A variational approach to the maximization of preferences without numerical representation

Monica Milasi*, Domenico Scopelliti†

Abstract

We introduce a variational approach to study a maximization problem of preferences that cannot be represented by a utility function. In such conditions, we reformulate the problem as a suitable variational problem and we give regularity properties of the solutions map. The theoretical results are applied in studying an equilibrium problem under uncertainty.

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1 Introduction

In decision theory, preferences characterize each individual's attitudes, perceptions, tastes, and inclinations with respect to the alternatives that are the object of choice. Once these are defined, the behavior of individuals considers the preference relation together with any other factor and/or constraint in order to make the best possible decision. This preference is described by means of a binary relation, and the individual does make the best according to it and the constraints in place. In [10], Debreu proved that, under suitable assumptions on the set of alternatives and/or the binary relation, a preference can be represented by means of a real function (see, e.g., Section 2). Hence, the preference maximization problem can be treated as the maximization problem of functions, and the literature contains several well-known approaches to deal with the resulting optimization problem. In particular, the variational inequalities theory provides powerful and flexible tools to deal with the above described class of problems, in both analysis and computations (see, e.g., [1, 9, 21, 27, 34]).
In [22], the authors gave a first example of variational inequality which arises from an optimization problem. Maximizing a concave differentiable function f on a closed convex set is equivalent to solving a variational inequality problem where the operator is the gradient of f. If the function is not differentiable the gradient can be replaced by the supergradient. In the setting of quasiconcave functions, necessary and sufficient conditions include the normal cone to

*Department of Economics, University of Messina.
†Department of Economics, University of Messina; Department of Economics and Managements, University of Pavia.

the superlevel set (see, e.g., [5]). Thanks to this characterization, the variational inequalities theory has been applied to a variety of equilibrium problems on convexity/concavity and quasiconvexity/quasiconcavity assumptions on the resulting utility/payoff functions. It is worth mentioning Nash equilibrium, competitive economic equilibrium problems, traffic equilibrium problems (see, e.g., [4, 14, 15, 16, 30]). In [28] and [29], the authors studied two economic equilibrium problems in which the consumers' preferences are described by a binary relation. In these papers, equilibria are studied using generalized quasi-variational inequalities without representing the preferences by a utility function. In particular, in the first one, the authors considered preferences to be complete, transitive, continuous, non-satiated, and semistrictly convex. In the latter, the assumption of completeness dropped. Hence, in [29], due to the lack of completeness on the considered preferences, a utility representation could not be possible. The present paper aims to provide a variational approach to study a general maximization problem of binary relations where a numerical representation of the preference relations is not given. After a theoretical study with strict preferences, in Section 4 we deal with an equilibrium problem under assumptions weaker than those considered in [28]. Moreover, to solve the variational problem here we propose a different approach to the one considered in [29]. Indeed, instead of solving a quasi-variational problem, we handle variational inequality problems. To this aim, we apply the regularity results shown in Section 3. To the best of our knowledge, there are no studies in which the preferences maximization is studied through a variational approach without numerical representation.

Usually, with preferences there are two different approaches. The first one is to define a weak relation $\succeq$, and then deduce a strict preference $\succ$ (see, e.g., [24]). The other approach instead considers a strict relation $\succ$ from which a weak preference relation is deduced (see, e.g., [20, 23]). Since in our study we use the strict upper counter set, we follow the latter approach. Moreover, we do not require that the strict preference is asymmetric and negatively transitive.

The assumption of completeness means that an individual should be able to compare any two possible alternatives. One can imagine real-life situations in which this assumption does not hold, for instance, when an individual is not able to rank his preferences between two or more choices. For instance, this can occur under uncertain conditions and/or with a lack of information. Furthermore, in many real-world situations, the set of alternatives is vectors, that is, the outcome of a choice involves different features to be evaluated separately. Hence, it is natural to work without the completeness requirement on the individuals' preferences. Indeed, from the second half of the last century, the study of incomplete preferences has attracted the attention of many scholars (see, e.g., [2, 25, 26]) and this interest has grown in recent years (see, e.g., [7, 13, 17, 18, 31, 32]). It was Von Neumann and Morgenstern, in [37], who first raised the question of the completeness assumption as a trait of rationality. Subsequently, Shapley, in [35], dropped the completeness assumption on the players' preferences. In [2], Aumann generalized Shapley's result. More recently, in [6], Bade extended Shapley and Aumann's results. In most of the quoted papers, scalarization ap-

proaches are used; in particular, decision-makers/players have to coordinate a suitable scalarization to get the desirable properties on the scalarized problem. In doing this, some questions can arise. For instance, in [13], the authors remarked the crucial role in the investigation about the stability of the solution with respect to perturbations on the selected scalarization. Instead, we point out that the strength of our variational approach is that it relies only on the study of the strict upper counter set and the associated normal cone.
The paper is organized as follows. Section 2 is devoted to some preliminary definitions and tools to deal with preference relations. Section 3 is dedicated to studying a maximization problem of a preference relation by using a variational approach. In particular, we characterize the problem as a variational inequality and we prove regularity properties on the map of solutions for a parametric variational inequality. In Section 4, we apply the theoretical results to an equilibrium problem under time and uncertainty. This application is constructive for future possible developments and applications. Finally, a section of the Conclusions is given.

2 Preference Relations

In this section, we recall definitions and properties of preference relations. For further details, the interested reader can refer to [23, 24].

We deal with the behavior of an individual, also called *the consumer*. Let $X \subseteq \mathbb{R}^C$, with $X \neq \emptyset$, be the set of alternatives and $\succ$ be a binary relation which describes the preferences of consumer over the set X. If the consumer says that x is better than y, we write $x \succ y$ and we read x *is strictly preferred to* y.

Definition 1. *Let* $\succ$ *be a preference relation over* X. *We say that* $\succ$ *is*

1. asymmetric: *there is no pair* x, y *from* X *such that* $x \succ y$ *and* $y \succ x$*;*
2. negatively transitive: *if* $x \succ y$*, then for any* $z \in X$*, either* $x \succ z$*, or* $z \succ y$*, or both;*
3. irreflexive: *for no* x *is* $x \succ x$.

We observe that, since in the definition we not require $x \neq y$, then if $\succ$ is asymmetric, it is irreflexive.

Definition 2. *Let* $\succ$ *be a preference relation over* X. *We say that* $\succ$ *is*

1. lower semicontinuous: *if* $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* x *with* $x \succ y$*, then there exists* $\nu \in \mathbb{N}$ *such that* $x_n \succ y$ *for all* $n > \nu$*;*
2. upper semicontinuous: *if* $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* x *with* $y \succ x$*, then there exists* $\nu \in \mathbb{N}$ *such that* $y \succ x_n$ *for all* $n > \nu$*;*

3. continuous: *it is lower and upper semicontinuous.*

Definition 3. *Let $\succ$ be a preference relation over X. We say that $\succ$ is non-satiated if for any $x \in X$ there exists $y \in X$ such that $y \succ x$.*

In the literature several kind of convexity of preferences are given; we introduce the following, as done, for instance, in [25].

Definition 4. *Let $\succ$ be a preference relation over X. We say that $\succ$ is* convex *if for every pair y and z with $y \succ x$ and $z \succ x$, for every $\lambda \in (0, 1)$ one has $\lambda y + (1 - \lambda)z \succ x$.*

A key concept in our approach is represented by the following sets. For all $x \in X$, we denote by $U(x)$ the *strict upper contour set*, that is the set of all elements of X strictly preferred to x, hence $U(x) := \{y \in X : \ y \succ x\}$. We can rewrite the properties of preference relations in terms of strict upper contour sets. Indeed, if $\succ$ is non-satiated, then $U(x) \neq \emptyset$ for all $x \in X$; $\succ$ is *lower semicontinuous* if and only if $U(x)$ is an open set for all $x \in X$; furthermore $\succ$ is convex if and only if $U(x)$ is a convex set for all $x \in X$.
An important tool in decision theory is to study when the preference relation can be represented by means of a real-valued function.

Definition 5. *Given $\succ$ on a set X, a* numerical representation *for those preferences is any function $u : X \to \mathbb{R}$ such that*

$$x \succ y \text{ if and only if } u(x) > u(y).$$

Debreu in [10] gave necessary and sufficient conditions to ensure that a preference relation is representable.

Theorem 1. *For $\succ$ to admit a numerical representation, it is necessary that $\succ$ is asymmetric and negatively transitive.*

Theorem 2. *Let $X = \mathbb{R}^C_+$. If $\succ$ is asymmetric, negatively transitive, and continuous, then $\succ$ can be represented by a continuous function u.*

Up to now, we have taken the strict preference relation as primitive. From it, two further relations can be derived.

Definition 6. *For x and y in X, write $x \succeq y$, which is read x is* weakly preferred *to y, if it is not the case that $y \succ x$. And we write $x \sim y$, read as x is* indifferent *to y, if it is not the case that either $x \succ y$ or $y \succ x$.*

Definition 7. *Let $\succeq$ be a preference relation over X. We say that $\succeq$ is*

1. complete*: for every pair $x, y \in X$ either $x \succeq y$ or $y \succeq x$ or both;*

2. transitive*: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.*

Proposition 1. *If strict preference $\succ$ is asymmetric and negatively transitive, the weak preference relation, defined from strict preference relation according to the Definition 6, is complete and transitive.*

In this way, thanks to Proposition 1, Theorem 2 can be reformulated in terms of a weak preference relation (see, e.g., Proposition 1.15 of [24]).

3 A Variational Approach for Preference Relations

The aim of this section is to study the problem of maximization of a strict preference relation by means of a variational inequality. From now on, let K be a non-empty, convex, and closed subset of X, with $X \subseteq \mathbb{R}^C$, and let $\succ$ be an irreflexive preference relation. We introduce the following preference maximization problem.

Problem 1. *Find $\tilde{x} \in K$ such that if $x \succ \tilde{x} \quad \Rightarrow \quad x \notin K$.*

A solution of Problem 1 is called a $\succ$ *maximal element* of K.

Remark 1. *The definition of a maximal element is more general than the definition of the* greatest element *of K, which requires that*

$$\tilde{x} \succ x \quad \textit{for all} \quad x \in K \quad \textit{with} \quad x \neq \tilde{x}\,.$$

The greatest element of K is still a maximal element; the converse, in general, is not true. Furthermore, if $\succ$ is asymmetric and negatively transitive, from Proposition 1 the weak preference relation $\succeq$ is complete; then, if $\tilde{x}$ is a $\succ$ maximal element of K, it is the $\succeq$ greatest element of K.

If the preference relation is representable by means of the utility $u : X \to \mathbb{R}$, the Problem 1 can be reformulated as the following utility maximization problem.

Problem 2. *Find $\tilde{x} \in K$ such that $u(\tilde{x}) = \max_{x \in K} u(x)$.*

Hence, if $\succ$ is asymmetric, negatively transitive, and continuous, from Theorem 2, it can be represented by a continuous function u, and Problems 1 and 2 are equivalent. Moreover, if $\succ$ is non-satiated and convex too, Problem 2 is equivalent to a suitable variational inequality (see, e.g., [5, 14, 16]).
Here, we want to operate in a general setting, where asymmetric and negatively transitive assumptions on $\succ$ are not required. Hence, we study Problem 1 by means of a variational approach without the numerical representation.
To our aim, we introduce the set-valued map $N : \mathbb{R}^C \rightrightarrows \mathbb{R}^C$ such that for all $x \in X$

$$N(x) := \{h \in \mathbb{R}^C : \langle h, y - x\rangle_C \leq 0 \quad \forall y \in U(x)\}$$

and $N(x) := \emptyset$ for all $x \notin X$. Let $S(0,1)$ and $\overline{B}(0,1)$, respectively, be the boundary and the closed unit ball of $\mathbb{R}^C$. Let $G : X \rightrightarrows \mathbb{R}^C$ be the set-valued map such that for all $x \in X$

$$G(x) := \begin{cases} conv\,(N(x) \cap S(0,1)) & if \;\; U(x) \neq \emptyset\,, \\ \\ \overline{B}(0,1) & if \;\; U(x) = \emptyset\,. \end{cases} \tag{1}$$

Remark 2. *If $\succ$ is a non-satiated preference relation on X, for all $x \in X$ one has that $U(x) \neq \emptyset$ and then $G(x) := conv\,(N(x) \cap S(0,1))$. Furthermore, if $\succ$ is convex, $N(x)$ is the normal cone to the convex set $U(x)$ at x.*

We introduce the following generalized variational inequality problem.

Problem 3. *Find* $\tilde{x} \in K$ *such that there exists* $h \in G(\tilde{x})$ *and* $\langle h, x - \tilde{x}\rangle_C \geq 0 \quad \forall x \in K$.

Theorem 3. *Let* $\succ$ *be a preference relation over* X.

(a) If $\succ$ *is convex, and* $x \in \overline{U(x)}$ *for all* x *such that* $U(x) \neq \emptyset$*, every solution to Problem 1 is a solution to Problem 3.*

(b) If $\succ$ *is lower semicontinuous, every solution to Problem 3 is a solution to Problem 1.*

Proof. **(a)** Let $\tilde{x}$ be a solution to Problem 1.
If $U(\tilde{x}) = \emptyset$, then $h = 0 \in G(\tilde{x})$ and $\langle h, x - \tilde{x}\rangle_C = 0 \quad \forall x \in K$.
If $U(\tilde{x}) \neq \emptyset$. For any $x \in U(\tilde{x})$ one has $x \notin K$. Hence $U(\tilde{x})$ and K are convex sets such that $U(\tilde{x}) \cap K = \emptyset$ and, from the separation theorem (see, e.g., [8]), there exists $h \in \mathbb{R}^C \setminus \{0_C\}$ such that

$$\langle h, r - s\rangle_C \geq 0 \qquad \forall s \in U(\tilde{x}) \; , \; \forall r \in K \, . \tag{2}$$

If we replace $r = \tilde{x}$ in (2), it follows that $\langle h, s - \tilde{x}\rangle_C \leq 0 \quad \forall s \in U(\tilde{x})$, hence $h \in N(\tilde{x}) \setminus \{0_C\}$ and $\tilde{h} = \frac{h}{\|h\|} \in G(\tilde{x})$. From (2), it follows

$$\langle \tilde{h}, r - s\rangle_C \geq 0 \qquad \forall s \in U(\tilde{x}) \; , \; \forall r \in K \, . \tag{3}$$

Being $U(\tilde{x}) \neq \emptyset$ and $\tilde{x} \in \overline{U(\tilde{x})}$, there exists $\{y_n\}_{n\in\mathbb{N}} \subseteq U(\tilde{x})$ such that $y_n \to \tilde{x}$. By replacing $s = y_n$ in (3), one has $\langle \tilde{h}, r - y_n\rangle_C \geq 0$ for all $r \in K$. Passing to the limit, we get

$$\langle \tilde{h}, r - \tilde{x}\rangle_C \geq 0 \quad \forall \, r \in K \, . \tag{4}$$

Hence, from (4) and being $\tilde{h} \in G(\tilde{x})$, we can conclude that $\tilde{x}$ is a solution to Problem 3.
(b) Let $\tilde{x}$ be a solution to Problem 3.
Clearly, if $U(\tilde{x}) = \emptyset$, $\tilde{x}$ is a solution to Problem 1. If $U(\tilde{x}) \neq \emptyset$, we suppose that there exists $x' \in K$ such that $x' \succ \tilde{x}$. Since $x' \in K$ and $\tilde{x}$ is a solution to Problem 3 one has $\langle h, x' - \tilde{x}\rangle_C \geq 0$. Moreover, being $h \in G(\tilde{x}) = conv\,(N(\tilde{x}) \cap S(0,1)) \subseteq conv\, N(\tilde{x}) = N(\tilde{x})$, one has $h \in N(\tilde{x})$, with $h \neq 0_C$, and from definition of the set-valued map N, it follows that $\langle h, x' - \tilde{x}\rangle_C \leq 0$. Hence $\langle h, x' - \tilde{x}\rangle_C = 0$.
Now, for all $n \in \mathbb{N}$, we pose $x_n := x' + \dfrac{1}{n}h$; since $x_n \to x'$, from lower semicontinuity of $\succ$, there exists $\nu \in \mathbb{N}$ such that $x_n \succ \tilde{x}$ for all $n \geq \nu$. Hence, since $x_n \in U(\tilde{x})$, one has $\langle h, x_n - \tilde{x}\rangle_C \leq 0$. Then

$$0 \geq \langle h, x_n - \tilde{x}\rangle_C = \langle h, x' - \tilde{x}\rangle_C + \frac{1}{n}\|h\|^2 = \frac{1}{n}\|h\|^2 \geq 0 \; ;$$

this contradicts the fact that $h \neq 0_C$. □

The existence of the solutions of Problems 1 and 3 are now investigated.

Theorem 4. *Let $\succ$ be upper semicontinuous and convex and K a compact set. Then, there exists $\tilde{x}$ solution to Problem 3.*

Proof. Firstly, we observe that for all $x \in K$ there exists $h \in N(x) \setminus \{0_C\}$, that is $h' = \dfrac{h}{\|h\|} \in G(x)$, then it follows that G is with non-empty values for all $x \in K$. Moreover, from definition, G is with compact, and convex values. We prove that G is a closed set-valued map. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq K$, $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^C$ be such that $h_n \in G(x_n)$ and $h_n \to h$ and $x_n \to x$. We have to verify that $h \in G(x)$.
If $U(x) = \emptyset$, being $h_n \in G(x_n) \subseteq \overline{B}(0,1)$, one has $h \in \overline{B}(0,1) = G(x)$.
If $U(x) \neq \emptyset$, there exists $x' \in K$ such that $x' \succ x$ and, from upper semicontinuity, there exists $\nu \in \mathbb{N}$ such that $x' \succ x_n$ for all $n > \nu$ and then $U(x_n) \neq \emptyset$ and $h_n \in conv(N(x_n) \cap S(0,1))$. Since $h_n \in conv(N(x_n) \cap S(0,1))$ there exist $g_n^k \in N(x_n) \cap S(0,1)$ with $k = 1, \dots, n+1$, and $\lambda_n^k \geq 0$ such that $\displaystyle\sum_{k=1}^{n+1} \lambda_n^k = 1$ and $h_n = \displaystyle\sum_{k=1}^{n+1} \lambda_n^k g_n^k$. Since for all $k = 1, \dots, n+1$, $\{g_n^k\}_{n\in\mathbb{N}} \subseteq S(0,1)$ one has $g_n^k \to g^k \in S(0,1)$. Moreover $g^k \in N(x)$; indeed, for all $y \in U(x)$, from upper semicontunity, there exists $\nu \in \mathbb{N}$ such that $y \succ x_n$, hence $y \in U(x_n)$ and, since $g_n^k \in N(x_n)$ one has $\langle g_n^k, y - x_n \rangle_C \leq 0$. Passing to the limit it follows $\langle g^k, y - x \rangle_C \leq 0$, that is $g^k \in N(x)$. Then, since $h = \sum_{k=1}^{n+1} \lambda^k g^k$ with $g^k \in N(x) \cap S(0,1)$, one has $h \in G(x)$; this proves that G is a closed set-valued map.
Then, being K a compact set and G a closed set-valued map and with non-empty, compact, and convex values, from existence theorem given in [36] there exists at least a solution to Problem 3. □

Theorem 5. *Let $\succ$ be continuous and convex and K a compact set. Then, there exists $\tilde{x}$ solution to Problem 1.*

Proof. Thesis follows from Theorems 3 and 4. □

To complete our theoretical study we consider the case in which the constraint set depends on a parameter, and we analyze the regularity of the solution map for the parametric problem.
Let $L \subset \mathbb{R}^M$ be non-empty and closed and $K : L \rightrightarrows \mathbb{R}^C$ be a set-valued map. Let us introduce the following parametric generalized variational inequality problem.

Problem 4. *Fixed $l \in L$. Find $\tilde{x} \in K(l)$ such that there exists $h \in G(\tilde{x})$ and $\langle h, x - \tilde{x} \rangle_C \geq 0 \quad \forall x \in K(l)$.*

Let $S : L \rightrightarrows \mathbb{R}^C$ be the set-valued map of solutions such that for all $l \in L$

$$S(l) := \{\tilde{x} \in K(l) : \ \tilde{x} \textit{ is a solution to Problem } 4\}.$$

We introduce the parametric maximization problem.

Problem 5. *Fixed $l \in L$. Find $\tilde{x} \in K(l)$ such that if $x \succ \tilde{x} \quad \Rightarrow \quad x \notin K(l)$.*

Clearly, if $\succ$ is lower semicontinuous, then every solution to Problem 4 is still a solution to Problem 5.

Theorem 6. *Let $\succ$ be upper semicontinuous and convex, and let K be a closed, lower semicontinuous set-valued map and with non-empty, closed, and convex values and such that $K(L)$ is a bounded set. Then the set-valued map of solutions S is upper semicontinuous and with non-empty and compact values.*

Proof. From Theorem 4, for all $l \in L$ it follows that $S(l) \neq \emptyset$. We prove that S is with closed values. For all $l \in L$, let $\{\tilde{x}_n\}_{n\in\mathbb{N}} \subseteq S(l)$ be a sequence converging to $\tilde{x}$. For all $n \in \mathbb{N}$, there exists $h_n \in G(\tilde{x}_n)$ such that $\langle h_n, x - \tilde{x}_n\rangle_C \geq 0$ for all $x \in K(l)$; the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges to h and, being G a closed set-valued map (as proved in Theorem 4), $h \in G(\tilde{x})$. Hence, passing to the limit, one has $\langle h, x - \tilde{x}\rangle_C \geq 0$ for all $x \in K(l)$, that is, $\tilde{x} \in S(l)$.
Since for all $l \in L$, $S(l)$ is a closed set and $S(l) \subseteq K(L)$ that is a bounded set, it follows that $S(l)$ is compact.
We prove that S is closed. Let $\{l_n\}_{n\in\mathbb{N}} \subseteq L$ and $\{\tilde{x}_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^C$ be two sequences with $\tilde{x}_n \in S(l_n)$ and such that $l_n \to l$ and $\tilde{x}_n \to \tilde{x}$. Being K a closed set-valued map, $\tilde{x} \in K(l)$. From lower semicontinuity of K, for all $x \in K(l)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x such that $x_n \in K(l_n)$. Since for all $n \in \mathbb{N}$, $\tilde{x}_n \in S(l_n)$, there exists $h_n \in G(\tilde{x}_n)$ such that $\langle h_n, x_n - \tilde{x}_n\rangle_C \geq 0$ and moreover, since $\{h_n\}_{n\in\mathbb{N}} \subseteq \overline{B}(0,1)$, one has $h_n \to h$ with $h \in G(\tilde{x})$, being G a closed set-valued map. Hence, passing to the limit, we get $\langle h, x - \tilde{x}\rangle_C \geq 0$, that is, $\tilde{x} \in S(l)$.
Finally, being $\overline{S(L)} \subseteq K(L)$, which is a bounded set, one has that S is compact. Hence, being S closed and compact, S is upper semicontinuous. □

4 Applications

In this section, we apply the theoretical results given in Section 3 to an economic equilibrium problem under time and uncertainty introduced by Debreu in [11]. Let us suppose that the trade takes place sequentially in two periods of time: $t = 0$, say today, and $t = 1$, say tomorrow. We set $\mathcal{T} := \{0, 1\}$. The uncertainty is expressed through a finite set of all possible states of the world, which can occur tomorrow. We denote by $\mathcal{S}_1 := \{1, \ldots, S\}$ the set of states of the world at time 1 and $\mathcal{S} := \{0\} \cup \mathcal{S}_1$. We pose $\Xi = \{\xi_0\} \cup \Xi_1$, with $|\Xi| = S + 1$; $\{\xi_0\}$ is the initial situation of the market and the set $\Xi_1 := \{\xi_1, \ldots, \xi_s, \ldots, \xi_S\}$ represents all possible situations that can occur tomorrow. Hence, the evolution of the market can be represented by means of the oriented graph $\mathcal{G}$ with nodes Ξ and root ξ_0. Each node $\xi_s \in \Xi$, with $s \in \mathcal{S}$, of the graph represents a *contingency* of the market structure, that is, it identifies time and information.
In this structure of time and uncertainty, we set a market economy in which a finite number of agents, with the same information, trade and consume a finite number of different commodities. The market opens only once, at date

$t = 0$, before the beginning of the physical history of the economic system. We denote by $\mathcal{I} := \{1, \ldots, i, \ldots, I\}$ and $\mathcal{H} := \{1 \ldots, h \ldots, H\}$, respectively, the set of agents and commodities.

At each contingency $\xi_s \in \Xi$, the agent i is endowed with every commodity $e_i^h(\xi_s) > 0$ and consumes the commodities $x_i^h(\xi_s) \geq 0$. Moreover, $p^h(\xi_s) \geq 0$ is the price of commodity h. Hence:

$$x_i := ((x_i^h(\xi_s))_{h\in\mathcal{H}})_{s\in\mathcal{S}} \in \mathbb{R}_+^D \qquad e_i := ((e_i^h(\xi_s))_{h\in\mathcal{H}})_{s\in\mathcal{S}} \in \mathbb{R}_{++}^D$$

$$p := ((p^h(\xi_s))_{h\in\mathcal{H}})_{s\in\mathcal{S}} \in \mathbb{R}_+^D \setminus \{0_D\},$$

where, for simplicity, we pose $D := H(S+1)$. Each agent is characterized by an irreflexive preference relation $\succ_i$ over the consumption set $\mathbb{R}_+^D$. The aim of the agent is to consume or trade according to his preference under the budget constraint:

$$\langle p, x_i \rangle_D \leq \langle p, e_i \rangle_D.$$

The constraint means that, if p is the price vector in the market, the value of the consumption plan of consumer i, $\langle p, x_i \rangle_D$, cannot exceed his wealth $\langle p, e_i \rangle_D$.

The vector $\mathcal{E} := (\mathcal{G}, (\succ_i, e_i)_{i\in\mathcal{I}})$ denotes the economy. The equilibrium conditions are given by the following mathematical formulation. [1]

Definition 8. *A vector $(\tilde{x}, \tilde{p}) \in \mathbb{R}_+^{ID} \times \mathbb{R}_+^D \setminus \{0_D\}$ is an* equilibrium *for the economy $\mathcal{E}$ if*

1. *for any $i \in \mathcal{I}$:*

$$\langle \tilde{p}, \tilde{x}_i \rangle_D \leq \langle \tilde{p}, e_i \rangle_D ;$$
$$\text{if } x_i \succ_i \tilde{x}_i \ \Rightarrow \ \langle \tilde{p}, x_i \rangle_D > \langle \tilde{p}, e_i \rangle_D ; \tag{5}$$

2. *for all $h \in \mathcal{H}$ and $s \in \mathcal{S}$:*

$$\sum_{i\in\mathcal{I}} \tilde{x}_i^h(\xi_s) \leq \sum_{i\in\mathcal{I}} e_i^h(\xi_s) \qquad \text{and} \qquad \langle \sum_{i\in\mathcal{I}} (\tilde{x}_i - e_i), \tilde{p} \rangle_D = 0. \tag{6}$$

We point out that in Definition 8 we introduce a *free-disposal* equilibrium. It relies on the fact that the prices are assumed to be nonnegative (see, e.g., [12]). In this assumption, at the equilibrium the second condition of (6) is added: if the total supply of some commodity $h \in \mathcal{H}$ in the market exceeds its total demand, then the corresponding price $\tilde{p}^h$ is zero. This means that it is allowed the excess supply of some commodities provided that they are free.

[1]Radner [33] presented an equilibrium model that generalizes the Debreu equilibrium to make the market institutions more realistic. The economy, evolving in T trade periods, is characterized by the possibility to trade, at each possible time and in each possible state that can occur, after the uncertainty is revealed and the market reopens and by the introduction of financial instruments that enable inter-temporal and insurance transfers of wealth through markets in each possible occurrence. The interested reader can refer to [29].

Without loss of generality, we can consider the prices in the following simplex-set on $\mathbb{R}^D$:

$$\Delta := \{p \in \mathbb{R}^D_+ : \sum_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} p^h(\xi_s) = 1\}\,.$$

For each $i \in \mathcal{I}$, let $\succ_i$ be non-satiated. Then, from Remark 2, the set-valued map (1) deduced from the preference $\succ_i$ becomes:

$$G_i : \mathbb{R}^D_+ \rightrightarrows \mathbb{R}^D \quad \text{s.t.} \quad G_i(x) := conv(N_i(x) \cap S(0,1)) \qquad \forall x \in \mathbb{R}^D_+\,,$$

and we pose $G(x) := \prod_{i\in\mathcal{I}} G_i(x_i)$ for all $x = (x_i)_{i\in\mathcal{I}} \in \mathbb{R}^{ID}_+$. Moreover, for all $i \in \mathcal{I}$, we set

$$B_i(p) := \big\{x_i \in \mathbb{R}^D_+ : \langle p, x_i\rangle_D \le \langle p, e_i\rangle_D\big\} \cap \Big[0, \sum_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} \sum_{i\in\mathcal{I}} e^h_i(\xi_s)\Big]^D$$

and $B(p) := \prod_{i\in\mathcal{I}} B_i(p)$. Now, the following generalized quasi-variational inequality problem is considered.

Problem 6. *Find $(\tilde{x}, \tilde{p}) \in B(\tilde{p}) \times \Delta$ for which there exists $h := (h_i)_{i\in\mathcal{I}} \in G(\tilde{x})$ such that*

$$\sum_{i\in\mathcal{I}} \langle h_i, x_i - \tilde{x}_i\rangle_D + \langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), p - \tilde{p}\rangle_D \ge 0 \qquad \forall (x,p) \in B(\tilde{p}) \times \Delta\,.$$

Theorem 7. *For all $i \in \mathcal{I}$, let $\succ_i$ be convex, lower semicontinuous, non-satiated, and such that $x_i \in \overline{U_i(x_i)}$ for all $x_i \in \mathbb{R}^D_+$. If $(\tilde{x}, \tilde{p}) \in B(\tilde{p}) \times \Delta$ is a solution to Problem 6, then it is an equilibrium vector for $\mathcal{E}$.*

Proof. Let $(\tilde{x}, \tilde{p}) \in B(\tilde{p}) \times \Delta$ be a solution to Problem 6.
1. *For any $i \in \mathcal{I}$, $\tilde{x}_i$ is a solution to the variational inequality:*

$$\langle h_i, x_i - \tilde{x}_i\rangle_D \ge 0 \qquad \forall x_i \in B_i(\tilde{p})\,. \tag{7}$$

Fixed $j \in \mathcal{I}$, let $(\hat{x}, \hat{p}) = (\hat{x}, \tilde{p})$ with $\hat{x}_i = \tilde{x}_i$ for all $i \ne j$ and $\hat{x}_j = x_j \in B_j(\tilde{p})$; by replacing $(\hat{x}, \hat{p})$ in the Problem 6 one has:

$$\sum_{i\in\mathcal{I}} \langle h_i, \hat{x}_i - \tilde{x}_i\rangle_D + \langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), \hat{p} - \tilde{p}\rangle_D = \langle h_j, x_j - \tilde{x}_j\rangle_D \ge 0 \quad \forall x_j \in B_j(\tilde{p})\,.$$

2. *For any $h \in \mathcal{H}$ and $s \in \mathcal{S}$, $\sum_{i\in\mathcal{I}} \tilde{x}^h_i(\xi_s) \le \sum_{i\in\mathcal{I}} e^h_i(\xi_s)$.*

Let $(\hat{x}, \hat{p}) = (\tilde{x}, \hat{p})$ with $\hat{p} \in \Delta$; by replacing $(\hat{x}, \hat{p})$ in Problem 6 one has:

$$\sum_{i\in\mathcal{I}} \langle h_i, \hat{x}_i - \tilde{x}_i\rangle_D + \langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), \hat{p} - \tilde{p}\rangle_D = \langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), \hat{p} - \tilde{p}\rangle_D \ge 0\,.$$

Since $\tilde{x}_i \in B_i(\tilde{p})$ for all $i \in \mathcal{I}$, one has $\langle \tilde{p}, \sum_{i\in\mathcal{I}} (\tilde{x}_i - e_i)\rangle_D \le 0$, hence:

$$\langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), \hat{p}\rangle_D \ge \langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), \tilde{p}\rangle_D \ge 0 \qquad \forall \hat{p} \in \Delta\,.$$

Fix $s^* \in \mathcal{S}$ and $h^* \in \mathcal{H}$, let

$$\hat{p} := \begin{cases} \hat{p}^{h^*}(\xi_{s^*}) = 1 \\ \hat{p}^h(\xi_s) = 0 \qquad \forall h \neq h^* \,,\; s \neq s^* \end{cases}$$

it follows

$$0 \leq \langle \sum_{i \in \mathcal{I}} (e_i - \tilde{x}_i), \hat{p} \rangle_D = \sum_{i \in \mathcal{I}} (e_i^{h^*}(\xi_{s^*}) - \tilde{x}_i^{h^*}(\xi_{s^*})) \,.$$

3. *For all $x_j \in \mathbb{R}_+^D$ such that $\langle p, x_j \rangle_D \leq \langle p, e_j \rangle_D$, it results: $\langle h_j, x_j - \tilde{x}_j \rangle_D \geq 0$.*
We suppose there exists $\hat{x}_j \in \mathbb{R}_+^D$ such that $\langle \tilde{p}, \hat{x}_j \rangle_D \leq \langle \tilde{p}, e_j \rangle_D$ and $\langle h_j, \hat{x}_j - \tilde{x}_j \rangle_D < 0$. From Step 2, for all $i \in \mathcal{I}$ and $h \in \mathcal{H}$, it follows that

$$\tilde{x}_i^h(\xi_s) \leq \sum_{i \in \mathcal{I}} \tilde{x}_i^h(\xi_s) \leq \sum_{i \in \mathcal{I}} e_i^h(\xi_s) < \sum_{s \in \mathcal{S}} \sum_{h \in \mathcal{H}} \sum_{i \in \mathcal{I}} e_i^h(\xi_s) \,.$$

Hence there exists $\delta > 0$ such that the intersection between $\mathbb{R}_+^D$ and the ball centered at $\tilde{x}_i$ and radius δ is in $\Big[0, \sum_{s \in \mathcal{S}} \sum_{h \in \mathcal{H}} \sum_{i \in \mathcal{I}} e_i^h(\xi_s)\Big]^D$. We pose:

$$y = \lambda \hat{x}_j + (1 - \lambda) \tilde{x}_j \qquad \text{such that} \qquad 0 < \lambda < \min \left\{ \frac{\delta}{\|\hat{x}_j - \tilde{x}_j\|}, 1 \right\} .$$

One has $\langle \tilde{p}, y \rangle_D \leq \langle \tilde{p}, e_j \rangle_D$ and $\|y - \tilde{x}_i\| < \delta$, that is $y \in B_i(\tilde{p})$. Moreover, one gets $\langle h_j, y - \tilde{x}_j \rangle_D < 0$ contradicting the condition (7).

4. *For any $i \in \mathcal{I}$ if $x_i \succ_i \tilde{x}_i \quad \Rightarrow \quad \langle \tilde{p}, x_i \rangle_D > \langle \tilde{p}, e_i \rangle_D$.*
It follows from Step 3 and Theorem 3.

5. *One has $\langle \sum_{i \in \mathcal{I}} (\tilde{x}_i - e_i), \tilde{p} \rangle_D = 0$.*
We suppose there exists $i \in \mathcal{I}$ such that $\langle \tilde{x}_i - e_i, \tilde{p} \rangle_D < 0$. Since $\tilde{x}_i \in \overline{U_i(\tilde{x}_i)}$ there exists $\{\tilde{x}_{i,n}\}_{n \in \mathbb{N}} \subseteq U_i(\tilde{x}_i)$ converging to $\tilde{x}_i$; hence, $\tilde{x}_{i,n} \succ_i \tilde{x}_i$ and $\langle \tilde{x}_{i,n} - e_i, \tilde{p} \rangle_D < 0$, which contradicts Step 3. Hence, $\langle \tilde{x}_i - e_i, \tilde{p} \rangle_D = 0$ for all $i \in \mathcal{I}$, and $\langle \sum_{i \in \mathcal{I}} (\tilde{x}_i - e_i), \tilde{p} \rangle_D = 0$. □

Lemma 1. *For all $i \in \mathcal{I}$, the set-valued map B_i is closed, lower semicontinuous and with non-empty, compact, and convex values.*

Proof. See, e.g., [28, 29]. □

In the next theorem, we give the existence of equilibrium by means of the Problem 6; the characteristics of our problem allow us to consider $I + 1$ variational inequalities instead of a single quasi-variational inequality. We will proceed in the following way: we fix a price $p \in \Delta$ and we study for all $i \in \mathcal{I}$ the first part of inequality of Problem 6 as a parametric generalized variational inequality. Once the existence of at least one solution is ensured, we introduce the set-valued map of the solutions. From this set-valued map, we define the operator of the second part of Problem 6; hence, we can solve this variational problem thanks to the properties of the solution map proven in Theorem 6. The vector of solutions given by the $I+1$ variational problems represents the solution to Problem 6.

Theorem 8. *For all $i \in \mathcal{I}$, let $\succ_i$ be convex, continuous, non-satiated, and such that $x_i \in \overline{U_i(x_i)}$ for all $x_i \in \mathbb{R}^D_+$. Then, the Problem 6 admits at least one solution, and there exists an equilibrium vector for $\mathcal{E}$.*

Proof. For each $i \in \mathcal{I}$ and $p \in \Delta$, we consider the parametric $GVI(G_i, B_i(p))$:

$$\textit{Find } \tilde{x}_i \in B_i(p) \textit{ for which there exists } h_i \in G_i(\tilde{x}_i) \textit{ such that}$$

$$\langle h_i, x_i - \tilde{x}_i \rangle_D \geq 0 \qquad \forall x_i \in B_i(p)\,. \tag{8}$$

We introduce the set-valued map of solutions $S_i : \Delta \rightrightarrows \mathbb{R}^D$ such that, for all $p \in \Delta$, it results

$$S_i(p) = \{\tilde{x}_i : \tilde{x}_i \;\; is \;\; solution \;\; of \;\; (8)\}\,.$$

From Lemma 1, being $B_i(\Delta) \subset \left[0, \sum_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} \sum_{i\in\mathcal{I}} e_i^h(\xi_s)\right]^D$ and from Theorem 6 it follows that S_i is upper semicontinuous and with non-empty, and compact values; hence, the set-valued map $S : \Delta \rightrightarrows \mathbb{R}^{ID}$ such that $S(p) := \prod_{i\in\mathcal{I}} S_i(p)$ for all $p \in \Delta$ has the same properties. Hence, thanks to Proposition 2.1 in [3], the set-valued map $conv\, S : \Delta \rightrightarrows \mathbb{R}^{ID}$ is upper semicontinuous. We introduce the following $GVI(conv\, S, \Delta)$:

$$\textit{Find } \tilde{p} \in \Delta \textit{ such that there exists } \tilde{x} \in conv\, S(\tilde{p}) \textit{ such that}$$

$$\langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), p - \tilde{p} \rangle_D \geq 0 \qquad \forall p \in \Delta\,.$$

Thanks to the properties of the set-valued map $conv\, S$ and the set Δ, from Theorem 2 in [36], there exists $\tilde{p} \in \Delta$ and $\tilde{x} \in conv\, S(\tilde{p})$ solution to $GVI(conv\, S, \Delta)$. Furthermore, since Δ is a convex set and the set-valued map S is with compact values, from Lemma 3.2 in [3], it follows that the set of solutions to $GVI(conv\, S, \Delta)$ coincides with the set of solutions of the following $GVI(S, \Delta)$:

$$\textit{Find } \tilde{p} \in \Delta \textit{ such that there exists } \tilde{x} \in S(\tilde{p}) \textit{ such that}$$

$$\langle \sum_{i\in\mathcal{I}} (e_i - \tilde{x}_i), p - \tilde{p} \rangle_D \geq 0 \qquad \forall p \in \Delta\,. \tag{9}$$

Then, from (8), with $\tilde{p} \in \Delta$, and (9), we get that $(\tilde{x}, \tilde{p})$ is a solution to the Problem 6. Hence, thanks to Theorem 7, we can conclude that $(\tilde{x}, \tilde{p}) \in B(\tilde{p})$ is an equilibrium vector for $\mathcal{E}$. □

Remark 3. *We observe that in Theorem 7 and Theorem 8 one can replace (see, e.g., [28]) the requirements that $\succ_i$ is non-satiated and $x_i \in \overline{U_i(x_i)}$ for all $x_i \in \mathbb{R}^D_+$ with the* locally non-satiated assumption*:*

$$\forall x_i \in \mathbb{R}^D_+\,,\ \forall \delta > 0\ \exists\, y_i \in \mathbb{R}^D_+ \cap B(x_i, \delta) \;\; \textit{such that} \;\; y_i \succ_i x_i\,.$$

5 Conclusions

In this paper, we made use of a variational approach to study a maximization problem based on a preference relation. A strict preference relation $\succ$ is considered. Under suitable assumptions $\succ$ is representable by means of a utility function; then, the preference maximization is equivalent to an optimization problem of a real values function and, it is well known that it can be studied by means of a variational inequality problem. However, here the considered assumptions are not sufficient to guarantee the existence of a utility function representing the preference relation, and then we can not apply the results known in the literature. In order to introduce the operator of a suitable variational problem, the main tool is represented by the strictly upper counter set and the normal cone associated with it. Hence, we opportunely characterize the preference maximization problem and we prove regularity properties on the map of solutions of the relative parametric variational problem.
We apply these results to an economic equilibrium problem. The aim of this application is manifold. From an economic point of view, considering incomplete preference relations may be an approach closer to the dynamics governing the choices in the real world. Instead, from a mathematical point of view, quasi-variational problems can be difficult to be solved. To overcome this difficulty, instead of solving a quasi-variational problem, we handle variational inequality problems so that a lot of algorithms are available to compute the solution (see, e.g, [19]). However, in doing this, we observe that the solution map studied in Theorem 6 is not with convex values. Nevertheless, in the proof of Theorem 8, this fact is opportunely overtaken thanks to the properties of the considered problem and Lemma 3.2 in [3].

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