*

- (i) *There exists a strictly convex $\tilde{F} \in C^1(\mathbb{R}^N)$ such that $\tilde{F}|_{B_{\sigma R}} = F$ and, for some C, α depending on F, R, N, σ , it holds*

$$(2.12) \quad |z|^2 \leq \alpha(\tilde{F}(z) + 1), \quad C^{-1}|z| - C \leq |D\tilde{F}(z)| \leq C(|z| + 1).$$

- (ii) *If F is K -q. u. c. in B_R as per Definition 1.1, and for some $\varepsilon > 0$ it holds $\lambda_{\min}(z) \geq \varepsilon$ in $B_R \setminus B_{\sigma R}$, in addition to (2.12) \tilde{F} can be chosen to be \tilde{K} -q. u. c., with $\tilde{K} = \tilde{K}(F, R, N, \sigma, \varepsilon)$.*

Proof. To prove (i), let $\tau = (1 + \sigma)/2$, choose a radial cut-off function $\eta \in C_c^\infty(B_R; [0, 1])$ such that $\eta \equiv 1$ in $B_{\tau R}$, and define

$$\tilde{F}(z) = \eta(z) F(z) + (1 - \eta(z)) \frac{|z|^2}{2} + C(|z| - \sigma R)_+^2,$$

where $C > 0$ is a constant to be chosen. Clearly $\tilde{F} \in C^1(\mathbb{R}^N)$ and obeys (2.12), so it remains to show that F is strictly convex for a suitable C . To this aim, set

$$A(z) := \frac{1}{2} D^2(|z| - \sigma R)^2 = \frac{\sigma R}{|z|} \frac{z}{|z|} \otimes \frac{z}{|z|} + \left(1 - \frac{\sigma R}{|z|}\right) I,$$

whose eigenvalues are 1 and $1 - \sigma R/|z|$. In particular, A is non-negative definite in $B_R \setminus B_{\sigma R}$, and its eigenvalues are uniformly bounded below in $\mathbb{R}^N \setminus B_{\tau R}$ by a positive constant; since \tilde{F} agrees with F in $B_{\sigma R}$, it follows that \tilde{F} is strictly convex in $B_{\tau R}$ and in $\mathbb{R}^N \setminus B_R$. A straightforward computation yields

$$D^2 \tilde{F} = \eta D^2 F + M + 2CA$$

$$M := (1 - \eta) I + D\eta \otimes DF + DF \otimes D\eta - 2D\eta \otimes z + \left(F - \frac{|z|^2}{2}\right) D^2 \eta$$

a. e. outside $B_{\sigma R}$, and we can choose C so that

$$\left(1 - \frac{\sigma}{\tau}\right) C = \max_{z \in B_R} |M(z)|_2,$$

ensuring

$$\lambda_{\min}(D^2\tilde{F}) \geq \eta \lambda_{\min}(D^2F) + \left(1 - \frac{\sigma}{\tau}\right)C \quad \text{in } B_R \setminus B_{\tau R}.$$

Summing up, \tilde{F} is globally strictly convex by an elementary argument.

To prove (ii), let $\tilde{\lambda}_{\min}(z)$ and $\tilde{\lambda}_{\max}(z)$ denote the minimum and maximum eigenvalues of $D^2\tilde{F}(z)$, and $\lambda_{\min}, \lambda_{\max}$ those for D^2F . It holds

$$\varepsilon \leq \lambda_{\min} \leq \tilde{\lambda}_{\min} \leq \tilde{\lambda}_{\max} \leq 3C + \lambda_{\max}$$

in $B_{\tau R} \setminus B_{\sigma R}$, so that from (1.9) we get

$$\frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}} \leq \frac{\tilde{\lambda}_{\max}}{\lambda_{\min}} \leq \frac{3C}{\lambda_{\min}} + K \leq \frac{3C}{\varepsilon} + K.$$

In $\mathbb{R}^N \setminus B_{\tau R}$ it holds

$$(1 - \sigma/\tau)C + \eta \lambda_{\min} \leq \tilde{\lambda}_{\min} \leq \tilde{\lambda}_{\max} \leq 3C + \eta \lambda_{\max},$$

so that

$$\frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}} \leq \frac{3C + K\eta\lambda_{\min}}{(1 - \sigma/\tau)C + \eta\lambda_{\min}} \leq \frac{3\tau}{\tau - \sigma} + K.$$

Since $\tilde{F} = F$ on $B_{\sigma R}$ and (1.9) holds for F there, the claim is proved. \square

3. DIVERGENCE FORM, QUASICONFORMAL EQUATIONS

3.1. The smooth setting. The core of our approach lies in the following elementary observation.

Lemma 3.1. *Let $X = PS$, where P and S are symmetric $N \times N$ matrices and P is positive definite with minimum and maximum eigenvalues λ_{\min} and λ_{\max} . Then*

$$(3.1) \quad |X - X^t|_2^2 \leq 2 \frac{(1 - \lambda_{\min}/\lambda_{\max})^2}{1 + (\lambda_{\min}/\lambda_{\max})^2} |X|_2^2.$$

Proof. Inequality (3.1) is invariant under rotations; thus, without loss of generality, we can suppose $P_{ij} = \lambda_i \delta_{ij}$ with $0 < \lambda_1 \leq \dots \leq \lambda_N$. Then from $X = PS$ we get

$$X_{ij} = \lambda_i S_{ij},$$

so that

$$|X - X^t|_2^2 = \sum_{ij} |X_{ji} - X_{ij}|^2 = 2 \sum_{i < j} |X_{ji} - X_{ij}|^2 = 2 \sum_{i < j} |\lambda_j S_{ji} - \lambda_i S_{ij}|^2,$$

and from the symmetry of S we conclude

$$(3.2) \quad |X - X^t|_2^2 = 2 \sum_{i < j} S_{ij}^2 (\lambda_j - \lambda_i)^2.$$

Similarly, we have

$$(3.3) \quad |X|_2^2 \geq \sum_{i < j} (X_{ij}^2 + X_{ji}^2) = \sum_{i < j} S_{ij}^2 (\lambda_i^2 + \lambda_j^2).$$

Let

$$\varphi(t) = \frac{(1-t)^2}{1+t^2},$$

which is decreasing in $[0, 1]$, and observe that for $j > i$ we have $\lambda_i/\lambda_j \in [0, 1]$. Therefore

$$(\lambda_j - \lambda_i)^2 = \frac{(\lambda_j - \lambda_i)^2}{\lambda_j^2 + \lambda_i^2} (\lambda_j^2 + \lambda_i^2) = \varphi\left(\frac{\lambda_i}{\lambda_j}\right) (\lambda_j^2 + \lambda_i^2) \leq \varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) (\lambda_j^2 + \lambda_i^2).$$

Inserting this estimate in (3.3) and recalling (3.2) we get

$$|X - X^t|_2^2 \leq 2\varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) \sum_{i < j} S_{ij}^2 (\lambda_i^2 + \lambda_j^2) = 2\varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) |X|_2^2.$$

□

Theorem 3.2. *Let $u \in C^2(B_{2R})$ solve*

$$\operatorname{div}(DF(Du)) = f \quad \text{in } B_{2R}$$

for a K -q. u. c. $F \in C^2(\mathbb{R}^N)$, and let

$$V(x) = DF(Du(x)).$$

Then for any $\theta \in (0, 2]$ there exist $C = C(N, K, \theta)$ and $C_R = C(N, K, \theta, R)$ such that

$$(3.4) \quad \|V\|_{W^{1,2}(B_R)} \leq C \|f\|_{L^2(B_{2R})} + C_R \|V\|_{L^\theta(B_{2R})}.$$

Proof. For any $\varepsilon > 0$ let

$$F_\varepsilon(z) = F(z) + \varepsilon \frac{|z|^2}{2}, \quad f_\varepsilon = f + \varepsilon \Delta u,$$

so that D^2F_ε is symmetric and positive definite. It holds

$$\lambda_{\min}(D^2F_\varepsilon(z)) = \lambda_{\min}(D^2F(z)) + \varepsilon, \quad \lambda_{\max}(D^2F_\varepsilon(z)) = \lambda_{\max}(D^2F(z)) + \varepsilon,$$

so that if (1.9) holds for F , it does so for D^2F_ε as well, with the same constant K .

Clearly u solves

$$\operatorname{div}(DF_\varepsilon(Du)) = f_\varepsilon$$

in B_{2R} . Letting

$$V_\varepsilon = DF_\varepsilon(Du),$$

it holds

$$DV_\varepsilon = D^2F_\varepsilon(Du) D^2u,$$

where the first matrix is symmetric positive definite and the second one is symmetric.

We thus apply Lemma 3.1 to the matrix

$$P = D^2F_\varepsilon(Du), \quad S = D^2u, \quad X = DV_\varepsilon = P S.$$

Recall that $D^2F_\varepsilon(Du)$ fulfills (1.9) with constant K , whence

$$\frac{(\lambda_{\max}(D^2F_\varepsilon(Du)) - \lambda_{\min}(D^2F_\varepsilon(Du)))^2}{\lambda_{\max}^2(D^2F_\varepsilon(Du)) + \lambda_{\min}^2(D^2F_\varepsilon(Du))} \leq \left(1 - \frac{1}{K}\right)^2.$$

From (3.1) we get

$$(3.5) \quad |\operatorname{curl} V_\varepsilon|_2^2 \leq 2 \left(1 - \frac{1}{K}\right)^2 |DV_\varepsilon|_2^2.$$

For any r, s with $R \leq r < s \leq 2R$, fix $\varphi \in C_c^\infty(B_s, [0, 1])$ such that

$$(3.6) \quad \varphi|_{B_r} \equiv 1, \quad |D\varphi| \leq \frac{C}{s-r}, \quad |D^2\varphi| \leq \frac{C}{(s-r)^2}.$$

This will allow to consider φV_ε as defined on the whole \mathbb{R}^N , so that (2.3) holds true. The stipulated properties of φ ensure that

$$\int 2(D\varphi^2, V_\varepsilon) f_\varepsilon + (D^2\varphi^2, V_\varepsilon \otimes V_\varepsilon)_2 dx \leq \frac{C}{(s-r)^2} \int_{B_s \setminus B_r} |V_\varepsilon|^2 dx + C \int_{B_{2R}} f_\varepsilon^2 dx,$$

where we used the Schwartz inequality on the first term and $s \leq 2R$. Using also (3.5) to control the curl term of Lemma 2.2 yields

$$\int \varphi^2 |DV_\varepsilon|_2^2 dx \leq \left(1 - \frac{1}{K}\right)^2 \int \varphi^2 |DV_\varepsilon|_2^2 dx + \frac{C}{(s-r)^2} \int_{B_s \setminus B_r} |V_\varepsilon|^2 dx + C \int_{B_{2R}} f_\varepsilon^2 dx.$$

We let $\varepsilon \rightarrow 0$ and bring the first term on the right to the left-hand side; recalling that $\varphi \equiv 1$ on B_r , we obtain

$$(3.7) \quad \int_{B_r} |DV|_2^2 dx \leq \frac{C_K}{(s-r)^2} \int_{B_s \setminus B_r} |V|^2 + C_K \int_{B_{2R}} f^2 dx$$

for any $R \leq r < s \leq 2R$. We next proceed as in [21]: by Lemma 2.1 with $m = 2$ and

$$\delta = \frac{s-r}{2\sqrt{C_K}R}$$

we get

$$\frac{C_K}{(s-r)^2} \int_{B_s \setminus B_r} |V|^2 dx \leq \frac{1}{4} \int_{B_s \setminus B_r} |DV|_2^2 dx + \frac{C_K R^{\frac{2-\theta}{\theta}}}{(s-r)^{2+\frac{2-\theta}{\theta}(N+1)}} \left(\int_{B_s \setminus B_r} |V|^\theta dx \right)^{2/\theta}$$

which inserted into (3.7) gives, for all $R \leq r < s \leq 2R$,

$$\int_{B_r} |DV|_2^2 dx \leq \frac{1}{4} \int_{B_s} |DV|_2^2 dx + C_K \int_{B_{2R}} f^2 dx + \frac{C_K R^{\frac{2-\theta}{\theta}}}{(s-r)^{2+\frac{2-\theta}{\theta}(N+1)}} \left(\int_{B_s \setminus B_r} |V|^\theta dx \right)^{2/\theta}.$$

A standard iteration lemma (see [32, Lemma 3.1, Ch 5]) improves the latter to

$$\int_{B_R} |DV|_2^2 dx \leq C_K \int_{B_{2R}} f^2 dx + \frac{C_K}{R^{2+\frac{2-\theta}{\theta}N}} \left(\int_{B_{2R}} |V|^\theta dx \right)^{2/\theta},$$

which is the desired estimate on the derivative of V . In order to control $\|V\|_{L^2(B_R)}^2$, we invoke the rescaled form of (2.1) which, in conjunction with the previous estimate, completes the proof of (3.4). \square

3.2. Local minimizers. For a bounded $\Omega \subseteq \mathbb{R}^N$ we let

$$J(w, \Omega) = \int_{\Omega} F(Dw) dx + \int_{\Omega} f w dx$$

whenever the two integrands are in $L^1(\Omega)$, sometimes omitting the dependence on Ω when this causes no confusion. We will consider J under p -coercivity assumptions on F and for $f \in W^{-1,p'}(\Omega)$, so that it is well defined on $W^{1,p}(\Omega)$.

Recall that $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a local minimizer for J in $W^{1,p}(\Omega)$ if, for any $B \Subset \Omega$,

$$(3.8) \quad J(u, B) = \inf \{ J(w, B) : w \in u + W_0^{1,p}(B) \}.$$

Theorem 3.3. *Let $F \in C^1(\mathbb{R}^N)$ be a q . u. c. function and $q > p > 1$ be given in Proposition 2.3, point (ii). For $f \in L^2(\Omega) \cap W^{-1,p'}(\Omega)$, let u be a local minimizer u of J in Ω . Then it holds*

$$(3.9) \quad \|DF(Du)\|_{W^{1,2}(B)} \leq C \left(\|f\|_{L^2(2B)} + \|DF(Du)\|_{L^\theta(2B)} \right)$$

for $4B \subseteq \Omega$ and any $\theta \in (0, 2]$, being $C = C(K, N, B, \theta)$ a suitable constant, and

$$(3.10) \quad \|DF(Du)\|_{W^{1,2}(B)} \leq C \left(1 + \|f\|_{L^2(2B)} + \|F(Du)\|_{L^1(2B)}^{(q-1)/p} \right)$$

for $C = C(K, N, B)$. Moreover, u satisfies the Euler-Lagrange equation

$$(3.11) \quad \int_{\Omega} (DF(Du), D\varphi) dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

Proof. Let $\text{Argmin}(F) = \{z_0\}$. By considering $\tilde{F}(z) = F(z + z_0) - F(z_0)$ and $\tilde{u}(x) = u(x) - (z_0, x)$ and noting that $f(\cdot)(z_0, \cdot) \in L_{\text{loc}}^1(\Omega)$, it is readily checked that \tilde{u} turns out to be a local minimizer of

$$\tilde{J}(w, \Omega) = \int_{\Omega} \tilde{F}(Dw) + f w dx.$$

Hence, hereafter we suppose $F(z) \geq F(0) = 0$. Finally, recalling Proposition 2.3, point (ii), we know that F is strictly convex and

$$(3.12) \quad C^{-1}|z|^p - C \leq F(z) \leq C(|z|^q + 1), \quad |DF(z)| \leq C(|z|^{q-1} + 1),$$

so that u is the unique minimizer locally, with respect to its own boundary values. We split the proof in several steps.

Step 1 (Approximating problems)

Fix $\varphi \in C_c^\infty(B_1, [0, +\infty[)$ such that $\|\varphi\|_1 = 1$ and let $\varphi_\varepsilon(x) = \varepsilon^{-N} \varphi(x/\varepsilon)$. For $B \Subset \Omega$ and $n \in \mathbb{N}$, let $f_n = f * \varphi_{\frac{1}{n}}$ and, for $\varepsilon_n, \mu_n \rightarrow 0^+$ to be chosen,

$$J_n(w) = \int_B F * \varphi_{\varepsilon_n}(Dw) + \frac{\mu_n}{2} |Dw|^2 + f_n w dx.$$

Set

$$\text{Lip}_\psi(B) = \{w \in \text{Lip}(\overline{B}) : w = \psi \text{ on } \partial B\}.$$

According to [45, Theorem 9.2], there is a solution $v_n \in \text{Lip}_{u * \varphi_{\frac{1}{n}}}(B)$ of

$$J_n(v_n) = \inf\{J_n(w) : w \in \text{Lip}_{u * \varphi_{\frac{1}{n}}}(B)\},$$

since $u * \varphi_{\frac{1}{n}}$ is smooth on ∂B (thus satisfying the Bounded Slope Condition) and also f_n is smooth. Moreover, for any *fixed* n there is no Lavrentiev gap for J_n ; see [12, p. 5923]. Hence v_n also solves

$$J_n(v_n) = \inf\{J_n(w) : w \in u * \varphi_{\frac{1}{n}} + W_0^{1,p}(B)\}.$$

Step 2 (Determining the parameters)

For any choice of ε_n , the integrand

$$F_n(z) := F * \varphi_{\varepsilon_n}(z) + \frac{\mu_n}{2} |z|^2$$

is μ_n -uniformly convex, hence [11, Theorem 4.1] ensures the existence of constants A_n (depending only on B and $\|f_n\|_\infty$, as well as on the regularity of $u * \varphi_{\frac{1}{n}}$, but not on ε_n , μ_n), such that

$$(3.13) \quad \text{Lip}(v_n) \leq \frac{A_n}{\mu_n} =: L_n.$$

We first choose $\mu_n \downarrow 0$ in such a way that

$$(3.14) \quad \lim_n \mu_n \int_B |Du * \varphi_{\frac{1}{n}}|^2 dx = 0, \quad \lim_n \mu_n^{p-1} A_n^{2-p} = 0,$$

determining a bound on L_n independent of ε_n .

In order to choose ε_n , let

$$M_n = 1 + \sup_B |Du * \varphi_{\frac{1}{n}}| + L_n,$$

which is independent of ε_n . Since $\varphi_\varepsilon * F \rightarrow F$ in $C_{\text{loc}}^1(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$, we can pick $(\varepsilon_n) \subseteq (0, 1)$, $\varepsilon_n \downarrow 0$, so that

$$(3.15) \quad \|F * \varphi_{\varepsilon_n} - F\|_{C^1(B_{M_n})} \leq \frac{1}{n}.$$

Clearly, it still holds

$$(3.16) \quad F_n \rightarrow F \quad \text{in } C_{\text{loc}}^1(\mathbb{R}^N).$$

Step 3 (The limsup inequality)

Testing the minimality of v_n against the admissible function $u * \varphi_{\frac{1}{n}}$ gives

$$J_n(v_n) \leq J_n(u * \varphi_{\frac{1}{n}}).$$

Owing to $u * \varphi_{\frac{1}{n}} \rightarrow u$ in $W^{1,p}(B)$ and (3.14), one has

$$(3.17) \quad \overline{\lim}_n J_n(u * \varphi_{\frac{1}{n}}) = \int_B f u dx + \overline{\lim}_n \int_B F * \varphi_{\varepsilon_n}(Du * \varphi_{\frac{1}{n}}) dx.$$

To estimate the last integral, we use (3.15) to get

$$\int_B F * \varphi_{\varepsilon_n}(Du * \varphi_{\frac{1}{n}}) dx \leq \frac{|B|}{n} + \int_B F(Du * \varphi_{\frac{1}{n}}) dx.$$

The vector-valued Jensen inequality then leads to

$$\begin{aligned} \overline{\lim}_n \int_B F * \varphi_{\varepsilon_n}(Du * \varphi_{\frac{1}{n}}) dx &\leq \overline{\lim}_n \int_B F(Du * \varphi_{\frac{1}{n}}) dx \\ &\leq \overline{\lim}_n \int_{(1+1/n)B} \varphi_{\frac{1}{n}} * F(Du) dx \\ &= \int_B F(Du) dx. \end{aligned}$$

Inserting the latter into (3.17) we conclude

$$(3.18) \quad \overline{\lim}_n J_n(v_n) \leq J(u).$$

Step 4 (Convergence of (v_n) to u)

From (3.18) we have that $(J_n(v_n))$ is bounded. By Jensen's inequality $F \leq F * \varphi_{\varepsilon_n}$ so that, through (3.12), for some constant $C = C(N, F, B) > 0$ we have

$$\begin{aligned} J_n(v_n) &\geq \int_B F(Dv_n) dx + \int_B f_n v_n dx \\ &\geq \frac{\|Dv_n\|_{L^p(B)}^p}{C} - C - \|f_n\|_{W^{-1,p'}(B)} \|D(v_n - u * \varphi_{\frac{1}{n}})\|_{L^p(B)} + \int_B f_n u * \varphi_{\frac{1}{n}} dx \\ &\geq \frac{\|Dv_n\|_{L^p(B)}^p}{C} - C - \|f_n\|_{W^{-1,p'}(B)} (\|Dv_n\|_{L^p(B)} + \|Du * \varphi_{\frac{1}{n}}\|_{L^p(B)}) + \int_B f_n u * \varphi_{\frac{1}{n}} dx. \end{aligned}$$

Since $f_n \rightarrow f$ in $W^{-1,p'}(B)$ and $u * \varphi_{1/n} \rightarrow u$ in $W^{1,p}(B)$, by (3.18) we deduce that (Dv_n) is bounded in $L^p(B)$. Moreover, by Poincaré's inequality,

$$\begin{aligned} \|v_n\|_{L^p(B)} &\leq \|v_n - u * \varphi_{\frac{1}{n}}\|_{L^p(B)} + \|u * \varphi_{\frac{1}{n}}\|_{L^p(B)} \\ &\leq C (\|D(v_n - u * \varphi_{\frac{1}{n}})\|_{L^p(B)} + \|u * \varphi_{\frac{1}{n}}\|_{L^p(B)}) \\ &\leq C (\|Dv_n\|_{L^p(B)} + \|u * \varphi_{\frac{1}{n}}\|_{W^{1,p}(B)}), \end{aligned}$$

so that (v_n) is bounded in $L^p(B)$ as well. Therefore the sequence (v_n) is bounded in $W^{1,p}(B)$, and hence possesses a (not relabeled) subsequence weakly convergent to some $v \in W^{1,p}(B)$; actually, it is readily checked that $v \in u + W_0^{1,p}(B)$. We claim that

$$\lim_n \mu_n \int_B |Dv_n|^2 dx = 0.$$

Indeed, this is obvious by Hölder's inequality when $p \geq 2$, while if $p < 2$ we use (3.13) and (3.14) to infer

$$\mu_n \int_B |Dv_n|^2 dx \leq \mu_n L_n^{2-p} \int_B |Dv_n|^p dx \leq \mu_n^{p-1} A_n^{2-p} \int_B |Dv_n|^p dx \rightarrow 0,$$

where we used the boundedness of (Dv_n) in $L^p(B)$. Thus

$$(3.19) \quad \lim_n \int_B \frac{\mu_n}{2} |Dv_n|^2 + f_n v_n dx = \int_B f v dx.$$

The functional

$$w \mapsto \int_B F(Dw) dx$$

is weakly lower semicontinuous in $W^{1,p}(B)$, whence (again by the Jensen inequality)

$$(3.20) \quad \begin{aligned} J(v) &\leq \underline{\lim}_n \left[\int_B F(Dv_n) dx + \int_B f v_n dx \right] \\ &\leq \underline{\lim}_n \left[\int_B F * \varphi_{\varepsilon_n}(Dv_n) dx + \frac{\mu_n}{2} \int_B |Dv_n|^2 + f_n v_n dx \right] = \underline{\lim}_n J_n(v_n). \end{aligned}$$

Coupling the latter with (3.18) we get $J(v) \leq J(u)$, implying $v = u$ by the strict convexity of J . In particular we obtain, up to subsequences,

$$(3.21) \quad Dv_n \rightharpoonup Du \quad \text{in } L^p(B),$$

and from (3.20), (3.18) we infer $J_n(v_n) \rightarrow J(u)$. Subtracting (3.19) we get

$$\int_B F * \varphi_{\varepsilon_n}(Dv_n) \rightarrow \int_B F(Du)$$

which, thanks to (3.15), implies

$$(3.22) \quad \int_B F(Dv_n) \rightarrow \int_B F(Du).$$

Step 5 (Uniform Sobolev bound on $DF(Dv_n)$)

By Proposition 2.3, point (iii), and the beginning of the proof of Theorem 3.2, F_n fulfills (1.9) with the same constant K ; since $F_n \in C^3(\mathbb{R}^N)$, (1.9) actually holds everywhere. Moreover, standard regularity theory ensures that $v_n \in C^2(B)$, so we can apply Theorem 3.2, and in particular (3.4), to obtain, for any $\theta \in (0, 2]$ and $r = 1/2, 1$,

$$(3.23) \quad \|DF_n(Dv_n)\|_{W^{1,2}(\frac{r}{2}B)} \leq C (\|f_n\|_{L^2(rB)} + \|DF_n(Dv_n)\|_{L^\theta(rB)}).$$

The first term on the right is clearly bounded by a multiple of $\|f\|_{L^2(B)}$. For the second one, we let p, q be given in (3.12) and choose $\theta = \min\{p/(q-1), 1\}$. By (3.12) we get

$$(3.24) \quad |DF(z)|^\theta \leq C (|z|^{\theta(q-1)} + 1) \leq C (|z|^p + 1)^{\theta(q-1)/p} \leq C (F(z) + 1)^{\theta(q-1)/p}.$$

Using (3.15) and (3.24) we obtain

$$(3.25) \quad \begin{aligned} \int_B |DF_n(Dv_n)|^\theta dx &\leq \int_B |DF(Dv_n)|^\theta dx + n^{-\theta} |B| + \mu_n^\theta \int_B |Dv_n|^\theta dx \\ &\leq C \int_B (F(Dv_n) + 1)^{\theta(q-1)/p} dx + n^{-\theta} |B| + \mu_n^\theta \|Dv_n\|_{L^p(B)}^\theta |B|^{1-\theta/p} \\ &\leq C |B|^{1-\theta(q-1)/p} \left(\int_B (F(Dv_n) + 1) dx \right)^{\theta(q-1)/p} + n^{-\theta} |B| + \mu_n^\theta \|Dv_n\|_{L^p(B)}^\theta |B|^{1-\theta/p}. \end{aligned}$$

The first integral is bounded by (3.22) and the remaining terms vanish, so that

$$(3.26) \quad \overline{\lim}_n \|DF_n(Dv_n)\|_{L^\theta(B)} \leq C \left(\int_B (F(Du) + 1) dx \right)^{(q-1)/p}.$$

Thanks to (3.23) for $r = 1$, (3.26) implies the Sobolev bound

$$(3.27) \quad \overline{\lim}_n \|DF_n(Dv_n)\|_{W^{1,2}(\frac{1}{2}B)} \leq C.$$

Step 6 (Passage to the limit)

Let $B' = \frac{1}{2}B$ and

$$V_n = DF_n(Dv_n).$$

Thanks to (3.27), (V_n) is bounded in $W^{1,2}(B')$, hence we can pick a subsequence satisfying

$$(3.28) \quad V_n \rightarrow V \quad \text{weakly in } W^{1,2}(B'), \text{ strongly in } L^2(B'), \text{ and pointwise a. e. in } B',$$

for a suitable $V \in W^{1,2}(B')$.

Each F_n is strictly convex and superlinear by construction, hence DF_n is a homeomorphism of \mathbb{R}^N . Moreover, by (3.16) we know that $DF_n \rightarrow DF$ locally uniformly. According to a theorem by Arens (see [29] for a modern exposition), this implies that $DF_n^{-1} \rightarrow DF^{-1}$ locally uniformly. Since $V_n \rightarrow V$ pointwise a. e., we infer that

$$Dv_n = DF_n^{-1}(V_n) \rightarrow DF^{-1}(V) \quad \text{pointwise a. e.,}$$

which, coupled with (3.21), allows the identification $Du = DF^{-1}(V)$. Therefore, $V_n \rightarrow DF(Du)$ in B' in all the senses prescribed in (3.28).

Let $B'' = \frac{1}{4}B$. By Lebesgue's theorem $\|V_n\|_{L^\theta(B')} \rightarrow \|V\|_{L^\theta(B')}$ hence, exploiting also the lower semi-continuity of the $W^{1,2}(B'')$ norm, we can pass to the limit in (3.23) with $r = 1/2$, yielding to (3.9). Estimate (3.10) similarly follows from (3.26). Finally, the validity of (3.11) can be checked only on balls B such that $4B \subseteq \Omega$, by a standard partition of unity argument. If B is such a ball, we can pass to the limit in the Euler Lagrange equations for the approximating problems constructed as before in $4B$, and since $DF_n(Dv_n) \rightarrow DF(Du)$, $f_n \rightarrow f$ strongly in $L^2(B)$, we get (3.11). \square

Remark 3.4. The previous theorem has an immediate consequence. Given a local minimizer of a convex functional of (p, q) growth, the a-priori regularity on Du is just $Du \in L_{\text{loc}}^p(\Omega)$, and the first step towards higher regularity is showing that actually $Du \in L_{\text{loc}}^q(\Omega)$. For general autonomous functionals of (p, q) -growth with $p \geq 2$, this is to be expected only when $q < p(N+2)/N$; see [33, 30]. For the class of q. u. c. integrands it holds $|DF(z)| \lesssim |z|^{p-1}$ so that, from $DF(Du) \in W_{\text{loc}}^{1,2}(\Omega)$, we infer by Sobolev's embedding that $DF(Du) \in L_{\text{loc}}^{2^*}(\Omega)$. Consequently $Du \in L_{\text{loc}}^{2^*(p-1)}(\Omega)$, hence for such class of integrands we obtain the larger range $q \leq 2^*(p-1)$. This range may not be optimal in the q. u. c. case, but on one hand it shows the advantages of considering the stress field instead of (1.5), while on the other hand it holds for any $f \in L_{\text{loc}}^2(\Omega) \cap W^{-1,p'}(\Omega)$. It is quite possible, in light of the results of [10], that, for f having a sufficiently high degree of summability, minimizers for q. u. c. integrands are automatically Lipschitz continuous, regardless of the largeness of the ratio q/p . This is actually the case for functionals with Uhlenbeck structure, see [19].

3.3. Examples.

Example 3.5 (On the assumption $F \in W^{2,1}$). In this example we show that, in order to obtain Sobolev regularity of $DF(Du)$, it is not sufficient to require that condition (1.9) holds at almost every point, but that Sobolev regularity of DF is a necessary assumption.

Let $N = 2$. For any ball $B \Subset \{(x, y) \in \mathbb{R}^2 : x > 0\}$, consider the smooth function

$$u(x, y) = \arctan(y/x).$$

We claim that, for any (not necessarily convex) C^2 function $F : \mathbb{R} \rightarrow \mathbb{R}$, u solves

$$(3.29) \quad \operatorname{div} DF(|Du|) = 0 \quad \text{in } B,$$

where we set $F(z) = F(|z|)$. Letting $z = (x, y)$, $z^\perp = (-y, x)$, it holds

$$Du(z) = \frac{z^\perp}{|z|^2}, \quad D^2u(z) = \frac{1}{|z|^4} \begin{pmatrix} 2xy & y^2 - x^2 \\ y^2 - x^2 & -2xy \end{pmatrix},$$

while

$$DF(|w|) = F'(|w|) \frac{w}{|w|}, \quad D^2F(|w|) = F''(|w|) \frac{w}{|w|} \otimes \frac{w}{|w|} + \frac{F'(|w|)}{|w|} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right).$$

An elementary computation then yields

$$D^2F(|Du|) D^2u(z) = \frac{1}{|z|^4} \begin{pmatrix} xy (F''(\frac{1}{|z|}) + F'(\frac{1}{|z|}) |z|) & F''(\frac{1}{|z|}) y^2 - F'(\frac{1}{|z|}) |z| x^2 \\ F'(\frac{1}{|z|}) |z| y^2 - F''(\frac{1}{|z|}) x^2 & -xy (F''(\frac{1}{|z|}) + F'(\frac{1}{|z|}) |z|) \end{pmatrix}$$

which has zero trace, proving the claim.

Now, let $h : [0, 1] \rightarrow [0, 1]$ denote the Cantor function. By abuse of notation, we still denote by h its extension to the whole \mathbb{R} defined as

$$h(t) = k + h(t - k) \quad \text{if } k \leq t < k + 1, \quad k \in \mathbb{Z},$$

and we also denote by \mathcal{C} the periodic extension of the Cantor set to the whole \mathbb{R} . Consider

$$F(|w|) := \frac{|w|^2}{2} + H(|w|), \quad H(t) := \int_0^t h(\tau) d\tau,$$

which is a strictly convex C^1 function with quadratic growth. Clearly, F can be approximated in C^1 by a sequence $\{F_n\}$ of smooth radial functions, so that we can pass to the limit into the corresponding weak formulations of (3.29) to obtain that u solves

$$\operatorname{div} DF(|Du|) = 0 \quad \text{weakly in } B.$$

However,

$$DF(|Du|(z)) = \frac{z^\perp}{|z|^2} + h(|z|^{-1}) \frac{z^\perp}{|z|}$$

is not even absolutely continuous in B , since its distributional derivative has a Cantor part concentrated on $\{z : 1/|z| \in \mathcal{C}\}$, which still has measure zero.

Since $h'(t) = 0$ in the classical sense for a. e. $t \in \mathbb{R}$, it is readily verified that F obeys (1.9) with $K = 1$ at every point of

$$\mathbb{R}^N \setminus \mathcal{C}_{\text{rad}}, \quad \mathcal{C}_{\text{rad}} = \{z \in \mathbb{R}^N : |z| \in \mathcal{C}\},$$

thus almost everywhere. Notice that DF indeed is not in $W_{\text{loc}}^{1,1}(\mathbb{R}^N)$, since it has a Cantor part concentrated on \mathcal{C}_{rad} .

Example 3.6 (Uhlenbeck structure). For divergence form equations having the Uhlenbeck structure

$$(3.30) \quad \operatorname{div}(a(|Du|) Du) = f$$

we can define F by

$$F(z) = \int_0^{|z|} t a(t) dt$$

to get $DF(z) = a(|z|) z$. If $a \in C^1(0, +\infty)$, it holds

$$D^2F(z) = a(|z|) I + |z| a'(|z|) \frac{z}{|z|} \otimes \frac{z}{|z|},$$

possessing the eigenvector $z/|z|$ with eigenvalue $a(|z|) + |z| a'(|z|)$, while its orthogonal eigenspace is relative to the unique eigenvalue $a(|z|)$. The equation is elliptic if and only if both eigenvalues are non-negative, and in order to bound the ratio between them we look at

$$K = \sup_{t>0} \max \left\{ \frac{a(t)}{a(t) + t a'(t)}, \frac{a(t) + t a'(t)}{a(t)} \right\}.$$

It is readily checked that if

$$i_a = \inf_{t>0} \frac{t a'(t)}{a(t)}, \quad s_a = \sup_{t>0} \frac{t a'(t)}{a(t)},$$

then

$$K = \max \left\{ \frac{1}{1 + i_a}, 1 + s_a \right\},$$

so that the q. u. convexity condition (1.9) is equivalent to the common requirement

$$(3.31) \quad -1 < i_a \leq s_a < \infty,$$

which is the one used, e. g., in [21]. Notice that when $a(z) = |z|^{p-2}$, which corresponds to the p -Poisson equation, we have $i_a = s_a = p - 2$, so that $K = \max\{p - 1, \frac{1}{p-1}\}$.

When f is sufficiently smooth (say, bounded), equation (3.30) can be seen as the Euler-Lagrange equation for the functional (1.2), which is strictly convex. Moreover, it holds (see [19, Proposition 2.15])

$$F(z) \geq a(1) \frac{|z|^{2+i_a}}{2+i_a}, \quad |z| \geq 1,$$

so that J is also coercive on $W^{1,2+i_a}(\Omega)$ when supplemented with reasonable boundary conditions. Since $2 + i_a > 1$ by (3.31), the variational treatment of (3.30) in standard Sobolev spaces is thus justified if one is not looking for optimal rearrangement invariant estimates.

Example 3.7 (Anisotropic examples). In [23, 3, 26, 25] anisotropic equations whose principal part arises as the Euler-Lagrange equation of

$$\int_{\Omega} G(H(Du)) \, dx$$

are considered, where $H \in C^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R}_+)$ is a convex, positively 1-homogeneous function and $G \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing, strictly convex function of p -growth. Clearly, H is fully determined by the unit “ball”

$$B_H = \{z \in \mathbb{R}^N : H(z) < 1\} \ni 0,$$

and any open, bounded, convex B with $0 \in B$ will uniquely determine such an H through its Minkowski functional. Notice that in general H may not even be a norm, due to the possible lack of symmetry of B_H .

The more general ellipticity assumption in the cited works reads as follows: H is said to be *uniformly convex* if the principal curvatures of ∂B_H are bounded from below by a positive constant. From [26, Appendix A], the uniform ellipticity of H amounts to

$$(H(z) D^2 H(z) \eta, \eta) \geq \delta |\eta|^2 \quad \forall z \in \mathbb{R}^N \setminus \{0\}, \eta \in DH(z)^\perp,$$

for some $\delta > 0$ (which is actually equivalent to the same type of inequality for $\eta \in z^\perp$). Under this assumption various results can be proved, and in particular the Sobolev regularity of the associated stress field is treated in [23, 3] as a stepping-stone to more general results. Here we show that anisotropic functionals of this kind fall within our general framework of q. u. c. functionals.

To this end, set $F(z) = G(H(z))$ so that

$$D^2 F(z) = G''(H(z)) DH(z) \otimes DH(z) + \frac{G'(H(z))}{H(z)} H(z) D^2 H(z)$$

for $z \neq 0$, and notice that both

$$DH(z) \otimes DH(z) \quad \text{and} \quad H(z) D^2 H(z)$$

are 0-homogeneous. Inspecting the proof of [26, Appendix A], we see that for any $z, \xi \in \mathbb{R}^N \setminus \{0\}$ it either holds

$$(DH(z) \otimes DH(z) \xi, \xi) \geq \lambda_1 |\xi|^2$$

or

$$(H(z) D^2 H(z) \xi, \xi) \geq \lambda_2 |\xi|^2$$

(with $\lambda_i = \lambda_i(H) > 0$), while altogether

$$(H(z) D^2 H(z) \xi, \xi) \leq \Lambda |\xi|^2, \quad (DH(z) \otimes DH(z) \xi, \xi) \leq \Lambda |\xi|^2$$

for some $\Lambda = \Lambda(H)$. It follows that the minimum eigenvalue of $D^2 F(z)$ is bounded from below by

$$\min \{ \lambda_1 G''(H(z)), \lambda_2 G'(H(z))/H(z) \},$$

while its maximum eigenvalue is bounded from above by

$$\Lambda (G''(H(z)) + G'(H(z))/H(z)).$$

Therefore F is K -q. u. c. for

$$K = \frac{\Lambda}{\min\{\lambda_1, \lambda_2\}} \sup_{t \in \mathbb{R}_+} \max \left\{ 1 + \frac{G'(t)}{G''(t)t}, 1 + \frac{G''(t)t}{G'(t)} \right\}.$$

The finiteness of the latter is compatible with the standard Uhlenbeck example: for $a(t) = G'(t)/t$, a straightforward calculation shows that

$$\sup_{t \in \mathbb{R}_+} \max \left\{ \frac{G'(t)}{G''(t)t}, \frac{G''(t)t}{G'(t)} \right\} < +\infty \quad \Leftrightarrow \quad -1 < i_a \leq s_a < +\infty.$$

Example 3.8 (Non-standard anisotropic growth). It is straightforward to check that if F_1 and F_2 are q. u. c. C^1 functions fulfilling (1.9) with constant K_1 and K_2 respectively, then $F_1 + F_2$ is $\max\{K_1, K_2\}$ -q. u. c.. This observation allows to consider anisotropic Euler-Lagrange equations arising from integrands of the form

$$F(z) = \sum_{i=1}^M |A_i(z - z_i)|^{p_i}, \quad z_i \in \mathbb{R}^N, \quad A_i \geq I, \quad p_i > 1 \quad \text{for } i = 1, \dots, M$$

As a more common example of anisotropic functionals, consider the weak solution of the Dirichlet problem

$$\begin{cases} \Delta_p u + D_1(|D_1 u|^{q-2} D_1 u) = f & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

for a ball B , $f \in L^\infty(B)$ and $q > p > 2$. This corresponds to the unique minimizer $u \in W_0^{1,p}(B)$ of

$$J(w) = \int_B \frac{1}{p} |Dw|^p + \frac{1}{q} |D_1 w|^q + f w \, dx,$$

and it is globally Lipschitz continuous, since

$$F(z) = \frac{1}{p} |z|^p + \frac{1}{q} |z_1|^q$$

is uniformly p -convex outside B_1 (see [11]), i. e.,

$$(D^2 F(z) \xi, \xi) \geq c_p (1 + |z|^2)^{\frac{p-2}{2}} |\xi|^2, \quad |z| \geq 1.$$

Moreover,

$$D^2 F(z) = |z|^{p-2} \left(I + (p-2) \frac{z}{|z|} \otimes \frac{z}{|z|} \right) + (q-1) |z_1|^{q-2} e_1 \otimes e_1,$$

so that the minimum and maximum eigenvalues of $D^2 F(z)$ fulfill

$$\lambda_{\min}(z) \geq |z|^{p-2}, \quad \lambda_{\max}(z) \leq (p-1) |z|^{p-2} + (q-1) |z|^{q-2}.$$

On the range of Du we have $|z|^{q-2} \leq C |z|^{p-2}$ for C depending on $\text{Lip } u$, leading to

$$\lambda_{\max}(z) / \lambda_{\min}(z) \leq p-1 + C(q-1),$$

so Theorem 3.3 applies thanks to the local nature of (1.9) described in Lemma 2.4.

Notice the relevance of the assumption $f \in L^\infty(B)$ (which could be weakened, but not down to $L^2(B)$; see [10]) in order to obtain Lipschitz continuity.

4. APPLICATIONS

4.1. Cordes-type conditions. We start with a generalization of (2.2). In this section, given a matrix field $M : \Omega \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$, its L^m -norm will be computed with respect to the Frobenius norm of $M = (m_{ij})$, i. e.,

$$\|M\|_m = \left(\int |M(x)|_2^m dx \right)^{1/m}, \quad |M(x)|_2^2 = \sum_{i,j=1}^N |m_{ij}(x)|^2.$$

Lemma 4.1. *Let $V \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$, $m > 1$. Then, for $m^* - 1 = \max\{m - 1, 1/(m - 1)\}$,*

$$(4.1) \quad \|DV\|_m \leq N^2 (m^* - 1) \left(\|\operatorname{div} V\|_m + \|\operatorname{curl} V\|_m \right).$$

Proof. Let R_j be the Riesz transform, defined as the Fourier multiplier with symbol $-i \xi_j / |\xi|$. Then it holds

$$(4.2) \quad D_h V^k = -R_h R_k \operatorname{div} V - \sum_{j=1}^N R_h R_j \operatorname{curl}_{kj} V,$$

which follows, taking the Fourier transform, from the identity

$$i \xi_h \hat{V}^k = \frac{\xi_h \xi_k}{|\xi|^2} \sum_{j=1}^N i \xi_j \hat{V}^j + \sum_{j=1}^N \frac{\xi_h \xi_j}{|\xi|^2} \left(i \xi_j \hat{V}^k - i \xi_k \hat{V}^j \right).$$

The second order Riesz transform has $L^m - L^m$ norm $m^* - 1$ on diagonal terms and $(m^* - 1)/2$ on off-diagonal terms, i. e.,

$$\begin{aligned} \|R_h R_k f\|_m &\leq \frac{m^* - 1}{2} \|f\|_m, & h \neq k, \\ \|R_h^2 f\|_m &\leq (m^* - 1) \|f\|_m \end{aligned}$$

(see [9, Theorem 2.4]). Thus from (4.2) we get

$$\|D_h V^k\|_m \leq (m^* - 1) \left(\|\operatorname{div} V\|_m + \sum_{j=1}^N \|\operatorname{curl}_{kj} V\|_m \right).$$

We sum over $k = 1, \dots, N$ and use the Hölder inequality to get

$$\sum_{k,j=1}^N \|\operatorname{curl}_{kj} V\|_m \leq N^{2(1-\frac{1}{m})} \left(\int \sum_{k,j=1}^N |\operatorname{curl}_{kj} V|^m \right)^{1/m}.$$

Since

$$\sum_{k,j=1}^N |\operatorname{curl}_{kj} V|^m \leq N^{2-m} \left(\sum_{k,j=1}^N |\operatorname{curl}_{kj} V|^2 \right)^{m/2}$$

we obtain

$$\sum_{k,j=1}^N \|\operatorname{curl}_{kj} V\|_m \leq N \|\operatorname{curl} V\|_m.$$

Summing also over $h = 1, \dots, N$ we finally get

$$\|DV\|_m \leq \sum_{h,k=1}^N \|D_h V^k\|_m \leq (m^* - 1) N^2 \left(\|\operatorname{div} V\|_m + \|\operatorname{curl} V\|_m \right).$$

□

Remark 4.2. Estimate (4.1) is very rough in its dependence on N . It is a common feature of L^m -bounds on Riesz transform that they do not depend on the dimension of the euclidean space. Up to our knowledge, optimal L^m estimates for the operator $D \operatorname{curl}^{-1}$ (let alone for the resolvent operator of the div – curl system) are not known.

Theorem 4.3. *Let F obey the assumptions of Theorem 3.3, in particular (1.9) with a constant $K \geq 1$ and p, q given in Proposition 2.3. Let furthermore $m > 1$ and $f \in L^m(\Omega) \cap W^{-1,p'}(\Omega)$. Then any local minimizer $u \in W_{\operatorname{loc}}^{1,p}(\Omega)$ for J given in (3.8) is such that $DF(Du) \in W_{\operatorname{loc}}^{1,m}(\Omega)$ and fulfills the estimate*

$$(4.3) \quad \|DF(Du)\|_{W^{1,m}(B)} \leq C \left(\|f\|_{L^m(2B)} + \|DF(Du)\|_{L^m(2B)} \right)$$

in each of the following cases:

- (1) $K \leq K_0$, with $K_0 > 1$ depending on N and m , with $C = C(N, m, B)$.
- (2) $|m - 2| \leq \delta_0$, for $\delta_0 > 0$ depending on N and K , with $C = C(N, K, B)$.

Proof. Given B such that $4B \Subset \Omega$, we follow the first four steps of the proof of Theorem 3.3, to find $u_n \in C^\infty(B)$, $f_n \in C^\infty(B)$, $F_n \in C^\infty(\mathbb{R}^N)$ and $v_n \in C^2(B)$ such that

- (i) $u_n \rightarrow u$ in $W^{1,p}(B)$, $f_n \rightarrow f$ in $L^m(B) \cap W^{-1,p'}(B)$;
- (ii) F_n is K -q. u. c.;
- (iii) $v_n \rightarrow u$ in $W^{1,p}(B)$, $v_n \rightarrow u$ in $L^p(B)$, and v_n solves

$$\operatorname{div}(DF_n(Dv_n)) = f_n;$$

- (iv) $\|F_n(Dv_n)\|_{L^1(B)} \rightarrow \|F(Du)\|_{L^1(B)}$.

We then proceed as in Theorem 3.2, in order to find a uniform bound for $DF_n(Dv_n)$ in $W^{1,m}(B')$, $B' = \frac{1}{2}B$. To simplify the notation, we omit for the moment the dependence of v , F , and f on n .

Let $V = DF(Dv)$ and observe that the assumptions of Lemma 3.1 hold true for the matrix $X = DV = D^2F(Dv) D^2v$; hence, pointwise,

$$(4.4) \quad |\operatorname{curl} V|_2 \leq \sqrt{2} \mathbf{e}(K) |DV|_2, \quad \mathbf{e}(K) = 1 - \frac{1}{K}.$$

Suppose $B = B_{2R}$ and, for any $R \leq r < s \leq 2R$, fix $\varphi \in C_c^\infty(B_s; [0, 1])$ as in (3.6).

We split the proof in two cases, according to the situations under consideration in the two statements. For the first assertion, we apply (4.1) to the field $W := \varphi V$ to get

$$\begin{aligned} \|\varphi DV\|_m &\leq \|DW\|_m + \|D\varphi \otimes V\|_m \\ &\leq N^2 (m^* - 1) (\|\operatorname{div} W\|_m + \|\operatorname{curl} W\|_m) + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)} \end{aligned}$$

for any $m > 1$. From

$$\operatorname{div} W = \varphi f + (V, D\varphi), \quad \operatorname{curl} W = \varphi \operatorname{curl} V + V \wedge D\varphi$$

we thus infer

$$(4.5) \quad \|\varphi DV\|_m \leq N^2 (m^* - 1) (\|\varphi \operatorname{curl} V\|_m + \|\varphi f\|_m) + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)}.$$

We then let $K_0 = K_0(N, m) > 1$ satisfying

$$(4.6) \quad \sqrt{2} N^2 (m^* - 1) \mathbf{e}(K_0) < 1$$

so that, if $K \leq K_0$, the curl term in (4.5) can be reabsorbed on the left. Thanks to the properties (3.6) of φ we deduce

$$(4.7) \quad \|DV\|_{L^m(B_r)} \leq C \|f\|_{L^m(B_R)} + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)}, \quad R \leq r < s \leq 2R,$$

for a constant $C = C(N, m)$.

Regarding the second assertion, we consider the linear operator $T(f, G) = DV$, where V solves

$$\begin{cases} \operatorname{div} V = f, \\ \operatorname{curl} V = \sqrt{2} G, \end{cases} \quad f \in C_c^\infty(\mathbb{R}^N), \quad G \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N \wedge \mathbb{R}^N),$$

which, as already noted, is represented in terms of Riesz transforms as

$$T(f, G)^{kh} = -R_h R_k f - \sqrt{2} \sum_{j=1}^N R_h R_j G^{kj}.$$

Estimate (4.1) implies that T has an extension $T : X_m \rightarrow Y_m$, where

$$X_m := L^m(\mathbb{R}^N) \times L^m(\mathbb{R}^N; \mathbb{R}^N \wedge \mathbb{R}^N), \quad Y_m := L^m(\mathbb{R}^N; \mathbb{R}^N \otimes \mathbb{R}^N),$$

with the norms

$$\|(f, G)\|_m = \left(\int |f|^m + |G|_2^m dx \right)^{1/m}, \quad \|M\|_m = \left(\int |M|_2^m dx \right)^{1/m}$$

on X_m and Y_m , respectively. For the complex interpolation spaces it holds

$$[X_{m_1}, X_{m_2}]_\theta = X_m, \quad \frac{1}{m} = \frac{1-\theta}{m_1} + \frac{\theta}{m_2}, \quad \theta \in [0, 1],$$

with equality of norms, and the same holds for the Y_m 's. On the other hand, Lemma 2.2 ensures that, with respect to these norms,

$$\|T\|_{\mathcal{L}(X_2, Y_2)} = 1.$$

Fix $\bar{m}' < 2 < \bar{m}$. The Riesz-Thorin interpolation theorem yields

$$\|T\|_{\mathcal{L}(X_m, Y_m)} \leq \|T\|_{\mathcal{L}(X_{\bar{m}}, Y_{\bar{m}})}^\theta, \quad \frac{1}{m} = \frac{1-\theta}{2} + \frac{\theta}{\bar{m}},$$

for any $2 \leq m \leq \bar{m}$, and a similar estimate holds also for $\bar{m}' \leq m \leq 2$. We infer that there exist $\eta : [\bar{m}', \bar{m}] \rightarrow [0, +\infty)$ such that

$$(4.8) \quad \lim_{m \rightarrow 2} \eta(m) = 0$$

and for all $m \in [\bar{m}', \bar{m}]$ it holds

$$\|T\|_{\mathcal{L}(X_m, Y_m)} \leq 1 + \eta(m).$$

To complete the choice of δ_0 in the second case, we proceed as in the proof of (4.5), setting $W = \varphi V$ and using the previous operator norm estimate, to get, for $\eta = \eta(m)$,

$$\begin{aligned} \|\varphi DV\|_m &\leq \|DW\|_m + \|V \otimes D\varphi\|_m \\ &\leq \|T(\operatorname{div} W, 2^{-1/2} \operatorname{curl} W)\|_m + \|V \otimes D\varphi\|_m \\ &\leq (1 + \eta) \|(\operatorname{div} W, 2^{-1/2} \operatorname{curl} W)\|_m + \|V \otimes D\varphi\|_m \\ &\leq (1 + \eta) \left(\|(\varphi f, \varphi 2^{-1/2} \operatorname{curl} V)\|_m + \|((V, D\varphi), V \wedge D\varphi)\|_m \right) + \|V \otimes D\varphi\|_m \\ &\leq \frac{1 + \eta}{\sqrt{2}} \|\varphi \operatorname{curl} V\|_m + C \|\varphi f\|_m + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)} \\ &\stackrel{(4.4)}{\leq} (1 + \eta) \mathbf{e}(K) \|\varphi DV\|_m + C \|f\|_{L^m(B_R)} + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)}. \end{aligned}$$

Since $\mathbf{e}(K) < 1$, thanks to (4.8) we can choose $\delta_0 = \delta_0(K, N)$ in such a way that

$$(1 + \eta(m)) \mathbf{e}(K) < 1, \quad \forall m \in [2 - \delta_0, 2 + \delta_0],$$

which again gives estimate (4.7) with a constant $C = C(N, K)$.

Proceeding as in the proof of Theorem 3.2, we see that estimate (4.7) improves to

$$(4.9) \quad \|V\|_{W^{1,m}(B_R)} \leq C \|f\|_{L^m(B_{2R})} + C_R \|V\|_{L^\theta(B_{2R})}, \quad \theta \in (0, m],$$

which therefore holds uniformly for all $V_n = DF_n(Dv_n)$ constructed at the beginning. If p, q are given in Proposition 2.3, point (ii), we set $\theta = \min\{p/(q-1), m\}$ and proceed as in (3.25) to get a uniform bound on $\|V_n\|_{L^\theta(B_{2R})}$. Thanks to (4.9), the latter in turn implies the compactness of the V_n in $L^m(B_R)$. The rest of the proof of Theorem 3.3 follows *verbatim*, providing estimate (4.3) in both the stated cases. We omit the details. \square

4.2. On the $C^{p'}$ conjecture. An immediate corollary of the Cordes condition proved in the previous section is the following one.

Corollary 4.4. *Any weak solution $u \in W_{\text{loc}}^{1,p}(\Omega)$ of $\Delta_p u = f \in L_{\text{loc}}^\infty(\Omega)$ belongs to $C^{2-\alpha}(\Omega)$, where $\alpha = \alpha(N, p) \leq C(N) |p - 2|$, provided p is sufficiently near 2.*

Proof. Recall that $|z|^p$ is K -q. u. c. with constant

$$K_p = \max\{p - 1, 1/(p - 1)\} = \begin{cases} p - 1 & \text{if } p \geq 2 \\ 1/(p - 1) & \text{if } p \in (1, 2). \end{cases}$$

We apply the Cordes estimates of the previous theorem, so that $|Du|^{p-2} Du \in W_{\text{loc}}^{1,m}(\Omega)$ for any $m \geq 2$ such that (see (4.6))

$$(4.10) \quad \sqrt{2} N^2 (m-1) (1 - 1/K_p) < 1.$$

Given p , we let

$$m_p = \frac{1}{2N^2 |p-2|}.$$

It is readily checked that, for all p such that $|p-2| < 1/(2N^3)$, inequality (4.10) holds for m_p and moreover $m_p > N$. By the Morrey embedding we thus have that $|Du|^{p-2} Du \in C^{1-N/m_p}(B)$.

The map $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined as

$$(4.11) \quad \Psi(y) = \begin{cases} |y|^{\frac{2-p}{p-1}} y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

is the inverse of the similarly defined map $z \mapsto z^{p-1} = |z|^{p-2} z$, for which the well-known inequalities

$$(z_1^{p-1} - z_2^{p-1}, z_1 - z_2) \geq \begin{cases} 2^{2-p} |z_1 - z_2|^p & \text{if } p \geq 2 \\ (p-1) |z_1 - z_2|^2 (1 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} & \text{if } 1 < p < 2 \end{cases}$$

hold true. By the Schwartz inequality we deduce

$$|z_1^{p-1} - z_2^{p-1}| \geq \begin{cases} 2^{2-p} |z_1 - z_2|^{p-1} & \text{if } p \geq 2 \\ c_M |z_1 - z_2| & \text{if } 1 < p < 2 \text{ and } |z_1| + |z_2| \leq M, \end{cases}$$

which means that Ψ is globally $1/(p-1)$ -Hölder-continuous if $p \geq 2$ and locally Lipschitz-continuous if $1 < p < 2$. In the second case, we observe that $f \in L_{\text{loc}}^\infty(\Omega)$ implies that $Du \in L_{\text{loc}}^\infty(\Omega)$, hence Ψ is Lipschitz-continuous on the range of Du . In both cases we thus have

$$Du \in C^{\alpha_p}(\Omega), \quad \alpha_p = \begin{cases} \frac{1}{p-1} \left(1 - \frac{N}{m_p}\right) = \frac{1 - 2N^3(p-2)}{p-1} & \text{if } p \geq 2 \\ 1 - 2N^3(2-p) & \text{if } 1 < p < 2, \end{cases}$$

giving the claim. \square

Remark 4.5. A similar conclusion can be drawn for $W_{\text{loc}}^{1,p}(\Omega)$ local minimizers of $J(\cdot, \Omega)$ when F is K -q. u. c. and $f \in L_{\text{loc}}^\infty(\Omega)$. Indeed, DF is K^{N-1} quasiconformal, hence so is DF^{-1} . In particular, DF^{-1} is $1/K$ -Hölder continuous and the α -Hölder regularity of $DF(Du)$ translates to α/K -Hölder regularity for Du . The dependence of the Hölder exponent of Du from K turns out to be $1 - c_N(K-1)$ for K sufficiently near 1.

Consider now a solution u of the inhomogeneous elliptic equations with Uhlenbeck structure

$$(4.12) \quad \text{div}(a(|Du|) Du) = f \in L_{\text{loc}}^m(\Omega), \quad m > 1,$$

where $a : (0, +\infty) \rightarrow (0, +\infty)$ is $C^1(0, +\infty)$ and fulfills the ellipticity condition $-1 < i_a \leq s_a < +\infty$ (see Example 3.6).

Solutions of (4.12) are to be meant in a generalized sense and, as in [21], we will use the notion of *approximable solutions* (or SOLA, solution obtained as limit of approximation) for (4.12): u is an approximable solution for (4.12) in Ω if $a(|Du|) Du \in L^1_{\text{loc}}(\Omega)$, (4.12) holds in the distributional sense in Ω , and there exists a sequence $(f_n) \subseteq C_c^\infty(\Omega)$ and corresponding weak solutions u_n of (4.12) in Ω with right-hand side f_n such that

$$f_n \rightarrow f \quad \text{in } L^m_{\text{loc}}(\Omega), \quad u_n \rightarrow u \quad \text{and} \quad Du_n \rightarrow Du \quad \text{a. e. in } \Omega,$$

and

$$\lim_n \int_{\Omega'} a(|Du_n|) |Du_n| dx = \int_{\Omega'} a(|Du|) |Du| dx$$

for all $\Omega' \Subset \Omega$. Notice that u may fail to belong to $W^{1,1}_{\text{loc}}(\Omega)$, but rather falls into the larger space

$$\mathcal{T}_{\text{loc}}^{1,1}(\Omega) = \left\{ v : T_k v \in W^{1,1}_{\text{loc}}(\Omega) \text{ for all } k > 0 \right\}, \quad T_k v = \max\{-k, \min\{v, k\}\},$$

for which a pointwise notion of Du is well-defined almost everywhere. Whenever $f \in W^{-1,p'}(\Omega)$, weak and SOLA solutions coincide, and in particular u belongs to the relevant Orlicz-Sobolev space. For the existence and uniqueness theory of SOLA we refer to [20].

We say that $u \in \mathcal{T}_{\text{loc}}^{1,1}(\Omega)$ is a cylindrical solution of (4.12) if there exists a point $x_0 \in \mathbb{R}^N$ and a k -dimensional vector subspace $V \subseteq \mathbb{R}^N$ with corresponding orthogonal projection $\pi_V : \mathbb{R}^N \rightarrow V$, such that

$$u(x) = v(|\pi_V(x - x_0)|)$$

for some $v : I \rightarrow \mathbb{R}$ with $I \subseteq [0, +\infty)$ open and $\Omega \subseteq \{x \in V : |x| \in I\} \times V^\perp$.

In order to study the regularity properties of a cylindrical solution of (4.12), we first perform some straightforward reductions. It is clear that we can assume

$$V = \{x \in \mathbb{R}^N : x_i \equiv 0 \ \forall i = k+1, \dots, N\}, \quad \Omega = A \times \mathbb{R}^{N-k}$$

with $A \subseteq \mathbb{R}^k$ invariant by the action of the orthogonal group O_k on \mathbb{R}^k . Actually, by the structure of equation (4.12), we can directly suppose that $k = N$, reducing to the case of radial solutions on a radial (meaning, invariant by O_N) domain Ω .

Theorem 4.6. *Let $a \in C^1((0, +\infty); (0, +\infty))$ fulfill (3.31) and $f \in L^m(\Omega)$ for some $m > 1$, where Ω is a radial domain. If u is a radial approximable solution of*

$$\operatorname{div}(a(|Du|) Du) = f$$

in Ω then, for any $m > 1$ and B_R such that $B_{2R} \Subset \Omega$, it holds

$$(4.13) \quad \|a(|Du|) Du\|_{W^{1,m}(B_{R/2})} \leq C_{m,R} (\|f\|_{L^m(B_{2R})} + \|a(|Du|) Du\|_{L^1(B_{2R})}).$$

Proof. The field $V = a(|Du|) Du \in L^1_{\text{loc}}(\Omega)$ is the pointwise limit of the fields $V_k = a(|DT_k u|) DT_k u$ for $k \rightarrow +\infty$, which satisfy

$$(4.14) \quad T \circ V_k = V_k \circ T \quad \forall T \in O_N, \quad |(V_k(x), x)| = |V(x)| |x|,$$

hence it satisfies the latter too. We extend V and f as zero outside Ω (thus keeping the previous properties for V) and let $V_\varepsilon = V * \varphi_\varepsilon$, $f_\varepsilon = f * \varphi_\varepsilon$, where φ_ε is a standard radial convolution kernel supported in B_ε . We claim that V_ε obeys (4.14). By changing variables and using the radially of φ_ε , it is readily checked that V_ε satisfies $T \circ V_\varepsilon = V_\varepsilon \circ T$ for all $T \in O_N$. Thus, in order to check the second condition in (4.14) it suffices to prove it at a point $x = r e_1$, where

$$V_\varepsilon(r e_1) = \int V(y) \varphi_\varepsilon(|r e_1 - y|) dy.$$

The integrand above is odd with respect to the reflections $y_k \mapsto -y_k$, $k = 2, \dots, N$, which implies that $V_\varepsilon(r e_1)$ is parallel to e_1 , and this concludes the proof of (4.14) for V_ε . It follows that for

$$h_\varepsilon(x) := (V_\varepsilon(x), x/|x|^2) \in C^\infty(\mathbb{R}^N)$$

it holds, with a slight abuse of notation, $h_\varepsilon(x) = h_\varepsilon(|x|)$ and

$$(4.15) \quad V_\varepsilon(x) = h_\varepsilon(|x|) x.$$

Given a radial subdomain $\Omega' \Subset \Omega$ and using Fubini's theorem, we have

$$\begin{aligned} \int_{\Omega'} (V_\varepsilon, D\psi) dx &= \int_{\Omega'} (V, \varphi_\varepsilon * D\psi) dx = \int_{\Omega'} (V, D(\psi * \varphi_\varepsilon)) dx \\ &= - \int_{\Omega'} f \psi * \varphi_\varepsilon dx = - \int_{\Omega'} f_\varepsilon \psi dx, \end{aligned}$$

thus V_ε fulfills $\operatorname{div} V_\varepsilon = f_\varepsilon$ weakly (and thus strongly) in Ω' for all sufficiently small $\varepsilon > 0$. From (4.15) we compute, for $x \neq 0$,

$$DV_\varepsilon(x) = h_\varepsilon(|x|) I + |x| h'_\varepsilon(|x|) \frac{x}{|x|} \otimes \frac{x}{|x|},$$

which is a symmetric matrix, so that

$$\operatorname{curl} V_\varepsilon = 0 \quad \text{in } \Omega'.$$

By the Helmholtz decomposition, for any $B_{2R} \subseteq \Omega'$ and sufficiently small $\varepsilon > 0$, we thus have $V_\varepsilon = Dv_\varepsilon$ for some $v_\varepsilon \in C^2(B_{2R})$, fulfilling weakly $\Delta v_\varepsilon = f_\varepsilon$. For $0 < r < s < 2R$ we choose cut-off functions as in (3.6) and suppose, without loss of generality, that v_ε has zero mean in B_s . By the standard Calderón-Zygmund estimates and Poincaré's inequality, it holds

$$\begin{aligned} \|D^2 v_\varepsilon\|_{L^m(B_r)}^m &\leq C_m \left(\|f_\varepsilon\|_{L^m(B_{2R})}^m + \frac{1}{(s-r)^m} \|Dv_\varepsilon\|_{L^m(B_s)}^m + \frac{1}{(s-r)^{2m}} \|v_\varepsilon\|_{L^m(B_s)}^m \right) \\ &\leq C_m \|f_\varepsilon\|_{L^m(B_{2R})}^m + C_m \left(\frac{1}{(s-r)^m} + \frac{R^m}{(s-r)^{2m}} \right) \|Dv_\varepsilon\|_{L^m(B_s)}^m. \end{aligned}$$

Hence, we can proceed as in the final part of the proof of Theorem 3.2 to improve the latter to

$$\|V_\varepsilon\|_{W^{1,m}(B_{R/2})} \leq C_m \|f_\varepsilon\|_{L^m(B_{2R})} + C_{m,R} \|V_\varepsilon\|_{L^1(B_{2R})}.$$

Since $V_\varepsilon \rightarrow V$ in $L^1(B_R)$, we obtain the claimed estimate (4.13) by lower semicontinuity. \square

Corollary 4.7. *Let $u \in W^{1,p}(B_R)$ be a cylindrical weak solution of*

$$\Delta_p u = f \in L^\infty(B_R).$$

Then

- (1) *If $p \geq 2$, $u \in C_{\text{loc}}^{p'-\varepsilon}(B_R)$ for all $\varepsilon > 0$, and also for $\varepsilon = 0$ if furthermore $f \in C_{\text{Dini}}^0(B_R)$.*
- (2) *If $p \leq 2$, $u \in C_{\text{loc}}^{2-\varepsilon}(B_R)$ for all $\varepsilon > 0$, and also for $\varepsilon = 0$ if furthermore $f \in C_{\text{Dini}}^0(B_R)$.*

Proof. According to Theorem 4.6, the field $V = |Du|^{p-2} Du$ belongs to $W_{\text{loc}}^{1,m}(B_R)$ for any $m > 1$; hence, by the Morrey embedding, it lies in $C^{1-\varepsilon}(B_R)$ for any $\varepsilon > 0$. Similarly, $V \in \text{Lip}_{\text{loc}}(B_R)$ if f is Dini-continuous, being the gradient of a solution of $\Delta v = f \in C_{\text{Dini}}^0(B_R)$. We conclude through the properties of the map Ψ in (4.11), as in the proof of Corollary 4.4. \square

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