A Stochastic Variational Approach to Study Economic Equilibrium Problems Under Uncertainty

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Abstract

This paper focuses on the analysis of an economic equilibrium model under time and uncertainty by using a stochastic variational inequality approach. Such an approach allows to capture, in a finite set of stages, the evolutionary aspects of the problem in response to an increasing level of information.

KeywordsStochastic variational inequality; Nonanticipativity; Equilibrium problem, Uncertainty

1 Introduction

In [5], Debreu introduced an economic equilibrium model which evolves in a sequence of markets under uncertainty on the future conditions. Subsequently, Radner in [13] generalized such equilibrium model by introducing the possibility of agents to transfer wealth among all possible future time. Throughout two different market structures, forward and spot markets, consumers' and firms' choices will depict not only their taste concerning the goods but also their beliefs regarding the event chosen by Nature.

The market evolves in a finite sequence of time and, at each future date, different states of the world are possible. At the beginning agents do not know the possible evolution of the market; the environment is progressively revealed, and, all information is revealed at the final time. Agents have to make their decisions under uncertainty conditions. In order to capture the essential dynamics of stochastic decision processes, it is needed an approach which encompasses *multistage* models responding to an increasing level of information.

The aim of this paper is to study an economic equilibrium problem under uncertainty by means of a stochastic variational inequality formulation. Thanks to the variational inequality theory, a large class of equilibrium problems has been studied (see, e.g., [1, 6, 7, 8, 11, 12]).

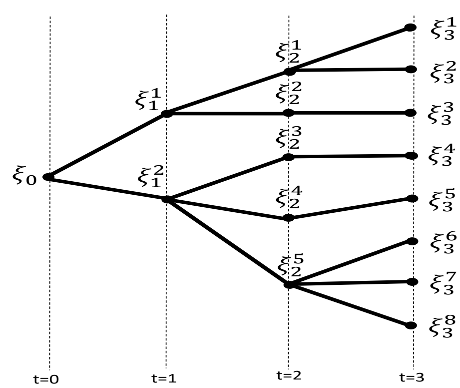
The stochastic variational inequalities have been introduced and studied in the last two decades as a natural extension of deterministic ones. In particular, Rockafellar and Wets in [16] introduced a formulation in an suitable functional setting relatively to a finite set of final possible states and certain information fields. A key concept of this approach is the presence of nonanticipativity constraints on the variables of the problem. Variables are not based on the information not yet known, but they are related to the information field up to the considered time. In addition, nonanticipativity constraints provide a powerful tool in both theoretical and computational aspects as they can be

dualized by multipliers, providing a tool for a point-wise decomposition of the original stochastic variational problem. The latter means that nonanticipativity formulation enables the decomposition of the original stochastic variational problem into a separate problem for each scenario.
The paper is organized as follows. In Section 2, we describe the general set up of a competitive equilibrium model under time and uncertainty. In Section 3, we introduce the main tools dealing with a scenario approach. Subsequently, in Section 4, we rewrite the problem introduced in Section 2 in a probabilistic setting. In Section 5, we reformulate the equilibrium problem in terms of a suitable stochastic quasi-variational inequality, both in basic and extensive form and, by using variational tools, we give the existence of equilibrium. Finally, in Section 6, we provide a procedure to compute the equilibrium solution using the Progressive Hedging Algorithm introduced in [17].

2 Set up of the Model

In this section we present a model of exchange and consumption under uncertainty, introduced in [9]. Let us suppose that the market starts at time $t = 0$ and evolves in a finite sequence of T future dates. The sets $\mathcal{T} := \{1, \ldots, T\}$ and $\mathcal{T}_0 := \{0\} \bigcup \mathcal{T}$ denote the sets of time periods, respectively, without and with the initial date. At each time $t \in \mathcal{T}$ one or more than one situations are possible; at the final time T, S states of the world are possible; we denote by $\Omega := \{\omega_1, \ldots, \omega_S\}$ the set of all alternative states at T. We can give a graphical representation of the evolution of the market through an *oriented graph* $\mathcal{G}$, consisting by a set of vertices $\Xi := \Xi_0 \cup \Xi_1 \cup \ldots \cup \Xi_T$, with $|\Xi_t| = k_t$ and $|\Xi| = N$, such that

- $\Xi_0 := \{\xi_0\}$ where ξ_0 is the root vertex: it represents the initial situation and it is the unique vertex without immediate predecessor.
- For all $t \in \mathcal{T}$, the set $\Xi_t := \left\{\xi_t^1, \ldots, \xi_t^{k_t}\right\}$ is a finite set of vertices and represents all possible situations at time t. Each ξ_t^j has a unique immediate predecessor in Ξ_{t-1}.
- $\Xi_T := \Omega$, that is $\xi_T^j = \omega_j$ for all $j = 1, \ldots, S$ are the terminal nodes of the graph.



Each node ξ_t^j of the graph represents a contingency of the market structure, that is, it identifies time and information. Now, in this structure of time and uncertainty, we can set an economy in which a finite number of agents, all with the same information, trade and consume a finite number of different commodities. We denote by $\mathcal{I} := \{1, \ldots, i, \ldots, I\}$ and $\mathcal{H} := \{1 \ldots, h \ldots, H\}$, respectively, the sets of agents and commodities. At each contingency ξ_t^j, each agent i is endowed with a strictly positive commodity vector $e_i(\xi_t^j) \in \mathbb{R}_{++}^H$, where the component $e_i^h(\xi_t^j)$ denotes the endowment of commodity h of agent i at the contingency ξ_t^j. Grouping in vectors, e_i represents the total endowment of the agent i:

$$e_i := (e_{i0}, e_{i1}, \ldots, e_{it}, \ldots, e_{iT}) \in \mathbb{R}_{++}^{HN},$$

where for all $t \in \mathcal{T}_0$, $e_{it} := (e_i(\xi_t^1), \ldots, e_i(\xi_t^{k_t})) = (e_i(\xi))_{\xi \in \Xi_t} \in \mathbb{R}_{++}^{Hk_t}$. The economy is characterized by two market structures: spot and forward markets.

Spot market: it opens at each contingency $\xi_t^j \in \Xi$ and agents consume or trade a certain amount of commodities $x_i(\xi_t^j) \in \mathbb{R}_+^H$ at prices $p(\xi_t^j) \in \mathbb{R}_+^H$. Grouping in vectors one has

$$x_i := (x_{i0}, x_{i1}, \ldots, x_{it}, \ldots, x_{iT}) \in \mathbb{R}_+^{HN}, \qquad p := (p_0, p_1, \ldots, p_t, \ldots, p_T) \in \mathbb{R}_+^{HN}$$

where each $x_{it} := (x_i(\xi_t^1), \ldots, x_i(\xi_t^{k_t})) = (x_i(\xi))_{\xi \in \Xi_t} \in \mathbb{R}_+^{Hk_t}$ represents the decisions that must be made at time t at prices $p_t := (p(\xi_t^1), \ldots, p(\xi_t^{k_t})) = (p(\xi))_{\xi \in \Xi_t} \in \mathbb{R}_+^{Hk_t}$ for each contingency in Ξ_t.

Forward market: at $t = 0$ a further market opens and offers participants the opportunity to reduce their exposure to future risks and randomness without, however, removing the incentive to trade and consume in the spot markets that opens at each time period after the uncertainty is revealed. Thanks to the forward market, agent can transfer wealth in terms of commodity-1 among all future contingencies for immediate cash that will be used for spot consumption goods or for future contracts in other contingencies. For each $i \in \mathcal{I}$, we denote the forward contracts and the relative prices through the vectors

$$z_i := (z_{i1}, \ldots, z_{it}, \ldots, z_{iT}) \in \mathbb{R}^{N-1}, \qquad q := (q_1, \ldots, q_t, \ldots, q_T) \in \mathbb{R}_+^{N-1}$$

so that $z_{it} := (z_i(\xi_t^1), \ldots, z_i(\xi_t^{k_t})) = (z_i(\xi))_{\xi \in \Xi_t} \in \mathbb{R}^{k_t}$ and $q_t := (q(\xi_t^1), \ldots, q(\xi_t^{k_t})) = (q(\xi))_{\xi \in \Xi_t} \in \mathbb{R}_+^{k_t}$, where $z_i(\xi_t^j)$ is the commodity-1 amount at ξ_t^j paid $q(\xi_t^j)$ at time 0. We observe that the components of z_i can be negative: if $z_i(\xi_t^j) < 0$, it is an amount to be delivered by agent i at ξ_t^j and $q(\xi_t^j)z_i(\xi_t^j)$ represents an *income* at ξ_0; while, if $z_i(\xi_t^j) > 0$, it is an amount to be received by agent i at ξ_t^j and $q(\xi_t^j)z_i(\xi_t^j)$ represents an *outcome* at ξ_0. Let, for all $\xi_t^j \in \Xi \setminus \{\xi_0\}$, $R(\xi_t^j) > \sum_{i \in \mathcal{I}} e_i^1(\xi_t^j)$; we pose $R := \prod_{\xi_t^j \in \Xi \setminus \{\xi_0\}} \Big[-R(\xi_t^j), R(\xi_t^j) \Big]$. Without loss of generality, we suppose that, for all $i \in \mathcal{I}$, $z_i \in R$ (see Radner [13]). Each agent i has a preference on the commodities which is expressed by means of a utility function $\mathcal{U}_i : \mathbb{R}_+^{HN} \to \mathbb{R}$. The aim of each agent is to maximize her own preferences on spot consumptions under the natural budget constraints set at the current price system (p, q):

$$\begin{aligned} M_i(p,q) :=\{&(x_i, z_i) \in \mathbb{R}_+^{HN} \times R : \\ &\langle p(\xi_0), x_i(\xi_0)\rangle_H + \langle q, z_i\rangle_{N-1} \leq \langle p(\xi_0), e_i(\xi_0)\rangle_H \\ &\langle p(\xi_t^j), x_i(\xi_t^j)\rangle_H \leq \langle p(\xi_t^j), e_i(\xi_t^j)\rangle_H + p^1(\xi_t^j) z_i(\xi_t^j) \quad \forall \xi_t^j \in \Xi_t, t \in \mathcal{T}\}. \end{aligned}$$

The first inequality represents the budget constraint at time 0 while, the second inequality represents the expected budget constraints at each contingency ξ_t^j, with $t \in \mathcal{T}$. Furthermore, market clearing conditions have to be satisfied: at each contingency ξ_t^j, the total spot consumption have not exceed the total endowment while the total forward contracts have the be zero. We denote by $\mathcal{E}$ the economy $\mathcal{E} := \left(\mathcal{G}, (\mathcal{U}_i, e_i)_{i\in\mathcal{I}}\right)$ and we can, now, formalize the equilibrium conditions.

Definition 1. *An* equilibrium *of plans, prices, and price expectations for the economy* $\mathcal{E}$ *is a vector* $\left((\bar{x}_i, \bar{z}_i)_{i\in\mathcal{I}}, \bar{p}, \bar{q}\right) \in \prod_{i\in\mathcal{I}} M_i(\bar{p}, \bar{q}) \times \mathbb{R}_+^{HN} \times \mathbb{R}_+^{N-1}$*, such that*

- *for any* $i \in \mathcal{I}$*:*

$$\max \mathcal{U}_i(x_i) = \mathcal{U}_i(\overline{x}_i) \qquad s.t. \quad (x_i, z_i) \in M_i(\bar{p}, \bar{q}) \; ; \tag{1}$$

- *for all* $t \in \mathcal{T}_0$*:*

$$\sum_{i\in\mathcal{I}} \bar{x}_i(\xi_t^j) \leq \sum_{i\in\mathcal{I}} e_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t; \tag{2}$$

- *for all* $t \in \mathcal{T}$*:*

$$\sum_{i\in\mathcal{I}} \bar{z}_i(\xi_t^j) = 0 \qquad \forall \xi_t^j \in \Xi_t. \tag{3}$$

3 Scenarios formulation: preliminary notions

This section is devoted to introduce the preliminary notions we need to study the economic equilibrium problem, introduced in Section 2, in a stochastic framework and throughout a variational approach. Firstly, we need to introduce the following information fields.

Definition 2. *A family of information-partitions of* Ω *is* $\mathcal{P} := \{F_t : t \in \mathcal{T}_0\}$ *where, for all* $t \in \mathcal{T}_0$*,* $F_t := \left\{F_t^1, \ldots, F_t^{k_t}\right\}$ *is a partition of* Ω *such that*

(i) $F_0 = \{\Omega\}$*;*

(ii) for all $t \in \mathcal{T}$*,* $F_{t+1} \subset F_t$*, that is: if* $F_{t+1}^j \in F_{t+1} \Rightarrow F_{t+1}^j \subset F_t^k$ *for some* $F_t^k \in F_t$*;*

(iii) $F_T = \Omega$*.*

For all $t \in \mathcal{T}_0$*, the set* F_t^j *is called* elementary event *and the partition* F_t *is called* event*.*

From an economic viewpoint, condition (i) means that at time $t = 0$ no uncertainty has resolved; condition (ii) means that information about the environment are progressively revealed, i.e. one has only partial information. Finally, (iii) tell us that all information are revealed at time T.

To link time-uncertainty structure introduced in Section 2 and the information-partitions, we can consider the oriented graph $\mathcal{G}$ as an *event-tree*: at each pair (ω, t) identified in $\mathcal{P}$ corresponds a contingency ξ_t^j and at each vertex ξ_t^j of the oriented graph $\mathcal{G}$ we tie the elementary event F_t^j, that is $F_t^j \cong \xi_t^j$. Each state of the world $\omega \in \Omega$ identifies a complete history of the environment up to time T

$$\omega \cong (\xi_0, \xi_1^j, \ldots, \xi_T^j)$$

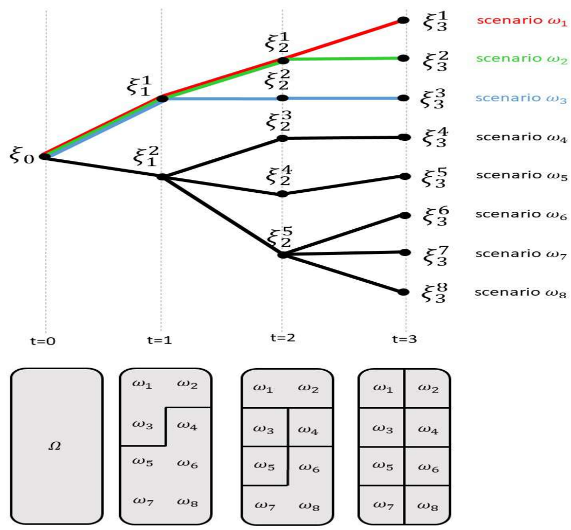


Figure 1: Example

that is called *scenario.* If two scenarios $\omega_s, \omega_c \in \Omega$ are in the same set $F_t^j \in F_t$, then they are indistinguishable at time t on the basis of available information: because they share the same path up to time t, the known information are the same, that is

$$\omega_s \cong (\xi_0, \xi_1^j, \dots, \xi_t^j, \xi_{t+1}^s, \dots, \xi_T^s) \qquad \omega_c \cong (\xi_0, \xi_1^j, \dots, \xi_t^j, \xi_{t+1}^c, \dots, \xi_T^c) \, .$$

In this approach, the key point is to consider the uncertainty quantities as function instead of vectors. We suppose that each scenario ω has a known probability $\pi(\omega)$ and, for $G \in \mathbb{R}$, we introduce $\mathcal{L}_G(\Omega, \pi) := \mathcal{L}_G$ the linear space of functions:

$$\mathcal{L}_G = \left\{ the \quad collection \quad of \quad all \quad functions \quad y : \Omega \to \mathbb{R}^G \right\}.$$

The space $\mathcal{L}_G$ is equipped with the following *expectational inner product* and the associated norm:

$$\langle\langle y, h \rangle\rangle_G := \mathbb{E}_\omega[\langle y, h \rangle_G] = \sum_{\omega \in \Omega} \pi(\omega) \langle y(\omega), h(\omega) \rangle_G, \qquad \|y\| := \left(\mathbb{E}_\omega[\langle y, y \rangle_G]\right)^{\frac{1}{2}} \tag{4}$$

where $\langle \cdot, \cdot \rangle_G$ is the usual inner product in $\mathbb{R}^G$. The structure (4) makes $\mathcal{L}_G$ a finite-dimensions Hilbert space. Moreover, if $G = G_0 + \ldots + G_t + \ldots + G_T$ one has $\mathcal{L}_G = \mathcal{L}_{G_0} \times \ldots \times \mathcal{L}_{G_t} \times \ldots \times \mathcal{L}_{G_T}$ where $\mathcal{L}_{G_t} = \{$ the collection of all functions $y_t : \Omega \to \mathbb{R}^{G_t}\}$; hence, for all $\omega \in \Omega$ we can consider $y(\omega) = (y_t(\omega))_{t \in \mathcal{T}_0}$.

Definition 3. *Given the information-partitions $\mathcal{P} = \{F_t : t \in \mathcal{T}_0\}$ of Ω, let $F_{\bar{t}} \in \mathcal{P}$; we say that $y \in \mathcal{L}_G$ is* $F_{\bar{t}}$-measurable *with respect to $\mathcal{P}$ if for all $j = 1, \ldots, k_{\bar{t}}$ one has:*

$$\forall \omega_s, \omega_c \in F_{\bar{t}}^j \qquad y_t(\omega_s) = y_t(\omega_c) \qquad \forall t = 0, \ldots, \bar{t} \, .$$

We say that $y \in \mathcal{L}_G$ is measurable *if it is F_t-measurable for all $F_t \in \mathcal{P}$ and $t \in \mathcal{T}_0$.*

We denote by $\mathcal{N}$ the set of measurable elements of $\mathcal{L}_G$:

$$\mathcal{N} := \{y \in \mathcal{L}_G : y \; is \quad F_t - measurable \quad \forall t \in \mathcal{T}_0\} \; .$$

$\mathcal{N}$ is called *nonanticipativity* constrains subspace of $\mathcal{L}_G$. We recall the following existence results for variational problems in the spaces $\mathbb{R}^G$ and $\mathcal{L}_G$.

Theorem 1 (See [4], Corollary 3.1). *Let $X \subseteq \mathbb{R}^G$ be a compact set and let $\Phi : X \rightrightarrows \mathbb{R}^G$ be an upper semicontinuous set-valued map on X with compact and convex values. Then, there exists $\bar{x} \in X$ and $\varphi \in \Phi(\bar{x})$ solution to $GVI(\Phi, X)$*

$$\langle \varphi, x - \bar{x} \rangle_G \geq 0 \quad \forall x \in X.$$

Theorem 2 (See [16], Theorem 3.5 and 3.6). *Let $\mathcal{C} = \{x \in \mathcal{L}_G : x(\omega) \in C(\omega) \quad \forall \omega \in \Omega\}$ be a nonempty, closed, and convex subspace of $\mathcal{L}_G$ and $\mathcal{F} : \mathcal{L}_G \to \mathcal{L}_G$ be a continuous operator. The set of solutions to the multistage stochastic variational inequality*

$$\langle\langle \mathcal{F}(\bar{x}), x - \bar{x} \rangle\rangle_G \geq 0 \qquad \forall x \in \mathcal{C} \cap \mathcal{N}$$

is always closed. It is sure to be bounded and nonempty if $\mathcal{C} \cap \mathcal{N} \neq \emptyset$ and the sets $C(\omega)$ are bounded. Furthermore, under monotonicity of $\mathcal{F}$ relatively to $\mathcal{C}$, the set of solutions to $SVI(\mathcal{F}, \mathcal{C})$ is convex; under strict monotonicity, if a solution to $SVI(\mathcal{F}, \mathcal{C})$ exists at all, it must be unique.

4 The equilibrium model by scenarios

The aim of this section is to reformulate the model introduced in the Section 2 in a scenarios setting. We consider an economy which is characterized by the information-partitions $\mathcal{P}$ of the set of scenarios Ω and by a probability measure on elements of Ω, $\Pi = (\pi(\omega))_{\omega \in \Omega}$. For each $i \in \mathcal{I}$, we suppose that $x_i, p, e_i \in \mathcal{L}_{H(T+1)}$ and $z_i, q \in \mathcal{L}_{(N-1)(T+1)}$. In particular, since z_i and q represent a decision in time 0, one has

$$z_{i0}(\omega) \in \mathbb{R}^{N-1}, \;\; z_{it}(\omega) = 0 \;\; \forall t \in \mathcal{T} \quad \text{and} \quad q_0(\omega) \in \mathbb{R}^{N-1}_+, \;\; q_t(\omega) = 0 \;\; \forall t \in \mathcal{T}$$

Hence, thanks to the above remark, we can consider $z_i, q \in \mathcal{L}_{N-1}$. Moreover, we require that all vectors x_i, p, e_i and z_i, q are measurable, that is for each F_t^j, $x_{it}(\omega)$ and $e_{it}(\omega)$ are constants for all $\omega \in F_t^j$. From an economic viewpoint, for all $\omega \in F_t^j$, $x_{it}(\omega)$ represents the bundle of spot consumption chosen by agent i at contingency ξ_t^j and $e_{it}(\omega)$ represents the initial endowment in contingency ξ_t^j. Moreover, for any $\omega \in F_t^j$, $p_t(\omega)$ is the spot price at time t and $\sum_{\omega \in F_t^j} p_t(\omega)$ represents the spot price vector at contingency $\xi_t^j = F_t^j$, see e.g. [3]. Hence, from F_t-measurability requirement, it follows that:

$$\forall \omega \in F_t^j \qquad x_{it}(\omega) = x_i(\xi_t^j) \;, \quad e_{it}(\omega) = e_i(\xi_t^j) \quad and \quad \sum_{\omega \in F_t^j} p_t(\omega) = p(\xi_t^j). \tag{5}$$

Furthermore, for each $\omega \in \Omega$, $z_i(\omega)$ represents the $N-1$ quantities sold or bought at $t = 0$ of commodity-1 eventually to be delivered or received by agent i in all possible contingencies ξ_t^j, with $t \in \mathcal{T}$ and $k = 1, \ldots, k_t$. Although we allow the decisions to depend on Ω, then the use of measurability constraints restricts the choice of z_i to the linear subspace of functions that are *constant* for each $\omega \in \Omega$. In this way, we pose that $z_i(\omega) = (z_{iF_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t}$ for each $\omega \in \Omega$. With similar comments, for each $\omega \in \Omega$, the vector $q(\omega) = (q_{F_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t}$ represents the forward prices at time 0 and it is such that, if we consider $\sum_{\omega\in\Omega} q(\omega) = |\Omega|\,(q_{F_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t}$, this sum represents the forward price vector as defined in Section 2. Summarizing, from F_0-measurability requirement, it follows that:

$$\forall\omega\in\Omega \qquad z_i(\omega) = (z_{iF_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t} = z_i \qquad \textit{and} \qquad \sum_{\omega\in\Omega} q(\omega) = |\Omega|\,(q_{F_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t} = q\,. \tag{6}$$

We point out that z_i can't really depend on ω, but the requirement that $z_i \in \mathcal{N}$ allow us to study the problem by events.
We use following notations for the nonanticipativity sets: $\mathcal{N}^1 \subseteq \mathcal{L}_{H(T+1)}$ and $\mathcal{N}^2 \subseteq \mathcal{L}_{N-1}$ respectively the sets of commodities x and e and contracts z which satisfies the first conditions (5) and (6); $\widetilde{\mathcal{N}}^1 \subseteq \mathcal{L}_{H(T+1)}$ and $\widetilde{\mathcal{N}}^2 \subseteq \mathcal{L}_{N-1}$ the sets of prices p and q which satisfies the second conditions (5) and (6). Hence, for sake of simplicity, we pose $C = H(T+1)$, $D = H(T+1)+N-1$ and

$$\mathcal{L} := \mathcal{L}_{H(T+1)} \times \mathcal{L}_{N-1}, \qquad \mathcal{N} := \mathcal{N}^1 \times \mathcal{N}^2, \qquad \widetilde{\mathcal{N}} := \widetilde{\mathcal{N}}^1 \times \widetilde{\mathcal{N}}^2\,.$$

In this setting, we suppose that the utility functions are represented by the expected utility

$$\mathcal{U}_i : \mathcal{L}_C \to \mathbb{R} \qquad \mathcal{U}_i\left(x_i\right) = \mathbb{E}_\omega\left[f_{i\omega}\left(x_i\right)\right] = \sum_{\omega\in\Omega} \pi\left(\omega\right) f_{i\omega}\left(x_i\left(\omega\right)\right)\ ,$$

where, for each $\omega \in \Omega$, $f_{i\omega} : \mathbb{R}_+^C \to \mathbb{R}$. Hence the economy is characterized by the vector $\mathcal{E} := \left(\mathcal{P}, \Pi, (\mathcal{U}_i, e_i)_{i\in\mathcal{I}}\right)$. The budget constraint space, at the price system $(p,q) \in \widetilde{\mathcal{N}}$, can be rewritten in the following form:

$$B_i\left(p,q\right) := \left\{(x_i, z_i) \in \mathcal{L} :\ \left(x_i\left(\omega\right), z_i\left(\omega\right)\right) \in B_{i\omega}\left(p,q\right) \quad \forall\omega\in\Omega\right\}$$

where, for all $\omega \in \Omega$

$$\begin{aligned} B_{i\omega}\left(p,q\right) :=\{&\left(x_i\left(\omega\right), z_i\left(\omega\right)\right) \in \mathbb{R}_+^C \times R(\omega) : \\ &\left\langle p_0\left(\omega\right), x_{i0}\left(\omega\right)\right\rangle_H + \left\langle q\left(\omega\right), z_i\left(\omega\right)\right\rangle_{N-1} \le \left\langle p_0\left(\omega\right), e_{i0}\left(\omega\right)\right\rangle_H \\ &\left\langle p_t\left(\omega\right), x_{it}\left(\omega\right)\right\rangle_H \le \left\langle p_t\left(\omega\right), e_{it}\left(\omega\right)\right\rangle_H + p_t^1\left(\omega\right) z_{it}(\omega) \qquad \forall t\in\mathcal{T}\}. \end{aligned} \tag{7}$$

The element $R(\omega)$ is introduced similarly as in Section 2. We pose $B\left(p,q\right) := \prod_{i\in\mathcal{I}} B_i\left(p,q\right)$. The aim of each agent is to maximize the expected utility on the set $B_i\left(p,q\right) \cap \mathcal{N}$, which is a nonempty, closed, and convex set of $\mathcal{L}$. Finally, we can reformulate the equilibrium from a viewpoint of scenarios and, then, we can set the problem in the space of functions $\mathcal{L}$.

Definition 4. *An* equilibrium of plans, prices, and price expectations *for the economy* $\mathcal{E} := \left(\mathcal{P}, \Pi, (\mathcal{U}_i, e_i)_{i\in\mathcal{I}}\right)$ *is a vector* $\left((\bar{x}_i, \bar{z}_i)_{i\in\mathcal{I}}, \bar{p}, \bar{q}\right) \in \prod_{i\in\mathcal{I}}(B_i\left(\bar{p},\bar{q}\right) \cap \mathcal{N}) \times \widetilde{\mathcal{N}}$, *such that*

- *for any* $i \in \mathcal{I}$:

$$\max_{(x_i,z_i)\in B_i(\bar{p},\bar{q})\cap\mathcal{N}} \mathbb{E}\left[f_{i\omega}\left(x_i\right)\right] = \mathbb{E}\left[f_{i\omega}\left(\bar{x}_i\right)\right]\ ; \tag{8}$$

- *for any* $\omega \in \Omega$
$$\sum_{i \in \mathcal{I}} \bar{x}_i(\omega) \leq \sum_{i \in \mathcal{I}} e_i(\omega); \tag{9}$$
- *for any* $\omega \in \Omega$
$$\sum_{i \in \mathcal{I}} \bar{z}_i(\omega) = 0. \tag{10}$$

Conditions (9) and (10) can be rewritten in terms of components of the vectors $\bar{x}_i(\omega)$, $e_i(\omega)$ and $\bar{z}_i(\omega)$:

$$\sum_{i \in \mathcal{I}} \bar{x}_{it}(\omega) \leq \sum_{i \in \mathcal{I}} e_{it}(\omega) \quad \forall t \in \mathcal{T}_0 \ , \qquad \sum_{i \in \mathcal{I}} \bar{z}_{iF_t^j} = 0 \quad \forall F_t^j \in \mathcal{P} \setminus F_0 \, .$$

Remark 1. *We introduce, for all* $i \in \mathcal{I}$ *and* $\omega \in \Omega$*, the maximization problem*

$$\max_{(x_i(\omega), z_i(\omega)) \in B_{i\omega}(\bar{p}, \bar{q})} f_{i\omega}(x_i(\omega)) = f_{i\omega}(\bar{x}_i(\omega)). \tag{11}$$

We observe that if $\bar{x}_i \in \mathcal{L}_C$ *is such that, for all* $\omega \in \Omega$*,* $\bar{x}_i(\omega)$ *is a solution to (11) and* $\bar{x}_i \in \mathcal{N}^1$*, then* $\bar{x}_i$ *is a solution to (8).*

Following proposition shows that the definitions in terms of contingencies and in terms of scenarios are equivalent.

Proposition 1. *The vector* $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \prod_{i \in \mathcal{I}} (B_i(\bar{p}, \bar{q}) \cap \mathcal{N}) \times \widetilde{\mathcal{N}}$ *is an equilibrium according to Definition 4 if and only if it is an equilibrium according to Definition 1.*

Proof. Since each pair (ω, t) identifies the contingency ξ_t^j, it follows that conditions (2), (3) and (9), (10) are equivalent. We have to prove that $B_i(\bar{p}, \bar{q}) \cap \mathcal{N} \cong M_i(\bar{p}, \bar{q})$. Let $(x_i, z_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$. For all $\omega \in \Omega$ one has:

$$\langle p_0(\omega), x_{i0}(\omega) \rangle_H + \langle q(\omega), z_i(\omega) \rangle_{N-1} \leq \langle p_0(\omega), e_{i0}(\omega) \rangle_H \ .$$

Summing up $\omega \in \Omega$, it follows that:

$$\sum_{\omega \in \Omega} \langle p_0(\omega), x_{i0}(\omega) \rangle_H + \sum_{\omega \in \Omega} \langle q(\omega), z_i(\omega) \rangle_{N-1} \leq \sum_{\omega \in \Omega} \langle p_0(\omega), e_{i0}(\omega) \rangle_H \ .$$

Since $(x_i, z_i) \in \mathcal{N}$, $(p, q) \in \widetilde{\mathcal{N}}$ and e_i measurable, from (5) and (6) we get

$$\langle p(\xi_0), x_i(\xi_0) \rangle_H + \langle q, z_i \rangle_{N-1} \leq \langle p(\xi_0), e_i(\xi_0) \rangle_H$$

that is the first inequality of the constraint set $M_i(\bar{p}, \bar{q})$. In similar way, we can prove that all constraints of $B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$ hold if and only if constraints in $M_i(\bar{p}, \bar{q})$ hold. We conclude that $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \mathcal{L}$ is an equilibrium according to Definition 4 if and only if it is an equilibrium according to Definition 1. □

We introduce, for all $i \in \mathcal{I}$, the following assumptions.

Assumptions F

(F.1) $f_{i\omega}$ is C^1 and concave.

(F.2) $f_{i\omega}$ is strictly increasing in commodity-1: $\forall \tilde{x}_i(\omega), \tilde{\tilde{x}}_i(\omega) \in \mathbb{R}^C_+$ with $\tilde{x}_i(\omega) \geq \tilde{\tilde{x}}_i(\omega)$, then

$$\tilde{x}^1_{it}(\omega) > \tilde{\tilde{x}}^1_{it}(\omega) \quad for \quad some \quad t \in \mathcal{T}_0 \qquad \Rightarrow \qquad f_{i\omega}(\tilde{x}_i) > f_{i\omega}\left(\tilde{\tilde{x}}_i\right).$$

(F.3) $f_{i\omega}$ is non-satiated: $\forall x_i(\omega) \in \mathbb{R}^C_+ \; \exists \tilde{x}_i(\omega) \in \mathbb{R}^C_+$ s.t. $f_{i\omega}(\tilde{x}_i) > f_{i\omega}(x_i)$.

Assumptions U

(U.1) $\mathcal{U}_i$ is C^1 and concave.

(U.2) $\mathcal{U}_i$ is strictly increasing in commodity-1: $\forall \tilde{x}_i, \tilde{\tilde{x}}_i \in \mathcal{L}_C$ with $\tilde{x}_i \geq \tilde{\tilde{x}}_i$, then

$$\tilde{x}^1_i(\omega) > \tilde{\tilde{x}}^1_i(\omega) \quad for \quad some \quad \omega \in \Omega \qquad \Rightarrow \qquad \mathcal{U}_i(\tilde{x}_i) > \mathcal{U}_i\left(\tilde{\tilde{x}}_i\right).$$

(U.3) $\mathcal{U}_i$ is non-satiated: $\forall x_i \in \mathcal{L}_C \; \exists \tilde{x}_i \in \mathcal{L}_C$ s.t. $\mathcal{U}_i(\tilde{x}_i) > \mathcal{U}_i(x_i)$.

Proposition 2. *Let $i \in \mathcal{I}$. If for each $\omega \in \Omega$, $f_{i\omega}$ satisfies Assumptions F, then the expected utility $\mathcal{U}_i$ satisfies Assumptions U. Moreover, the gradient of $\mathcal{U}_i$ is monotone decreasing.*

Proof. Firstly, we introduce the gradient operator $\nabla\mathcal{U}_i : \mathcal{L}_C \to \mathcal{L}_C$, such that for all $x_i \in \mathcal{L}_C$ associates the map $\nabla\mathcal{U}_i(x_i)$, with

$$\begin{aligned}\nabla\mathcal{U}_i(x_i) :& \Omega \to \mathbb{R}^C \\ & \omega \to \nabla f_{i\omega}(x_i(\omega)).\end{aligned}$$

It follows that $\mathcal{U}_i$ and $\nabla\mathcal{U}_i$ are continuous (see Section 4 in [16]). The concavity and the strictly increasing in commodity-1 of $\mathcal{U}_i$ are immediate consequences of Assumptions (F.1) and (F.2). Furthermore, for all $x_i, \tilde{x}_i \in \mathcal{L}_C$, since from Assumption (F.1) $f_{i\omega}$ is concave, so $\nabla f_{i\omega}$ is monotonic decreasing. For all $\omega \in \Omega$ one has:

$$\langle \nabla f_{i\omega}(x_i) - \nabla f_{i\omega}(\tilde{x}_i), x_i(\omega) - \tilde{x}_i(\omega)\rangle_C \leq 0 \qquad \forall x_i(\omega), \tilde{x}_i(\omega) \in \mathbb{R}^C_+.$$

Hence:

$$\sum_{\omega\in\Omega} \pi(\omega)\langle \nabla f_{i\omega}(x_i) - \nabla f_{i\omega}(\tilde{x}_i), x_i(\omega) - \tilde{x}_i(\omega)\rangle_C = \langle\langle \nabla\mathcal{U}_i(x_i) - \nabla\mathcal{U}_i(\tilde{x}_i), x_i - \tilde{x}_i\rangle\rangle_C \leq 0$$

that is $\nabla\mathcal{U}_i$ is a monotone operator. Now, we prove that $\mathcal{U}_i$ is non-satiated. Let $x_i \in \mathcal{L}_C$, $\tilde{\omega} \in \Omega$ such that $\pi(\tilde{\omega}) > 0$ and $x_i(\tilde{\omega}) \in \mathbb{R}^C_+$. From Assumption (F.3), there exists $\tilde{x}_i(\tilde{\omega}) \in \mathbb{R}^C_+$ such that $f_{i\tilde{\omega}}(\tilde{x}_i) > f_{i\tilde{\omega}}(x_i)$. Let $\tilde{\tilde{x}}_i \in \mathcal{L}_C$ be such that $\tilde{\tilde{x}}_i(\omega) = x_i(\omega)$ for all $\omega \neq \tilde{\omega}$ and $\tilde{\tilde{x}}_i(\tilde{\omega}) = \tilde{x}_i(\tilde{\omega})$. One has:

$$\begin{aligned}\mathcal{U}_i(\tilde{\tilde{x}}_i) &= \sum_{\omega\in\Omega} \pi(\omega) f_{i\omega}\left(\tilde{\tilde{x}}_i\right) = \sum_{\omega\neq\tilde{\omega}} \pi(\omega) f_{i\omega}(x_i) + \pi(\tilde{\omega}) f_{i\tilde{\omega}}(\tilde{x}_i) > \\ &> \sum_{\omega\neq\tilde{\omega}} \pi(\omega) f_{i\omega}(x_i) + \pi(\tilde{\omega}) f_{i\tilde{\omega}}(x_i) = \sum_{\omega\in\Omega} \pi(\omega) f_{i\omega}(x_i) = \mathcal{U}_i(x_i) \qquad \Rightarrow \quad \mathcal{U}_i(\tilde{\tilde{x}}_i) > \mathcal{U}_i(x_i).\end{aligned}$$

□

We observe that, in order to have the non-satiated assumption of $\mathcal{U}_i$, it is sufficient that there exists at least one ω, with $\pi(\omega) > 0$, such that $f_{i\omega}$ satisfies Assumption (F.3).

Proposition 3. *Let Assumption (U.2) be satisfied. Let* $(\bar{x},\bar{z},\bar{p},\bar{q}) \in (B(\bar{p},\bar{q}) \cap \mathcal{N}) \times \widetilde{\mathcal{N}}$ *and* $(\bar{x},\bar{z})$ *be such that for all* $i \in \mathcal{I}$, $\bar{x}_i$ *is maximal for* $\mathcal{U}_i$ *in* $B_i(\bar{p},\bar{q}) \cap \mathcal{N}$*. Then, for any* $\omega \in \Omega$ *and* $t \in \mathcal{T}_0$, $\bar{p}^1_t(\omega) > 0$ *and* $\bar{q}_{F^j_t} > 0$ *for each* $F^j_t \in \mathcal{P} \setminus F_0$*.*

Proof. We assume that there exist $\omega \in \Omega$ and $t^* \in \mathcal{T}_0$, with $\omega \in F^j_{t^*}$, such that $\bar{p}^1_{t^*}(\omega) = 0$. Fixed $i \in \mathcal{I}$, we define $\hat{x}_i \in \mathcal{L}_C$ such that

$$\begin{aligned}\forall \omega \in \Omega: \quad & if \quad \omega \notin F^j_{t^*} \qquad \hat{x}_i(\omega) = \bar{x}_i(\omega)\\ & if \quad \omega \in F^j_{t^*} \qquad \hat{x}_i(\omega) = \begin{cases}\bar{x}_{it^*}(\omega) + K\underline{e}_1\\ \bar{x}_{it}(\omega) \qquad \forall t \neq t^*.\end{cases}\end{aligned}$$

where $K > 0$ and $\underline{e}_1 = (1,0,\dots,0) \in \mathbb{R}^H_+$. Then, $(\hat{x}_i,\bar{z}_i) \in B_i(\bar{p},\bar{q}) \cap \mathcal{N}$ and since $\mathcal{U}_i$ is strictly increasing in commodity-1 and $\hat{x}_i > \bar{x}_i$ we have that $\mathcal{U}_i(\hat{x}_i) > \mathcal{U}_i(\bar{x}_i)$ which contradicts the fact that $\bar{x}_i$ is a maximum point of $\mathcal{U}_i$ in $B_i(\bar{p},\bar{q}) \cap \mathcal{N}$.
The proof of $\bar{q}_{F^j_t} > 0$, for all $F^j_t \in \mathcal{P} \setminus F_0$, is close to the latter.

□

Thanks to the Proposition 3, without loss of generality, for all $\omega \in \Omega$ and $t \in \mathcal{T}_0$, we pose:

- $\Delta^0_\omega := \left\{(p_0(\omega), q(\omega)) \in \mathbb{R}^H_+ \times \mathbb{R}^{N-1}_+ : \sum_{h \in \mathcal{H}} p^h_0(\omega) + \sum_{F^j_t \in \mathcal{P}\setminus F_0} q_{F^j_t} = \frac{1}{|\Omega|}\right\}$
  and $\Delta_{F_0} := \{(p_0,q) \in \mathcal{L} : \quad (p_0(\omega), q(\omega)) \in \Delta^0_\omega \quad \forall \omega \in \Omega\}$;
- $\Delta^t_\omega := \left\{p_t(\omega) \in \mathbb{R}^H_+ : \sum_{h \in \mathcal{H}} p^h_t(\omega) = \frac{1}{\left|F^j_t\right|} \quad with \quad F^j_t \subseteq \Omega \quad s.t. \quad \omega \in F^j_t\right\}$
  and $\Delta_{F^j_t} := \left\{p_t \in \mathcal{L}_H : \quad p_t(\omega) \in \Delta^t_\omega \quad \forall \omega \in F^j_t\right\}$.

Therefore, by considering $\Delta_\omega := \prod_{t \in \mathcal{T}_0} \Delta^t_\omega$, the following simplex subspace is obtained

$$\Delta := \{(p,q) \in \widetilde{\mathcal{N}} : \quad \left((p_0(\omega), q(\omega)), (p_t(\omega))_{t \in \mathcal{T}}\right) \in \Delta_\omega \qquad \forall \omega \in \Omega\}. \tag{12}$$

5 A stochastic variational formulation

In this section, our aim is to reformulate the equilibrium problem as a suitable *stochastic quasi-variational problem* (SQVI). To this end, we follow the approach used in [16]. We introduce the following problem:

$$\text{Find } (\bar{x},\bar{z},\bar{p},\bar{q}) \in (B(\bar{p},\bar{q}) \cap \mathcal{N}) \times \Delta \text{ such that}$$

$$\sum_{i \in \mathcal{I}} \langle\langle \nabla \mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C + \langle\langle (\sum_{i \in \mathcal{I}} (\bar{x}_i - e_i), \sum_{i \in \mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q}) \rangle\rangle_D \leq 0 \tag{13}$$

$$\forall (x,z,p,q) \in (B(\bar{p},\bar{q}) \cap \mathcal{N}) \times \Delta\,.$$

Remark 2. *The vector* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *is a solution of the* $SQVI$ *(13) if and only if following inequalities simultaneously hold:*

(i) for each $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i)$ *is a solution to*

$$\langle\langle \nabla \mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C \leq 0 \qquad \forall (x_i, z_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}. \tag{14}$$

(ii) $(\bar{p}, \bar{q})$ *is a solution to*

$$\langle\langle (\sum_{i\in\mathcal{I}} (\bar{x}_i - e_i), \sum_{i\in\mathcal{I}} \bar{z}_i), (p, q) - (\bar{p}, \bar{q}) \rangle\rangle_D \leq 0 \qquad \forall (p, q) \in \Delta\,. \tag{15}$$

The following proposition will be useful to obtain the characterization.

Proposition 4. *Let* $(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$ *be a solution to (8). Then, for each* $\omega \in \Omega$ *one has:*

$$\langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega) \rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega) \rangle_{N-1} = 0\,, \tag{16}$$

$$\langle \bar{p}_t(\omega), \bar{x}_{it}(\omega) \rangle_H = \langle \bar{p}_{it}(\omega), e_{it}(\omega) \rangle_H + \bar{p}^1_t(\omega)\, \bar{z}_{iF^j_t} \qquad \forall t \in \mathcal{T}. \tag{17}$$

Proof. If there exists $\tilde{\omega} \in \Omega$ such that $\langle \bar{p}_0(\tilde{\omega}), \bar{x}_{i0}(\tilde{\omega}) - e_{i0}(\tilde{\omega}) \rangle_H + \langle \bar{q}(\tilde{\omega}), \bar{z}_i(\tilde{\omega}) \rangle_{N-1} < 0$, from F_0-measurability the strict inequality holds for each $\omega \in \Omega$. We define $\hat{x}_i \in \mathcal{L}_C$ such that, for all $\omega \in \Omega$:

$$\hat{x}_{it}(\omega) := \begin{cases} \bar{x}_{i0}(\omega) + K\underline{e}_1 \\ \bar{x}_{it}(\omega) \quad \forall t \in \mathcal{T} \end{cases} \qquad \textit{with} \quad 0 < K \leq -\frac{\langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega) \rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega) \rangle_{N-1}}{\bar{p}^1_0(\omega)}\,. \tag{18}$$

Since

$$\begin{aligned} &\langle \bar{p}_0(\omega), \hat{x}_{i0}(\omega) - e_{i0}(\omega) \rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega) \rangle_{N-1} = \\ &= \langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega) \rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega) \rangle_{N-1} + K\bar{p}^1_0(\omega) \leq \\ &\leq \langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega) \rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega) \rangle_{N-1} + \\ &\quad + \left(-\frac{\langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega) \rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega) \rangle_{N-1}}{\bar{p}^1_0(\omega)} \right) \bar{p}^1_0(\omega) = 0\,, \end{aligned}$$

one has that $(\hat{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$. Since $\mathcal{U}_i$ is strictly increasing in commodity-1 and $\hat{x}_i > \bar{x}_i$ we have that $\mathcal{U}_i(\hat{x}_i) > \mathcal{U}_i(\bar{x}_i)$, contradicting the fact that $\bar{x}_i$ is a maximum point of $\mathcal{U}_i$ in $B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$. In similar way, we get relation (17). □

Remark 3. *Let* $(\overline{x}, \overline{z}) = (\overline{x}_i, \overline{z}_i)_{i\in\mathcal{I}}$ *be such that* $(\overline{x}_i, \overline{z}_i)$ *is a solution to (8); then from Proposition 4, summing up to i inequalities (16) and (17) one has*

$$\langle \bar{p}_0(\omega), \sum_{i\in\mathcal{I}} (\bar{x}_{i0}(\omega) - e_{i0}(\omega)) \rangle_H + \langle \bar{q}(\omega), \sum_{i\in\mathcal{I}} \bar{z}_i(\omega) \rangle_{N-1} = 0\,, \tag{19}$$

$$\langle \bar{p}_t(\omega), \sum_{i\in\mathcal{I}} (\bar{x}_{it}(\omega) - e_{it}(\omega)) \rangle_H = \bar{p}^1_t(\omega) \sum_{i\in\mathcal{I}} \bar{z}_{iF^j_t}\,. \tag{20}$$

Theorem 3. *For all $i \in \mathcal{I}$, let $\mathcal{E}$ be an economy which satisfies the Assumptions U. Then, $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to the SQVI (13) if and only if it is an equilibrium vector of plans, prices, and price expectations for $\mathcal{E}$.*

Proof.
Claim 1 *For all $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i)$ is a solution of the maximization problem (8) if and only if it is a solution of (14).*
It follows from Proposition 2 and from Example 1 of Section 4 in [16], where $\mathcal{G} = -\mathcal{U}$ and $\mathcal{C} = B(\bar{p}, \bar{q})$.
Claim 2 *If $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to the SQVI (13), then $\forall \omega \in \Omega$ $\sum_{i \in \mathcal{I}} \bar{z}_i(\omega) \leq 0$ and $\sum_{i \in \mathcal{I}} (\bar{x}_i(\omega) - e_i(\omega)) \leq 0$.*
Let $G_0 = H + N - 1$ and $G_t = H$ for each $t \in \mathcal{T}$, from Remark 2, it follows that the following inequalities simultaneously hold

$$\langle\langle (\sum_{i \in \mathcal{I}} (\bar{x}_{i0} - e_{i0}), \sum_{i \in \mathcal{I}} \bar{z}_i), (p_0, q) - (\bar{p}_0, \bar{q}) \rangle\rangle_{H+N-1} \leq 0 \qquad \forall (p_0, q) \in \Delta_{F_0} \tag{21}$$

and for all $t \in \mathcal{T}$

$$\langle\langle \sum_{i \in \mathcal{I}} (\bar{x}_{it} - e_{it}), p_t - \bar{p}_t \rangle\rangle_H \leq 0 \qquad \forall p_t \in \Delta_{F_t^j}. \tag{22}$$

Since, for all $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$, summing up i the inequalities of (7), one has:

(i) $\left\langle \bar{q}(\omega), \sum_{i \in \mathcal{I}} \bar{z}_i(\omega) \right\rangle_{N-1} + \left\langle \bar{p}_0(\omega), \sum_{i \in \mathcal{I}} (\bar{x}_{i0}(\omega) - e_{i0}(\omega)) \right\rangle_H \leq 0$ for all $\omega \in \Omega$, that is

$$\langle\langle \bar{q}, \sum_{i \in \mathcal{I}} \bar{z}_i \rangle\rangle_{N-1} + \langle\langle \bar{p}_0, \sum_{i \in \mathcal{I}} (\bar{x}_{i0} - e_{i0}) \rangle\rangle_H \leq 0\,, \tag{23}$$

(ii) $\left\langle \bar{p}_t(\omega), \sum_{i \in \mathcal{I}} (\bar{x}_{it}(\omega) - e_{it}(\omega)) \right\rangle_H - \bar{p}_t^1(\omega) \left(\sum_{i \in \mathcal{I}} \bar{z}_{iF_t^j} \right) \leq 0$ for all $t \in \mathcal{T}$ and $\omega \in \Omega$, that is

$$\langle\langle \bar{p}_t, \sum_{i \in \mathcal{I}} (\bar{x}_{it} - e_{it}) \rangle\rangle_H - \langle\langle \bar{p}_t^1, \sum_{i \in \mathcal{I}} \bar{z}_{iF_t^j} \rangle\rangle_1 \leq 0\,. \tag{24}$$

From (21) and (23), we get

$$\langle\langle q, \sum_{i \in \mathcal{I}} \bar{z}_i \rangle\rangle_{N-1} + \langle\langle p_0, \sum_{i \in \mathcal{I}} (\bar{x}_{i0} - e_{i0}) \rangle\rangle_H \leq 0 \qquad \forall (p_0, q) \in \Delta_{F_0}. \tag{25}$$

For all $h^* \in \mathcal{H}$, we pose $(\tilde{p}_0, \tilde{q}) \in \mathcal{L}_{H+N-1}$ such that

$$\forall \omega \in \Omega: \qquad \tilde{p}_0^h(\omega) := \begin{cases} \frac{1}{|\Omega|} & if\ h = h^* \\ 0 & \forall h \neq h^* \end{cases} \quad , \quad \tilde{q}(\omega) = 0_{N-1}\,.$$

Being $(\tilde{p}_0, \tilde{q}) \in \Delta_{F_0}$, we can replace it in (25) and since x_i and e_i are F_0-measurable one has:

$$\sum_{\omega \in \Omega} \left(\pi(\omega) \tilde{p}_0^{h^*}(\omega) \sum_{i \in \mathcal{I}} \left(\bar{x}_{i0}^{h^*}(\omega) - e_{i0}^{h^*}(\omega) \right) \right) = \left(\sum_{\omega \in \Omega} \pi(\omega) \frac{1}{|\Omega|} \right) \sum_{i \in \mathcal{I}} \left(\bar{x}_{i0}^{h^*}(\omega) - e_{i0}^{h^*}(\omega) \right) =$$
$$= \sum_{i \in \mathcal{I}} \left(\bar{x}_{i0}^{h^*}(\omega) - e_{i0}^{h^*}(\omega) \right) \leq 0.$$

Hence, it follows that

$$\sum_{i\in\mathcal{I}}\left(\bar{x}^h_{i0}\left(\omega\right)-e^h_{i0}\left(\omega\right)\right)\leq 0 \qquad \forall\omega\in\Omega \;\; and \;\; \forall h\in\mathcal{H}.$$

Further, fixed $F^{j^*}_{t^*}$, we pose $(\tilde{p}_0,\tilde{q})\in\mathcal{L}_{H+N-1}$ such that

$$\forall\omega\in\Omega: \qquad \tilde{p_0}(\omega):=0_H \;\;,\;\; \tilde{q}\left(\omega\right):=\begin{cases}\frac{1}{|\Omega|} & if \;\; F^j_t=F^{j^*}_{t^*}\\ 0 & \forall F^j_t\neq F^{j^*}_{t^*}\end{cases}$$

Being $(\tilde{p}_0,\tilde{q})\in\Delta_{F_0}$, we can replace it in (25) and from measurability of z_i one has:

$$\sum_{\omega\in\Omega}\left(\pi(\omega)\tilde{q}(\omega)\sum_{i\in\mathcal{I}}\bar{z}_i\left(\omega\right)\right)=\left(\sum_{\omega\in\Omega}\pi(\omega)\frac{1}{|\Omega|}\right)\sum_{i\in\mathcal{I}}\bar{z}_{iF^j_t}=\sum_{i\in\mathcal{I}}\bar{z}_{iF^j_t}\leq 0\,.$$

Moreover, from the previous result and from (24), we have for all $t\in\mathcal{T}$

$$\langle\langle\bar{p}_t,\sum_{i\in\mathcal{I}}\left(\bar{x}_{it}-e_{it}\right)\rangle\rangle_H\leq\langle\langle\bar{p}^1_t,\sum_{i\in\mathcal{I}}\bar{z}_{iF^j_t}\rangle\rangle_1\leq 0$$

so that by (22) we get for all $t\in\mathcal{T}$

$$\langle\langle p_t,\sum_{i\in\mathcal{I}}\left(\bar{x}_{it}-e_{it}\right)\rangle\rangle_H\leq 0 \qquad \forall p_t\in\Delta_{F^j_t}\,. \tag{26}$$

Fixed a F^j_t, we pose $\tilde{p}_t\in\mathcal{L}_H$ such that

$$\forall\omega\in F^j_t, \qquad \tilde{p}^h_t(\omega):=\begin{cases}\frac{1}{|F^j_t|} & if \;\; h=h^*\\ 0 & \forall h\neq h^*\,.\end{cases}$$

Being $\tilde{p}_t\in\Delta_{F^j_t}$, we can replace it in (26) and since x_i and e_i are F_t-measurable one has:

$$\sum_{\omega\in F^j_t}\left(\pi(\omega)\tilde{p}_t(\omega)\sum_{i\in\mathcal{I}}(\bar{x}^{h^*}_{it}\left(\omega\right)-e^{h^*}_{it}\left(\omega\right))\right)=\left(\sum_{\omega\in F^j_t}\pi(\omega)\frac{1}{\left|F^j_t\right|}\right)\sum_{i\in\mathcal{I}}\left(\bar{x}^{h^*}_{it}\left(\omega\right)-e^{h^*}_{it}\left(\omega\right)\right)=$$
$$=\sum_{i\in\mathcal{I}}\left(\bar{x}^{h^*}_{it}\left(\omega\right)-e^{h^*}_{it}\left(\omega\right)\right)\leq 0.$$

Hence, it follows that for all $t\in\mathcal{T}$ and for all $h\in\mathcal{H}$

$$\sum_{i\in\mathcal{I}}\left(\bar{x}^h_{it}\left(\omega\right)-e^h_{it}\left(\omega\right)\right)\leq 0 \qquad \forall\omega\in\Omega\,.$$

Claim 3 *If $(\bar{x},\bar{z},\bar{p},\bar{q})$ is a solution to the SQVI (13), then $\forall\omega\in\Omega$ $\sum_{i\in\mathcal{I}}\bar{z}_i\left(\omega\right)=0$.*
From Claim 1, Proposition 3, and Claim 2, one has $\bar{q}(\omega)>0$ and $\sum_{i\in\mathcal{I}}\bar{z}_i(\omega)\leq 0$ $\forall\omega\in\Omega$, hence $\langle\langle\bar{q},\sum_{i\in\mathcal{I}}\bar{z}_i\rangle\rangle_{N-1}\leq 0$. If we suppose that $\left\langle\bar{q}\left(\omega\right),\sum_{i\in\mathcal{I}}\bar{z}_i\left(\omega\right)\right\rangle_{N-1}<0$ for some $\omega\in\Omega$, from Proposition 4 one has

$$\langle\bar{p}_0\left(\omega\right),\sum_{i\in\mathcal{I}}\left(\bar{x}_{i0}\left(\omega\right)-e_{i0}\left(\omega\right)\right)\rangle_H>0$$

which, being $p_0 \in \Delta_{F_0}$, contradicts Claim 2. Then, one has $\langle\langle \bar{q}, \sum_{i \in \mathcal{I}} \bar{z}_i \rangle\rangle_{N-1} = 0$ and since $\bar{q}(\omega) > 0$ for all $\omega \in \Omega$, we get $\sum_{i \in \mathcal{I}} \bar{z}_i(\omega) = 0$ for all $\omega \in \Omega$.

Then, thanks to Claims 1, 2 and 3, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to SQVI (13), then it is an equilibrium solution. Moreover, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium solution, from (19) and (20), condition (15) hold, and from Claim 1 (14) is satisfied. Then $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to (13). □

From theoretical and computational viewpoints, sometimes it will be useful to relax the nonanticipativity constraints of the decision variables. In doing this, we get the tools to formulate an equivalent problem allowing for point-wise optimization (see [14]). We pose $\mathcal{M}^1 := \left(\mathcal{N}^1\right)^\perp$ and $\mathcal{M}^2 := \left(\mathcal{N}^2\right)^\perp$ respectively the subspaces of the nonanticipativity multipliers relative to x and z and we pose $\mathcal{M} := \mathcal{M}^1 \times \mathcal{M}^2$, so that $\rho = \left(\rho^1, \rho^2\right) \in \mathcal{M}$.

Hence, for the Riesz orthogonal decomposition, one has $\mathcal{L}_C = \mathcal{N}^1 + \left(\mathcal{N}^1\right)^\perp$ and $\mathcal{L}_{N-1} = \mathcal{N}^2 + \left(\mathcal{N}^2\right)^\perp$, that is

$$\mathcal{L}_C = \mathcal{N}^1 + \mathcal{M}^1 \qquad \mathcal{L}_{N-1} = \mathcal{N}^2 + \mathcal{M}^2. \tag{27}$$

We fix $(\bar{p}, \bar{q}) \in \Delta$ and we introduce, as in [16], the following stochastic variational inequality in *extensive form*

$$\text{Find } (\bar{x}_i, \bar{z}_i) \in \mathcal{N} \text{ such that exists } \bar{\rho}_i \in \mathcal{M} \text{ and for all } \omega \in \Omega \text{ one has}$$

$$\left\langle \left(\nabla f_{i\omega}(\bar{x}_i) + \bar{\rho}_i^1(\omega), \bar{\rho}_i^2(\omega)\right), (x_i(\omega), z_i(\omega)) - (\bar{x}_i(\omega), \bar{z}_i(\omega)) \right\rangle_D \leq 0 \quad \forall (x_i(\omega), z_i(\omega)) \in B_{i\omega}(\bar{p}, \bar{q}). \tag{28}$$

Proposition 5. *The stochastic variational problems (28) and (14) are equivalent.*

Proof. We suppose that $(\bar{x}_i, \bar{z}_i)$ is a solution to (28); for each $\omega \in \Omega$, it follows that

$$\begin{aligned} &\langle \left(\nabla f_{i\omega}(\bar{x}_i) + \bar{\rho}_i^1(\omega), \bar{\rho}_i^2(\omega)\right), (x_i(\omega), z_i(\omega)) - (\bar{x}_i(\omega), \bar{z}_i(\omega)) \rangle_D = \\ =&\langle \nabla f_{i\omega}(\bar{x}_i), x_i(\omega) - \bar{x}_i(\omega) \rangle_C + \langle \bar{\rho}_i^1(\omega), x_i(\omega) - \bar{x}_i(\omega) \rangle_C + \langle \bar{\rho}_i^2(\omega), z_i(\omega) - \bar{z}_i(\omega) \rangle_{N-1} \leq 0 \\ &\qquad \forall (x_i(\omega), z_i(\omega)) \in B_{i\omega}(\bar{p}, \bar{q}). \end{aligned}$$

We multiply for $\pi(\omega)$ and we sum up to ω; one has

$$\langle\langle \nabla \mathcal{U}(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C + \langle\langle \bar{\rho}_i^1, x_i - \bar{x}_i \rangle\rangle_C + \langle\langle \bar{\rho}_i^2, z_i - \bar{z}_i \rangle\rangle_{N-1} \leq 0 \quad \forall (x_i, z_i) \in B_i(\bar{p}, \bar{q}). \tag{29}$$

Moreover, since $\bar{\rho}_i^1 \in \mathcal{M}^1 = \left(\mathcal{N}^1\right)^\perp$ and $\bar{\rho}_i^2 \in \mathcal{M}^2 = \left(\mathcal{N}^2\right)^\perp$, from (29) one has

$$\langle\langle \nabla \mathcal{U}(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C \leq 0 \quad \forall (x_i, z_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}. \tag{30}$$

Hence, $(\bar{x}_i, \bar{z}_i)$ is a solution to (14).

Being $B_{i\omega}(\bar{p}, \bar{q})$ a polyhedron for each $\omega \in \Omega$, from Theorem.3.2 in [16], the converse still holds. □

Thanks to Proposition 5 we can characterize the equilibrium vector as a solution to a variational problem in extensive form.

Corollary 1. *For all $i \in \mathcal{I}$ and $\omega \in \Omega$, let Assumptions F be satisfied. Then, $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (B(\bar{p}, \bar{q}) \cap \mathcal{N}) \times \Delta$ is a solution of the stochastic variational problem*

$$\sum_{i \in \mathcal{I}} \left\langle \left(\nabla f_{i\omega}\left(\bar{x}_i\right) + \bar{\rho}_i^1\left(\omega\right), \bar{\rho}_i^2\left(\omega\right)\right), \left(x_i\left(\omega\right), z_i\left(\omega\right)\right) - \left(\bar{x}_i\left(\omega\right), \bar{z}_i\left(\omega\right)\right)\right\rangle_D + \\ + \langle (\sum_{i \in \mathcal{I}} \left(\bar{x}_i(\omega) - e_i(\omega)\right), \sum_{i \in \mathcal{I}} \bar{z}_i(\omega)), (p(\omega), q(\omega)) - (\bar{p}(\omega), \bar{q}(\omega)) \rangle_D \leq 0 \\ \forall \left(x_i\left(\omega\right), z_i\left(\omega\right), p(\omega), q(\omega)\right) \in B_{i\omega}\left(\bar{p}, \bar{q}\right) \times \Delta_\omega \tag{31}$$

for all $\omega \in \Omega$ and for some $(\bar{\rho}^1, \bar{\rho}^2) \in \mathcal{M}$ if and only if it is an equilibrium vector of plans, prices, and price expectations for $\mathcal{E}$.

Proof. From Proposition 5 condition (28) is equivalent to the variational problem (14) which is equivalent to the equilibrium conditions. □

Proposition 6. *For each $i \in \mathcal{I}$, the set-valued map $B_i : \Delta \rightrightarrows \mathcal{L}$ is lower semicontinuous, closed and with nonempty, closed, and convex values.*

Proof. *B_i is a closed map.*
Let $\{(p_n, q_n)\}_{n \in \mathbb{N}} \subseteq \Delta$ and $\{(x_{i,n}, z_{i,n})\}_{n \in \mathbb{N}} \subseteq \mathcal{L}$ be such that $(x_{i,n}, z_{i,n}) \in B_i(p_n, q_n)$ for all n, $(p_n, q_n) \xrightarrow{\mathcal{L}} (p, q)$ and $(x_{i,n}, z_{i,n}) \xrightarrow{\mathcal{L}} (x_i, z_i)$.
Firstly, we observe that since $(p_n, q_n) \subseteq \Delta$, one has, for each $\omega \in \Omega$, $\{(p_{0,n}(\omega), q_n(\omega))\}_{n \in \mathbb{N}} \subseteq \Delta_\omega^0$ and $\{p_{t,n}(\omega)\}_{n \in \mathbb{N}} \subseteq \Delta_\omega^t$ for each $t \in \mathcal{T}$; hence this sequence converges to $(p(\omega), q(\omega))$. For each $n \in \mathbb{N}$, when $(x_{i,n}, z_{i,n}) \in B_i(p_n, q_n)$ one has $(x_{i,n}(\omega), z_{i,n}(\omega)) \in B_{i\omega}(p_n, q_n)$ for each $\omega \in \Omega$, that is

$$0 \leq \langle p_{0,n}\left(\omega\right), x_{i0,n}\left(\omega\right)\rangle_H \leq -\left\langle q_n\left(\omega\right), z_{i,n}\left(\omega\right)\right\rangle_{N-1} + \langle p_{0,n}\left(\omega\right), e_{i0}\left(\omega\right)\rangle_H \\ 0 \leq \langle p_{t,n}\left(\omega\right), x_{it,n}\left(\omega\right)\rangle_H \leq \langle p_{t,n}\left(\omega\right), e_{it}\left(\omega\right)\rangle_H + p_{t,n}^1\left(\omega\right) z_{it,n}(\omega) \qquad \forall t \in \mathcal{T}. \tag{32}$$

Since $z_{i,n}(\omega) \in R(\omega)$ one has that, for all $\omega \in \Omega$, $\{z_{i,n}(\omega)\}_{n \in \mathbb{N}}$ converges to $z_i(\omega)$. Hence, from (32), one has that the sequence $\{x_{i,n}(\omega)\}_{n \in \mathbb{N}}$ is bounded and converges to $x_i(\omega)$. Then $(x_i(\omega), z_i(\omega)) \in B_{i\omega}(p, q)$, for all $\omega \in \Omega$, and $(x_i, z_i) \in B_i(p, q)$; hence, B_i is a closed map.

B_i is with nonempty, closed, and convex values.
We fix $(p, q) \in \Delta$. Since $(e_i, 0_{\mathcal{L}}) \in B_i(p, q)$, it follows that $B_i(p, q)$ is nonempty and, from definition, $B_i(p, q)$ is a convex set. Being B_i a closed map, then its values are necessarily closed.

B_i is lower semicontinuous.
Let $\{(p_n, q_n)\}_{n \in \mathbb{N}} \subseteq \Delta$ be converging to (p, q); for all $(x_i, z_i) \in B_i(p, q)$ we have to prove that there exists a sequence $\{(x_{i,n}, z_{i,n})\}_{n \in \mathbb{N}} \subseteq \mathcal{L}$ such that $(x_{i,n}, z_{i,n}) \in B_i(p_n, q_n)$ and $(x_{i,n}, z_{i,n}) \xrightarrow{\mathcal{L}} (x_i, z_i)$. It is clear that, for all $\omega \in \Omega$, we can consider $(x_i(\omega), z_i(\omega))$ and it is sufficient to find $(x_{i,n}(\omega), z_{i,n}(\omega)) \in B_{i\omega}(p_n, q_n)$ such that $(x_{i,n}(\omega), z_{i,n}(\omega)) \to (x_i(\omega), z_i(\omega))$. Fixed $\omega \in \Omega$, if

$$\langle p_0\left(\omega\right), x_{i0}\left(\omega\right)\rangle_H + \langle q\left(\omega\right), z_i\left(\omega\right)\rangle_{N-1} < \langle p_0\left(\omega\right), e_{i0}\left(\omega\right)\rangle_H \\ \langle p_t\left(\omega\right), x_{it}\left(\omega\right)\rangle_H < \langle p_t\left(\omega\right), e_{it}\left(\omega\right)\rangle_H + p_t^1\left(\omega\right) z_{it}(\omega) \qquad t \in \mathcal{T} \tag{33}$$

then

$$\langle p_{0,n}\left(\omega\right), x_{i0}\left(\omega\right)\rangle_H + \langle q_n\left(\omega\right), z_i\left(\omega\right)\rangle_{N-1} < \langle p_{0,n}\left(\omega\right), e_{i0}\left(\omega\right)\rangle_H \\ \langle p_{t,n}\left(\omega\right), x_{it}\left(\omega\right)\rangle_H < \langle p_{t,n}\left(\omega\right), e_{it}\left(\omega\right)\rangle_H + p_{t,n}^1\left(\omega\right) z_{it}(\omega) \qquad t \in \mathcal{T}.$$

Hence $(x_i(\omega), z_i(\omega)) \in B_{i\omega}(p_n, q_n)$ and then $(x_i(\omega), z_i(\omega)) \in LiB_{i\omega}(p_n, q_n)$, where we identify with $LiB_{i\omega}(p_n, q_n)$ the lower limit, in Kuratowski sense, of the sequence $B_{i\omega}(p_n, q_n)$. We suppose that $(x_i(\omega), z_i(\omega))$ is such that at least one inequality of (33) is not satisfied. Being $e_i(\omega) \in \mathbb{R}^C_{++}$, there exists $x_i(\omega) \in \mathbb{R}^C_{++}$ such that $(x_i(\omega), 0_{N-1})$ satisfies the strict inequalities of (33), hence it belongs to the interior of $B_{i\omega}(p, q)$. Then, from Proposition 1.1.14 (v) of [10] and being $B_{i\omega}(p, q)$ a closed set, one has $cl\ int\ B_{i\omega}(p, q) = cl\ B_{i\omega}(p, q) = B_{i\omega}(p, q)$. Clearly, from definition of $LiB_{i\omega}(p_n, q_n)$, one has $int\ B_{i\omega}(p, q) \subset LiB_{i\omega}(p_n, q_n)$ and, from Proposition 8.2.1 of [10], $LiB_{i\omega}(p_n, q_n)$ is a closed set. Hence:

$$B_{i\omega}(p, q) = cl\ int\ B_{i\omega}(p, q) \subset cl\ LiB_{i\omega}(p_n, q_n) = LiB_{i\omega}(p_n, q_n).$$

We can conclude that B_i is lower semicontinuous. □

Theorem 4. *Let Assumptions F be satisfied for each $\omega \in \Omega$ and $i \in \mathcal{I}$. Then, there exists an equilibrium vector of plans, prices, and price expectations for $\mathcal{E}$.*

Proof. In order to prove the existence of equilibrium, thank to Theorem 3, we prove that the SQVI (13) admits at least one solution. For each $\omega \in \Omega$ and $(p(\omega), q(\omega)) \in \Delta_\omega$, we introduce the bounded set

$$\widetilde{B}_{i\omega}(p, q) := \prod_{i \in \mathcal{I}} \left[B_{i\omega}(p, q) \cap \left(\left[0, \sum_{i \in \mathcal{I}} e_i(\omega) + \widetilde{M} \right] \times \mathbb{R}^{N-1} \right) \right] \tag{34}$$

where $\widetilde{M} \in \mathbb{R}_+$. We observe that from properties of map $B_{i\omega}$, proved in Proposition 6, the map $\widetilde{B}_{i\omega}$ is lower semicontinuous, closed, and with nonempty, closed, and convex values. We denote by $SQVI(\widetilde{B})$ the variational problem (13) in the convex set $\widetilde{B}(p, q)$.

There exists the solution of $SQVI(\widetilde{B})$.

For each $i \in \mathcal{I}$ and $(p, q) \in \Delta$, we consider the parametric stochastic variational inequality SVI(p, q):

$$\text{Find } (\bar{x}_i, \bar{z}_i) \in \widetilde{B}_i(p, q) \cap \mathcal{N} \text{ such that}$$

$$\langle\langle \nabla \mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C \leq 0 \qquad \forall (x_i, z_i) \in \widetilde{B}_i(p, q) \cap \mathcal{N}\,. \tag{35}$$

We introduce the map of the solutions $\Phi_i : \Delta \rightrightarrows \mathcal{L}$ such that, for all $(p, q) \in \Delta$,

$$\Phi_i(p, q) := \{(\bar{x}_i, \bar{z}_i) : (\bar{x}_i, \bar{z}_i) \ \ is \ \ solution \ \ of \ \ SVI(p, q)\ (35)\}\,.$$

From Proposition 2, it follows that operator $\nabla \mathcal{U}_i$ is continuous and monotone; moreover, since $(e_i, 0) \in \widetilde{B}_i(p, q)$, which is measurable, we get $\widetilde{B}_i(p, q) \cap \mathcal{N} \neq \emptyset$. Thanks to Theorem 2, it follows that, for all $(p, q) \in \Delta$, $\Phi_i(p, q)$ is nonempty, bounded, closed, and convex. We prove that Φ_i is closed. Let $\{(p_n, q_n)\}_{n \in \mathbb{N}} \subseteq \Delta$ and $\{(\bar{x}_{i,n}, \bar{z}_{i,n})\}_{n \in \mathbb{N}} \subseteq \mathcal{L}$ be two sequences with $(\bar{x}_{i,n}, \bar{z}_{i,n}) \in \Phi_i(p_n, q_n)$, and such that $(p_n, q_n) \xrightarrow{\mathcal{L}} (p, q)$ and $(\bar{x}_{i,n}, \bar{z}_{i,n}) \xrightarrow{\mathcal{L}} (\bar{x}, \bar{z})$, we have to prove that $(\bar{x}_i, \bar{z}_i) \in \Phi_i(p, q)$. Since $\widetilde{B}_i$ is a closed map then $(\bar{x}_i, \bar{z}_i) \in \widetilde{B}_i(p, q)$. Being $\widetilde{B}_i$ is lower semicontinuous, it follows that for each $(x_i, z_i) \in \widetilde{B}_i(p, q)$ there exists a sequence $\{(x_{i,n}, z_{i,n})\}_{n \in \mathbb{N}}$ converging to (x_i, z_i) such that $(x_{i,n}, z_{i,n}) \in \widetilde{B}_i(p_n, q_n)$ for all n. Since $(\bar{x}_{i,n}, \bar{z}_{i,n}) \in \Phi_i(p_n, q_n)$, then

$$\langle\langle \nabla \mathcal{U}_i(\bar{x}_{i,n}), x_{i,n} - \bar{x}_{i,n} \rangle\rangle_C \leq 0 \text{ and passing to the limit } \langle\langle \nabla \mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C \leq 0\,,$$

that is $(\bar{x}_i, \bar{z}_i) \in \Phi_i(p, q)$. Hence, for each $(p, q) \in \Delta$, since $\Phi(p, q) \subseteq \left(\left[0, \sum_{i \in \mathcal{I}} e_i(\omega) + \widetilde{M} \right] \times R \right)$, it follows that $\Phi_i(p, q)$ is also a compact map. Being $\Phi_i(p, q)$ a closed and compact map, it is upper semicontinuous. Furthermore the map $\Phi(p, q) := \prod_{i \in \mathcal{I}} \Phi_i(p, q)$ is upper semicontinuous.

Now, we consider the following stochastic generalized variational inequality:

Find $(\bar{p},\bar{q}) \in \Delta$ such that there exists $(\bar{x},\bar{z}) \in \Phi(\bar{p},\bar{q})$ and

$$\langle\langle(\sum_{i\in\mathcal{I}} (\bar{x}_i - e_i)\, , \sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q})\rangle\rangle_D \le 0 \qquad \forall\, (p,q) \in \Delta\, . \tag{36}$$

From properties of Δ and Φ and thanks to Theorem 1, there exists $(\bar{p},\bar{q}) \in \Delta$ and $(\bar{x},\bar{z}) \in \Phi(\bar{p},\bar{q})$ solutions to (36). So, $(\bar{x},\bar{z},\bar{p},\bar{q}) \in \left(\widetilde{B}\,(\bar{p},\bar{q}) \cap \mathcal{N}\right) \times \Delta$, with $(\overline{p},\overline{q})$ solution to (36) and $(\overline{x},\overline{z}) \in \Phi(\overline{p},\overline{q})$, is a solution to $SQVI(\widetilde{B})$.

Any solution of the $SQVI(\widetilde{B})$ is a solution of $SQVI$ (13).
Let $(\bar{x},\bar{z},\bar{p},\bar{q})$ be a solution of $SQVI(\widetilde{B})$. Thanks to Remark 2, it is sufficient to prove that $(\bar{x}_i,\bar{z}_i)$ is a solution to (14). We suppose that there exists $(\hat{x}_i,\hat{z}_i) \in B_i\,(\bar{p},\bar{q}) \cap \mathcal{N}$ such that

$$\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \hat{x}_i - \bar{x}_i\rangle\rangle_C > 0. \tag{37}$$

Let $\lambda \in [0,1]$ be such that

$$0 < \lambda < \min\left\{1; \frac{\sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \bar{x}_i^h(\omega)}{\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega)}\;,\quad \textit{with}\;\; h \in \mathcal{H}\;\; \textit{s.t.}\;\; \hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) > 0\right\} \tag{38}$$

and we pose $(\tilde{x}_i,\tilde{z}_i) = \lambda(\hat{x}_i,\hat{z}_i) + (1-\lambda)(\bar{x}_i,\bar{z}_i)$. From convexity of $B\,(\bar{p},\bar{q}) \cap \mathcal{N}$ one has $(\tilde{x}_i,\tilde{z}_i) \in B_i\,(\bar{p},\bar{q}) \cap \mathcal{N}$ and it results that $(\tilde{x}_i,\tilde{z}_i)$ is still in (34). Indeed, for each $\omega \in \Omega$ and $h \in \mathcal{H}$, one has:

$$\begin{aligned}\sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) &= \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \lambda\hat{x}_i^h(\omega) - (1-\lambda)\bar{x}_i^h(\omega) = \\ &= \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \lambda\left[\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega)\right] - \bar{x}_i^h(\omega).\end{aligned}$$

Hence, for all $h \in \mathcal{H}$, one has:

(i) if $\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) = 0 \Rightarrow \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) = \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \bar{x}_i^h(\omega) \ge 0$;

(ii) if $\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) < 0 \Rightarrow \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) > \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \bar{x}_i^h(\omega) \ge 0$;

(iii) if $\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) > 0 \Rightarrow$ from (38) one has $\sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) > 0$.

Hence $(\tilde{x}_i,\tilde{z}_i) \in \widetilde{B}\,(\bar{p},\bar{q}) \cap \mathcal{N}$ and moreover, from inequality (37)

$$\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \tilde{x}_i - \bar{x}_i\rangle\rangle_C = \langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \lambda\hat{x}_i + (1-\lambda)\bar{x}_i - \bar{x}_i\rangle\rangle_C = \lambda\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \hat{x}_i - \bar{x}_i\rangle\rangle_C > 0\,.$$

This contradicts the fact that $(\bar{x}_i,\bar{z}_i)$ is a solution to SVI (35). Thus, we can conclude that $(\bar{x},\bar{z},\bar{p},\bar{q})$ is still a solution of $SQVI$ (13). □
We point out that a quasi-variational inequalities problem is characterized by the fact that the convex set depends on the solution of the problem; this fact can represent a difficulty to solve the problem. With the operative approach used to prove Theorem 4, we overcame this difficulty: instead of solving a stochastic quasi-variational inequality, we handle stochastic variational inequalities problems. From this, the next remark follows.

Remark 4. *In the assumption (F.1) if we replace the concavity of $f_{i\omega}$ with the strict concavity, with similar arguments of Proposition 2, one has that $\mathcal{U}_i$ is strictly concave and $\nabla\mathcal{U}_i$ is a strictly monotone operator. Then the parametric stochastic variational inequality (35) admits a unique solution and the map of the solutions Φ_i reduces to a single-valued map. Hence, the problem (36) becomes the variational inequality*

$$\textit{Find} \quad (\bar{p},\bar{q}) \in \Delta \quad \textit{such that} \quad \langle\langle (\sum_{i\in\mathcal{I}} (\bar{x}_i - e_i)\,, \sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q})\rangle\rangle_D \leq 0 \qquad \forall\,(p,q) \in \Delta\,.$$

However, we observe that in order to guarantee the uniqueness of equilibrium we need to prove that the map of solution Φ is strictly monotone, so that we obtain the uniqueness of solution of (36) too.

6 Computation procedure

In this section, we present a computational procedure to find the equilibrium solution by solving the $SQVI$ (31). To this aim we use the same procedure used to prove Theorem 4. Under Assumptions F, for all $i \in \mathcal{I}$ and $\omega \in \Omega$, we build two sequences $\{(\hat{x}^\nu, \hat{z}^\nu)\}_{\nu\in\mathbb{N}} \subseteq \mathcal{L}$ and $\{(\hat{p}^n, \hat{q}^n)\}_{n\in\mathbb{N}} \subseteq \Delta$ which converge to a solution of (31).
The procedure is structured in two sequential phases. At each phase, we split the stochastic variational problem into a finite number of deterministic ones and we solve them in *parallel*. This allow us to deal efficiently with large-scale problems arising from real-world applications in a dynamic-stochastic framework.

Procedure: Phase 1

In the first phase, we fix $(p,q) \in \Delta$ and we solve the parametric stochastic variational inequality (35). We use the procedure known in literature as Progressive Hedging Algorithm, which allows us to split the variational problem, which is set in the space of functions $\mathcal{L}$, into $|\mathcal{I}| \cdot |\Omega| = IS$ variational problems $[SVI\,(i,\omega)]$ in $\mathbb{R}^D$.

Progressive Hedging Algorithm
We introduce two sequences $\{(\hat{x}^\nu, \hat{z}^\nu)\}_{\nu\in\mathbb{N}} \subseteq \mathcal{L}$ and $\{\hat{\rho}^\nu\}_{\nu\in\mathbb{N}} \subseteq \mathcal{M}$:
let $\hat{\rho}^0 = 0$ as starting point, $r > 0$ a fixed parameter and $\nu \in \mathbb{N}$ an iteration index.

$$\boxed{\nu = 1}$$

(i) `Choice of` $(\hat{x}^1, \hat{z}^1) \in \mathcal{L}$. For all $i \in \mathcal{I}$ and $\omega \in \Omega$, we consider the $[SVI\,(i,\omega)]$:

$$\left\langle \nabla f_{i\omega}\left(\hat{x}_i^1\right), x_i(\omega) - \hat{x}_i^1(\omega)\right\rangle_C \leq 0 \qquad \forall\,(x_i\,(\omega)\,, z_i\,(\omega)) \in \widetilde{B}_{i\omega}(p,q) \tag{39}$$

Since the operator is continuous and $\widetilde{B}_{i\omega}(p,q)$ is a bounded set, there exists at least one solution of (39). We choose $\left(\hat{x}_i^1\,(\omega)\,, \hat{z}_i^1\,(\omega)\right)$ arbitrarily, among the solution set of (39).

(ii) We pose $\left(\tilde{x}_i^1, \tilde{z}_i^1\right) = P_{\mathcal{N}}\left(\hat{x}_i^1, \hat{z}_i^1\right)$ and $\hat{\rho}_i^1 = rP_{\mathcal{M}}\left(\hat{x}_i^1, \hat{z}_i^1\right)$. We denote by $P_{\mathcal{N}}\left(\hat{x}_i^1, \hat{z}_i^1\right)$ and $P_{\mathcal{M}}\left(\hat{x}_i^1, \hat{z}_i^1\right)$ the projection of $\left(\hat{x}_i^1, \hat{z}_i^1\right)$ to sets, respectively, $\mathcal{N}$ and $\mathcal{M}$.

$$\boxed{\forall\, \nu \in \mathbb{N}}$$

(i) `Choice of` $(\hat{x}^\nu, \hat{z}^\nu) \in \mathcal{L}$. For all $i \in \mathcal{I}$ and $\omega \in \Omega$, we consider the stochastic variational problem

$$\begin{gathered} \langle \nabla f_{i\omega}\left(\hat{x}_i^\nu, \hat{z}_i^\nu\right) + \hat{\rho}_i^{\nu-1}(\omega) \\ + r\left[\left(\hat{x}_i^\nu(\omega), \hat{z}_i^\nu(\omega)\right) - \left(\tilde{x}_i^{\nu-1}(\omega), \tilde{z}_i^{\nu-1}(\omega)\right)\right], \left(x_i(\omega), z_i(\omega)\right) - \left(\hat{x}_i(\omega), \hat{z}_i(\omega)\right)\rangle_D \leq 0 \\ \forall (x_i(\omega), z_i(\omega)) \in \widetilde{B}_{i\omega}(p, q)\,. \end{gathered} \tag{40}$$

The operator is strongly monotone, then there exists a unique solution $\left(\hat{x}_i^\nu(\omega), \hat{z}_i^\nu(\omega)\right)$. Hence, we set $(\hat{x}_i^\nu, \hat{z}_i^\nu) \in \mathcal{L}$ such that, for all $\omega \in \Omega$, $\left(\hat{x}_i^\nu(\omega), \hat{z}_i^\nu(\omega)\right)$ is the unique solution to (40).

(ii) We pose $(\tilde{x}_i^\nu, \tilde{z}_i^\nu) = P_{\mathcal{N}}\left(\hat{x}_i^\nu, \hat{z}_i^\nu\right)$ and $\hat{\rho}_i^\nu = \hat{\rho}_i^{\nu-1} + rP_{\mathcal{M}}\left(\hat{x}_i^{\nu-1}, \hat{z}_i^{\nu-1}\right)$.

Convergence

From Theorem 2 of [17] it follows that $(\hat{x}^\nu, \hat{z}^\nu) \xrightarrow{\mathcal{L}} (\bar{x}, \bar{z}) \in \mathcal{N}$ *and* $\hat{\rho}^\nu \xrightarrow{\mathcal{L}} \bar{\rho} \in \mathcal{M}$. Moreover, $(\bar{x}, \bar{z})$ is a solution to the parametric SVI in extensive form (28) and, thanks to Proposition 5, $(\bar{x}, \bar{z}) \in \widetilde{B}(p, q) \cap \mathcal{N}$ is a solution to (35). We call $(\bar{x}, \bar{z})$ as *optimal strategy solution.*

Procedure: Phase 2

In this phase we use the Projected Subgradient Algorithm to solve the SVI (36), where for all $(p, q) \in \Delta$, $(\bar{x}(p, q), \bar{z}(p, q))$ is the optimal strategy solution obtained in Phase 1. We pose

$$\varphi(p,q) := -(\varphi_1(p,q), \varphi_2(p,q)) \qquad \varphi_1(p,q) := \sum_{i \in \mathcal{I}} (\bar{x}_i(p,q) - e_i) \quad , \quad \varphi_2(p,q) := \sum_{i \in \mathcal{I}} \bar{z}_i(p,q)$$

and, for each $\omega \in \Omega$, we consider the problem

$$\begin{gathered} \text{Find } (\bar{p}(\omega), \bar{q}(\omega)) \in \Delta_\omega \text{ such that} \\ \langle \varphi_\omega(\bar{p}, \bar{q}), (p(\omega), q(\omega)) - (\bar{p}(\omega), \bar{q}(\omega))\rangle_D \geq 0 \qquad \forall (p(\omega), q(\omega)) \in \Delta_\omega\,. \end{gathered} \tag{41}$$

Thanks to the structure of Δ and the measurability of $(\bar{x}(p, q), \bar{z}(p, q))$, we can consider the S deterministic variational problems (41) in $\mathbb{R}^D$ and solving them in parallel. We introduce the Auslender's gap function (see, e.g. [2]):

$$\begin{array}{rcl} \Psi_\omega: & \Delta_\omega \to & \mathbb{R} \\ (\tilde{p}(\omega), \tilde{q}(\omega)) \to & \Psi_\omega(\tilde{p}, \tilde{q}) = & \displaystyle\max_{(p(\omega), q(\omega)) \in \Delta_\omega} \langle \varphi_\omega(\tilde{p}, \tilde{q}), (\tilde{p}(\omega), \tilde{q}(\omega)) - (p(\omega), q(\omega))\rangle_D \end{array} \tag{42}$$

For this map following properties hold. From Theorem 4, since φ_ω is a single-valued map, it follows that φ_ω is continuous; hence, from compactness of Δ_ω, one has that Ψ_ω is well posed. Moreover, from Theorem 8.3. in [15], it follows that operator Ψ_ω is proper, convex, and lower semicontinuous being the maximum of a family of affine continue functions and $\Psi_\omega(\tilde{p}, \tilde{q}) \geq 0$ for all $(\tilde{p}, \tilde{q})$. We pose $\partial\, \Psi_\omega$ the subdifferential of Ψ_ω:

$$\partial \Psi_\omega(\tilde{p}, \tilde{q}) = \left\{\tau \in \mathbb{R}^D \;:\; \Psi_\omega(p, q) - \Psi_\omega(\tilde{p}, \tilde{q}) \geq \langle \tau_\omega, (p(\omega), q(\omega)) - (\tilde{p}(\omega), \tilde{q}(\omega))\rangle_D \quad \forall (p(\omega), q(\omega)) \in \Delta_\omega\right\}.$$

From Theorem 3.2.15 in [10], $\partial \Psi_\omega(p, q) \neq \emptyset$ for all $(p, q) \in$ *ri dom* Ψ_ω. Moreover, one has:

$\Psi_\omega(\tilde{p},\tilde{q}) = 0$ if and only if $(\tilde{p}(\omega),\tilde{q}(\omega))$ is a solution to (41).

Projected Subgradient Algorithm
We introduce the sequence $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\} \subseteq \Delta_\omega$. We fix a starting point $(\hat{p}^1(\omega),\hat{q}^1(\omega)) \in \Delta_\omega$; it is usual to consider the centroid of Δ_ω. Clearly, if $\Psi_\omega(\hat{p}^1,\hat{q}^1) = 0$, one has that $(\hat{p}^1(\omega),\hat{q}^1(\omega))$ is a solution to (41). We suppose that $\Psi_\omega(\hat{p}^1,\hat{q}^1) > 0$.

$$\boxed{n \in \mathbb{N}}$$

`Choice of` $(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega)) \in \Delta_\omega$. For all $n \in \mathbb{N}$:

$$\left(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega)\right) = P_{\Delta_\omega}\left(\left(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega)\right) - \tau_\omega^n\rho_\omega^n\right) \tag{43}$$

where

$$\tau_\omega^n \in \partial\Psi_\omega\left(\hat{p}^n,\hat{q}^n\right) \quad \textit{and} \quad \rho_\omega^n = \frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)}{\|\tau_\omega^n\|^2}$$

Also in this case, at each iteration $n \in \mathbb{N}$, the variational sub-problems are solved in *parallel* through a *warm start* procedure, until a suitable solution of (36) is obtained, that is up to we get for each $\omega \in \Omega$ a limit point $(\hat{p}(\omega),\hat{q}(\omega))$, of the approximating sequence $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\}_{n\in\mathbb{N}}$, such that $\Psi_\omega(\hat{p},\hat{q}) = 0$.

Convergence
Let $\{(\hat{p}^n,\hat{q}^n)\}_{n\in\mathbb{N}} \subseteq \Delta$ be the sequence such that for all $\omega \in \Omega$, $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\}_{n\in\mathbb{N}} \subseteq \Delta_\omega$ is given by (43) with $\{\tau_\omega^n\}_{n\in\mathbb{N}}$ bounded. We prove that the sequence converges to the solution to SVI (36).
Let $(\bar{p}(\omega),\bar{q}(\omega))$ be a solution to (41); it is sufficient to prove that for all $\omega \in \Omega$, $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\}_{n\in\mathbb{N}}$ converges to $(\bar{p}(\omega),\bar{q}(\omega))$. Firstly, we observe that

$$\langle\tau_\omega^n,(\bar{p}(\omega),\bar{q}(\omega)) - (\hat{p}^n(\omega),\hat{q}^n(\omega))\rangle_D \leq \Psi_\omega(\bar{p},\bar{q}) - \Psi_\omega(\hat{p}^n,\hat{q}^n) = -\Psi_\omega(\hat{p}^n,\hat{q}^n)\,. \tag{44}$$

Hence, from (44), (43) and from nonexpansivity of projection mapping, it follows that

$$\begin{aligned}
&\left\|(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 = \\
&= \left\|P_{\Delta_\omega}\left((\hat{p}^n(\omega),\hat{q}^n(\omega)) - \tau_\omega^n\rho_\omega^n\right) - P_{\Delta_\omega}(\bar{p}(\omega),\bar{q}(\omega))\right\|^2 \leq \left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - \tau_\omega^n\rho_\omega^n - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 = \\
&= \left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 + (\rho_\omega^n)^2\left\|\tau_\omega^n\right\|^2 + 2\rho_\omega^n\langle\tau_\omega^n,(\bar{p}(\omega),\bar{q}(\omega)) - (\hat{p}^n(\omega),\hat{q}^n(\omega))\rangle \leq \\
&\leq \left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 + (\rho_\omega^n)^2\left\|\tau_\omega^n\right\|^2 - 2\rho_\omega^n\Psi_\omega(\hat{p}^n,\hat{q}^n) = \\
&= \left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 + \frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)^2}{\|\tau_\omega^n\|^4}\left\|\tau_\omega^n\right\|^2 - 2\frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)}{\|\tau_\omega^n\|^2}\Psi_\omega(\hat{p}^n,\hat{q}^n) = \\
&= \left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 - \frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)^2}{\|\tau_\omega^n\|^2} \leq \left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2.
\end{aligned}$$

Hence, the sequence $\left\{\left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2\right\}_{n\in\mathbb{N}}$ is decreasing, and we get

$$0 \leq \Psi_\omega(\hat{p}^n,\hat{q}^n)^2 \leq \|\tau_\omega^n\|^2\left(\left\|(\hat{p}^n(\omega),\hat{q}^n(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2 - \left\|(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega)) - (\bar{p}(\omega),\bar{q}(\omega))\right\|^2\right)$$

and since $\{\tau_\omega^n\}_{n\in\mathbb{N}}$ is bounded, it follows that $\lim_{n\to+\infty} \Psi_\omega(\hat{p}^n, \hat{q}^n) = \Psi_\omega(\hat{p}, \hat{q}) = 0$, hence $(\hat{p}(\omega), \hat{q}(\omega))$ is a solution to (41). Then, we can conclude that $(\hat{p}, \hat{q})$ is a solution to (36).

So, when $\nu \to \infty$ and $n \to \infty$, we get that the sequences converge to $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (\widetilde{B}(\bar{p}, \bar{q}) \cap \mathcal{N}) \times \Delta$. This limit point is still a solution of (13) and thanks to Theorem 3, it is an equilibrium of plans, prices, and price expectations for $\mathcal{E}$.

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