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DECAY IN CHEMOTAXIS SYSTEMS WITH A LOGISTIC TERM

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Abstract. This paper is concerned with a general fully parabolic Keller-Segel system, defined in a convex bounded and smooth domain Ω of $\mathbb{R}^N$, for $N \in \{2,3\}$, with coefficients depending on the chemical concentration, perturbed by a logistic source and endowed with homogeneous Neumann boundary conditions. For each space dimension, once a suitable energy function in terms of the solution is defined, we impose proper assumptions on the data and an exponential decay of such energies is established.

1. **Introduction.** There is a great interest in biology in studying bacteria and their behavior. A key property of a large class of bacteria (as for instance the E. coli) is that in presence of certain chemicals they move preferentially towards higher concentration of the chemical when it is a chemoatractant, or towards a lower concentration when it is a repellent. If Ω represents the environment occupied by cells and chemicals and $t > 0$ the time, the mathematical model proposed by Keller and Segel (see [7]) which precisely idealizes such a phenomenon in $Q = \Omega \times (t > 0)$ is the following

$$\begin{cases} \overbrace{u_t}^{cell-rate\ of\ change} = \overbrace{\operatorname{div}(g(v)\nabla u)}^{u-diffusion} - \overbrace{\operatorname{div}(u\chi(v)\nabla v)}^{chemotaxis} + \overbrace{f(u),}^{source} \text{ in } Q, \\ \overbrace{v_t}^{chemical-rate\ of\ change} = \overbrace{\alpha\Delta v}^{v-diffusion} + \overbrace{h(u,v),}^{proliferation} \quad \text{in } Q, \\ \frac{\partial u}{\partial n} = 0 \ \text{ and } \ \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial\Omega, \ t > 0, \\ u = u_0(x) \geq 0 \ \text{ and } \ v = v_0(x) \geq 0, \quad \text{in } \Omega, \end{cases} \tag{1}$$

where, as usual, $u = u(x,t)$ and $v = v(x,t)$ represent respectively the cell density and the concentration of the chemical substance at time t and position x, $\frac{\partial}{\partial n}$ is the outward normal derivative to the boundary $\partial\Omega$ and $u_0(x) = u(x,0)$ and $v_0(x) = v(x,0)$ are the initial distributions for u and v. The homogeneous Neumann boundary conditions (zero-flux boundary conditions) indicate that the domain is totally insulated.

In 1, the chemoattractant spreads with rate $\alpha > 0$ and its proliferation is modeled by the function h, while f is a source which controls the evolution of the cell

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distribution; in particular, the case $f(u) = 0$ corresponds to the phenomenon in which the amoeba production is negligible. On the other hand, both the chemotatic sensitivity χ and mobility g may be a constant or functions depending on the chemical v, according to biological situations: in particular as to the sensitivity it may assume different forms as

$$\chi(v) = \frac{\chi_0}{v}, \quad \chi(v) = \frac{k\,\chi_0}{(k+v)^2}, \quad v > 0, \tag{2}$$

k, χ_0 positive constants. Note that in each case as v decreases, $\chi(v)$ increases (for other expressions of the chemotatic sensitivity, see Murray [12] and [13]).

We note the difference in sign of the terms $\operatorname{div}(g(v)\nabla u)$ and $\operatorname{div}(u\chi(v)\nabla v)$, corresponding to the diffusion of the cells and to the cross diffusion term; precisely, the effect of the former is to stabilize the distribution of the cells in the environment, while the latter tends to break this stabilization working like a negative diffusion. Exactly in line with such competition between these different effects, it is desirable that the final balance between stabilizing and destabilizing forces in model 1 can infer a global and bounded solutions (u, v) which further results in some steady state for u and v. Conversely, this optimal situation is not always attained and some singularities formations (the so called *chemotactic collapse*) in which u may possibly become unbounded in a certain instant (blow-up time) may appear: for instance in the N-dimensional setting, with $N \geq 2$, and g and χ constants, $f \equiv 0$ and h behaving as $h(u, v) = -u + v$, blowing up solutions to 1 have been detected in [4] and [23] (see also [8], [9] and [10] for the analysis of techniques concerning estimates of blow-up time to some parabolic problems).

With the aim of suppressing the aforementioned singularities, the introduction of a *logistic-type* source $f(u) = au - bu^2$ (a, b positive) in the model may avoid this scenario. In fact, since the positive contribution $+au$ corresponds to a birth rate while the negative $-bu^2$ to a death rate, if the last one prevails on the first, an uncontrolled growth of the population u can be prevented. For instance, for the problem

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u & x \in \Omega,\ t > 0, \end{cases} \tag{3}$$

under zero-flux boundary conditions and defined in a convex smooth and bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$, where f verifies $f(0) \geq 0$ and $f(s) \leq a - bs^2$, for $s \geq 0$, and $a \geq 0$, b, χ, τ positive constants, in [21] the author proves that if b is big enough, for all sufficiently smooth and nonnegative initial data u_0 and v_0, it possesses a unique bounded and global-in-time classical solution. Additionally, for the same problem 3, also defined in a convex smooth and bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$, but with source term f controlled, respectively from below and above, by $-c_0(s + s^\alpha)$ and $a - bs^\alpha$, for $s \geq 0$, and with some $\alpha > 1$, $a \geq 0$ and $b, c_0 > 0$, global existence of very weak solutions, as well their boundedness properties and long time behavior are discussed in [18], [19] and also [20]. Nevertheless, if the death rate does not sufficiently strong contribute in the source f once again blow-up scenarios may manifest (see for instance [11], [17] and [22] for models related to this aspect).

2. **Main results and assumptions.** In accordance with all of the above, the present work considers the following chemotaxis system

$$\begin{cases} u_t = \operatorname{div}(g(v)\nabla u) - \operatorname{div}(u\chi(v)\nabla v) + au - bu^2, \quad x \in \Omega,\ t > 0, \\ v_t = \alpha \Delta v - k(v)v + uh(v), \quad x \in \Omega,\ t > 0, \end{cases} \tag{4}$$

in a convex smooth and bounded domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$. Equations 4 are equipped with homogeneous Neumann boundary conditions and initial data:

$$\begin{cases} \dfrac{\partial u}{\partial n} = 0 \ \text{ and } \ \dfrac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \ t > 0, \\ u = u_0(x) \geq 0 \ \text{ and } \ v = v_0(x) \geq 0, \quad x \in \Omega. \end{cases} \tag{5}$$

Inspired by the contribution by Payne and Straughan, where in [14] they discuss the decay of the solution to 4-5 in the case $a = b = 0$, the aim of our paper is to derive sufficient conditions on the data in order to establish the same long time behavior for the solution but in the case where the logistic term strongly affects the u-evolution. Precisely, for both $N = 2$ and $N = 3$, we define a suitable energy function in terms of the solution (u, v) and successively we deduce that, in the sense of this measure-function, (u, v) approaches to a steady solution $(\bar{u}, \bar{v})$, that we suppose constant; as a consequence of that we can conclude that not only no formation of spatial instabilities is plausible but also that (u, v) decays to a constant distribution. Hence, the novelty of our result is the investigation of the Keller-Segel system when the coefficients depend on v and in the presence of a source term of logistic type. In fact a large part of the articles on Keller-Segel-type systems focus their analysis when the mobility and the chemotactic sensitivity are constants or depend on u (see for instance [1], [2], [15] and [16]). Moreover, the logistic type source makes the analysis more complex with respect to that in the original model and it is herein addressed by means of some mathematical deductions linked to technical Sobolev-type inequalities. In this way, we get a decay result for a larger classes of Keller-Segel system, since in [14] no source term is present.

Our conclusions rely on some assumptions and statements: throughout the paper we assume that $\chi(v)$ is a regular function such that

$$|\chi| \leq \chi_1, \quad |\chi'| \leq \chi_2, \quad v \geq 0, \tag{6}$$

for constants $\chi_1, \ \chi_2$, where $\chi' = \frac{\partial \chi}{\partial v}$. We note that the assumptions 6 are verified by both choices in 2.

Moreover α, a and b are positive constants and the function $k(v)$ is assumed to satisfy

$$k(v) = \frac{m}{1 + Kv}, \quad v \geq 0, \tag{7}$$

where $m, \ K$ are positive constants (view the notation of Keller and Segel in [6]). Moreover, $g(v)$ and $h(v)$ are regular functions which satisfy

$$\begin{aligned} & g \geq g_0 > 0, \quad v \geq 0, \\ & |h| \leq h_1, \quad |h'| \leq h_2, \quad v \geq 0, \end{aligned} \tag{8}$$

where g_0, h_1 and h_2 are constants and $h' = \frac{\partial h}{\partial v}$. We will denote by $\| \cdot \|_2$ the $L^2(\Omega)$-norm, with

$$\bar{u}(x, t) = \frac{1}{|\Omega|} \int_\Omega u(x, t) \, dx, \tag{9}$$

and $\bar{v}$ the solution of the equation

$$\bar{v} \ k(\bar{v}) = \bar{u} \ h(\bar{v}). \tag{10}$$

Since we are interested in constant steady solution of 4, we observe that if $\bar{u} = \frac{a}{b}$ by integration of the first equation in 4 we find

$$\begin{aligned} \bar{u} &= \bar{u}_0, \\ \bar{v}k(\bar{v}) &= \bar{u}_0 h(\bar{v}). \end{aligned} \tag{11}$$

From the above remarks we easily derive the following

Lemma 2.1. *Assume that the $\int_\Omega u_0(x)\,dx$ is bounded. Then there exists a positive constant $C_{\bar{u}}$ depending on u_0, such that for all $t > 0$*

$$\int_\Omega u(x,t)\,dx \le C_{\bar{u}}. \tag{12}$$

Finally, we also need some geometrical definitions connected to the domain Ω: for some origin inside in Ω we write

$$\rho_0 = \min_{\partial\Omega} x_i n_i > 0, \quad d^2 = \max_{\bar{\Omega}} x_i x_i, \tag{13}$$

n_i being the i-th component of the unit normal vector directed outward on $\partial\Omega$.

We consider classical solutions of 4-5 in the sense that the solution belongs the space $C^0(\bar{\Omega}\times[0,\infty))\bigcap C^{2,1}(\Omega\times(0,\infty))$.

Now, our main results are given by the following theorems.

For $\Omega \in \mathbb{R}^2$, $\sigma > 0$ constant and $t > 0$, let us introduce the energy function

$$\Phi(t) := \frac{1}{2}\|u-\bar{u}\|_2^2 + \frac{\sigma}{2}\|\nabla v\|_2^2, \quad \Phi(0) := \Phi_0. \tag{14}$$

Theorem 2.2. *Let (u,v) be a classical solution of 4-5 in $\Omega\times(t>0)$, Ω bounded and convex domain in $\mathbb{R}^2$. Assume that χ, $k(v)$, $g(v)$ and $h(v)$ satisfy 6-8. Then, for any $\sigma > 0$ and $M > 0$ it is possible to find α and g_0 sufficiently large such that there exist positive constants α_i, $i = 1, \dots, 4$ to have*

$$\Phi'(t) \le -\Big(\alpha_1\|\Delta v\|_2^2 + \alpha_2\|\nabla u\|_2^2\Big)\Big(1 - \alpha_3\Phi - \alpha_4\Phi^{\frac{1}{2}}\Big), \quad t > 0 \tag{15}$$

with Φ defined in 14, provided that $\int_\Omega u_0 dx = M$.

Moreover if the initial data satisfy

$$\alpha_3\Phi_0 + \alpha_4\Phi_0^{\frac{1}{2}} < 1, \tag{16}$$

then $\Phi(t)$ is exponentially decreasing in time.

For $\Omega \subset \mathbb{R}^3$, let us introduce now another energy function:

$$\Psi(t) := \tfrac{1}{2}\|u-\bar{u}\|_2^2 + \tfrac{\sigma}{2}\|\nabla v\|_2^2 + \tfrac{\sigma_1}{2}\|\Delta v\|_2^2, \quad \Psi(0) := \Psi_0, \tag{17}$$

with σ and σ_1 positive constants.

Theorem 2.3. *Let (u,v) be a classical solution of 4-5 in $\Omega\times(t>0)$, Ω bounded and convex domain in $\mathbb{R}^3$. Assume that χ, $k(v)$, $g(v)$ and $h(v)$ satisfy 6-8. Then, for any σ, σ_1 and $M > 0$ it is possible to find α and g_0 sufficiently large such that there exist positive constants ξ_i, $i = 1, \dots, 5$ to have*

$$\Psi'(t) \le -\mathcal{A}\Big(1 - \xi_4\Psi^{\frac{1}{2}} - \xi_5\Psi\Big), \quad t > 0, \tag{18}$$

where

$$\mathcal{A} = \xi_1\|\nabla u\|_2^2 + \xi_2\|\Delta v\|_2^2 + \xi_3\|\nabla\Delta v\|_2^2,$$

and Ψ defined in 17, provided that $\int_\Omega u_0 dx = M$.

Moreover if the initial data satisfy

$$\xi_4\Psi_0^{\frac{1}{2}} + \xi_5\Psi_0 < 1, \tag{19}$$

then $\Psi(t)$ *is exponentially decreasing in time.*

Remark 1. As a consequence of the expressions 15 and 18, the solution (u, v) of problem 4-5 decays to a constant steady state $(\bar{u}, \bar{v})$, respectively in Φ- and Ψ-measure. As a consequence, blow up cannot occur.

3. Auxiliary materials.

We state some inequalities in the form that will be suitable for our purposes later. The following two inequalities are well known, but we include them since they will be used through the paper.

Lemma 3.1. *Let* Ω *be a convex domain in* $\mathbb{R}^N$, $N \geq 1$, *and* w *be a solution of the free membrane problem*

$$\begin{cases} \Delta w + \mu w = 0, & in\ \Omega, \\ \frac{\partial w}{\partial n} = 0, & on\ \partial\Omega. \end{cases} \tag{20}$$

Then the first non zero eigenvalue μ_2 *satisfies*

$$\mu_2\|\nabla w\|_2^2 \leq \|\Delta w\|_2^2, \tag{21}$$

and

$$\mu_2||\Delta w||_2^2 \leq ||\nabla \Delta w||_2^2. \tag{22}$$

Values for μ_2 *are given in* [3] *and* [5].

Lemma 3.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$, w *a regular function in* Ω *with* $\frac{\partial w}{\partial n} = 0$ *on* $\partial\Omega$. *Then there exists a positive constant* c_1 *such that*

$$\int_\Omega w^4 dx \leq c_1 \int_\Omega w^2 dx \int_\Omega |\nabla w|^2 dx, \quad if\ w\ with\ mean\ value\ zero. \tag{23}$$

Moreover there exists a positive constant c_2 *such that*

$$\int_\Omega |\nabla w|^4 dx \leq c_2 \int_\Omega |\nabla w|^2 dx \int_\Omega |\Delta w|^2 dx. \tag{24}$$

For the proof of 23 we use a combination of Gagliardo-Nirenberg and Poincaré's inequalities

$$\|w\|_4^4 \leq C_{GN}\|w\|_{W^{1,2}}^2\|w\|_2^2 \leq C_{GN}C_P\|\nabla w\|_2^2\|w\|_2^2,$$

where C_{GN} and C_P are the constants in the Gagliardo-Nirenberg and Poincaré's inequalities. Then, with $c_1 = C_{GN}C_P$, 23 holds. For the proof of 24 see (49) in [14].

Lemma 3.3. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$, w *regular function in* Ω *with* $\frac{\partial w}{\partial n} = 0$ *on* $\partial\Omega$. *Then there exist positive constants* $\bar{c}_1$, $\bar{c}_2$ *such that*

$$\int_\Omega (w - \bar{w})^4 dx \leq \bar{c}_1\Big(\int_\Omega (w - \bar{w})^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{3}{2}}, \tag{25}$$

and

$$\int_\Omega |\nabla w|^4 dx \leq \bar{c}_2\Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\Delta w|^2 dx\Big)^{\frac{3}{2}}. \tag{26}$$

For the proof of Lemma 3.3 see par. 6.3 and 6.4 in [14].

4. **Proof of Theorem 2.2.**

Proof. Differentiating 14 we obtain

$$\begin{aligned}\Phi'(t) &= \int_\Omega (u-\bar{u})u_t dx + \sigma \int_\Omega \nabla v \nabla v_t dx \\ &= \int_\Omega (u-\bar{u})\nabla\cdot\Big(g(v)\nabla u\Big)dx - \int_\Omega (u-\bar{u})\nabla\cdot\Big(u\chi(v)\nabla v\Big)dx \\ &\quad + \int_\Omega (u-\bar{u})(au-bu^2)dx + \sigma\int_\Omega \nabla v\nabla v_t dx = I_1+I_2+I_3+I_4.\end{aligned} \tag{27}$$

Now, we estimate the four terms of 27 separately.

By the divergence theorem, the boundary conditions 5 and the first assumption in 8, we have

$$I_1 = \int_\Omega (u-\bar{u})\nabla\cdot\Big(g(v)\nabla u\Big)dx = -\int_\Omega g(v)|\nabla u|^2 dx \le -g_0\int_\Omega |\nabla u|^2 dx, \tag{28}$$

and

$$\begin{aligned} I_2 &= -\int_\Omega (u-\bar{u})\nabla\cdot\Big(u\chi(v)\nabla v\Big)dx = \int_\Omega u\chi(v)\nabla u\nabla v dx \\ &= \int_\Omega (u-\bar{u})\chi(v)\nabla u\nabla v dx + \bar{u}\int_\Omega \chi(v)\nabla u \nabla v dx.\end{aligned} \tag{29}$$

In the first term at the right hand side of 29 we now use the divergence theorem, the Cauchy-Schwarz inequality and 6 to obtain

$$\begin{aligned} &\int_\Omega (u-\bar{u})\chi(v)\nabla u\nabla v dx \\ =&\frac{1}{2}\int_\Omega \nabla\cdot\Big[(u-\bar{u})^2\chi(v)\nabla v\Big]dx - \frac{1}{2}\int_\Omega (u-\bar{u})^2\nabla\cdot\Big[\chi(v)\nabla v\Big]dx \\ =&-\frac{1}{2}\int_\Omega (u-\bar{u})^2\chi'(v)|\nabla v|^2 dx - \frac{1}{2}\int_\Omega (u-\bar{u})^2\chi(v)\Delta v dx \\ \le&\frac{\chi_2}{2}\Big(\int_\Omega (u-\bar{u})^4 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla v|^4 dx\Big)^{\frac{1}{2}} \\ &+\frac{\chi_1}{2}\Big(\int_\Omega (u-\bar{u})^4 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\Delta v|^2 dx\Big)^{\frac{1}{2}}.\end{aligned} \tag{30}$$

In 30 we use the Sobolev inequalities 23 and 24 with respectively $w = u-\bar{u}$ and $w = v$ and we get

$$\begin{aligned} &\int_\Omega (u-\bar{u})\chi(v)\nabla u\nabla v dx \\ \le&\frac{\chi_2}{2}\Big(c_2\int_\Omega |\nabla v|^2 dx\int_\Omega |\Delta v|^2 dx\Big)^{\frac{1}{2}}\Big(c_1\int_\Omega (u-\bar{u})^2 dx\int_\Omega |\nabla u|^2 dx\Big)^{\frac{1}{2}} \\ &+\frac{\chi_1}{2}\Big(\int_\Omega |\Delta v|^2 dx\Big)^{\frac{1}{2}}\Big(c_1\int_\Omega (u-\bar{u})^2 dx\int_\Omega |\nabla u|^2 dx\Big)^{\frac{1}{2}} \\ \le&\frac{\chi_2\sqrt{c_1c_2}}{\sqrt{\sigma}}\Phi\|\Delta v\|_2\|\nabla u\|_2 + \frac{\chi_1\sqrt{c_1}}{\sqrt{2}}\Phi^{\frac{1}{2}}\|\Delta v\|_2\|\nabla u\|_2 \\ =&\Big(\frac{\chi_2\sqrt{c_1c_2}}{\sqrt{\sigma}}\Phi + \frac{\chi_1\sqrt{c_1}}{\sqrt{2}}\Phi^{\frac{1}{2}}\Big)\|\Delta v\|_2\|\nabla u\|_2.\end{aligned} \tag{31}$$

In the last term at right hand of 29 we use 6, the Schwarz inequality and relation 21, to infer

$$\begin{aligned} \bar{u}\int_\Omega \chi(v)\nabla u\nabla v dx \leq& \bar{u}\chi_1\|\nabla u\|_2\|\nabla v\|_2 \leq \bar{u}\chi_1\mu_2^{-\frac{1}{2}}\|\nabla u\|_2\|\Delta v\|_2 \\ \leq& \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}}{2\varepsilon}\|\nabla u\|_2^2 + \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}\varepsilon}{2}\|\Delta v\|_2^2, \end{aligned} \tag{32}$$

where in the last inequality we have used the arithmetic-geometric inequality

$$a^r b^s \leq ra + sb, \quad a>0,\ b>0,\ r>0,\ s>0,\ r+s=1, \tag{33}$$

and with ϵ a positive constant. Substituting 31 and 32 in 29 we can write

$$\begin{aligned} I_2 \leq& \Big(\frac{\chi_2\sqrt{c_1c_2}}{\sqrt{\sigma}}\Phi + \frac{\chi_1\sqrt{c_1}}{\sqrt{2}}\Phi^{\frac{1}{2}}\Big)\|\Delta v\|_2\|\nabla u\|_2 + \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}}{2\varepsilon}\|\nabla u\|_2^2 \\ &+ \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}\varepsilon}{2}\|\Delta v\|_2^2. \end{aligned} \tag{34}$$

By Hölder's inequality, $|\Omega|\bar{u}^3 \leq \int_\Omega u^3 dx$. Hence concerning I_3 we can deduce

$$\begin{aligned} I_3 =& \int_\Omega (u-\bar{u})(au - bu^2)dx = a\int_\Omega u(u-\bar{u})dx - b\int_\Omega u^2(u-\bar{u})dx \\ \leq& a\int_\Omega u^2 dx + b\bar{u}\int_\Omega u^2 dx - a\int_\Omega u\bar{u}dx - b\int_\Omega u^3 dx \\ \leq& (a+b\bar{u})\int_\Omega u^2 dx - a|\Omega|\bar{u}^2 - b|\Omega|\bar{u}^3. \end{aligned} \tag{35}$$

By Poincaré's inequality,

$$\int_\Omega w^2 dx \leq C_P\int_\Omega |\nabla w|^2 dx + |\Omega|\bar{w}^2, \text{ for all } w \in W^{1,2}.$$

Therefore,

$$I_3 \leq \gamma\textstyle\int_\Omega |\nabla u|^2 dx + (a+b\bar{u})|\Omega|\bar{u}^2 - a|\Omega|\bar{u}^2 - b|\Omega|\bar{u}^3 = \gamma\int_\Omega |\nabla u|^2 dx, \tag{36}$$

where $\gamma = C_P(a + bC_{\bar{u}})$, $C_{\bar{u}}$ in 12.

Finally, by using the second equation of 4, the divergence theorem and 5, the last term I_4 of 27 can be estimated as

$$\begin{aligned} I_4 =& \sigma\int_\Omega \nabla v\nabla v_t dx \\ =& -\sigma\alpha\int_\Omega |\Delta v|^2 dx + \sigma\int_\Omega vk(v)\Delta v dx - \sigma\int_\Omega uh(v)\Delta v dx. \end{aligned} \tag{37}$$

In the second and third terms of the last inequality of 37 we use the divergence theorem, 7 and 8 to obtain

$$\int_\Omega vk(v)\Delta v dx = -\int_\Omega (k+vk')|\nabla v|^2 dx = -m\int_\Omega \frac{|\nabla v|^2}{(1+Kv)^2}dx \tag{38}$$

and

$$\begin{aligned} -\int_\Omega uh(v)\Delta v dx \leq& \frac{1}{2}h_1\|u-\bar{u}\|_2^2 + \frac{1}{2}h_1\|\Delta v\|_2^2 + \bar{u}h_2\|\nabla v\|_2^2 \\ \leq& \frac{h_1}{2\mu_2}\|\nabla u\|_2^2 + \frac{1}{2}h_1\|\Delta v\|_2^2 + \frac{\bar{u}h_2}{\mu_2}\|\Delta v\|_2^2, \end{aligned} \tag{39}$$

where in the last inequality we have used 21 and the following inequality

$$\mu_2 \int_\Omega (u-\bar{u})^2 dx \le \int_\Omega |\nabla u|^2 dx, \tag{40}$$

proved by Hardy et al. in [3] (note that $u-\bar{u}$ has zero mean value).

Replacing 38 and 39 in 37 we arrive at

$$I_4 \le -\Big(\sigma\alpha - \sigma\Big[\frac{h_1}{2} + \frac{\bar{u}h_2}{\mu_2}\Big]\Big)\|\Delta v\|_2^2 + \sigma\frac{h_1}{2\mu_2}\|\nabla u\|_2^2. \tag{41}$$

Combining 27 with 28, 34 36 and 41 we deduce

$$\begin{aligned}
\Phi'(t) \le &- \Big(\sigma\alpha - \sigma\frac{h_1}{2} - \sigma\frac{\bar{u}h_2}{\mu_2} - \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}\varepsilon}{2}\Big)\|\Delta v\|_2^2 \\
&- \Big(g_0 - \gamma - \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}}{2\varepsilon} - \sigma\frac{h_1}{2\mu_2}\Big)\|\nabla u\|_2^2 \\
&+ \Big(\frac{\chi_2\sqrt{c_1c_2}}{\sqrt{\sigma}}\Phi + \frac{\chi_1\sqrt{c_1}}{\sqrt{2}}\Phi^{\frac{1}{2}}\Big)\|\Delta v\|_2\|\nabla u\|_2 \\
= &- \alpha_1\|\Delta v\|_2^2 - \alpha_2\|\nabla u\|_2^2 + \Big(\frac{\chi_2\sqrt{c_1c_2}}{\sqrt{\sigma}}\Phi + \frac{\chi_1\sqrt{c_1}}{\sqrt{2}}\Phi^{\frac{1}{2}}\Big)\|\Delta v\|_2\|\nabla u\|_2,
\end{aligned} \tag{42}$$

with

$$\begin{aligned}
\alpha_1 = &\sigma\alpha - \sigma\frac{h_1}{2} - \sigma\frac{\bar{u}h_2}{\mu_2} - \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}\varepsilon}{2}, \\
\alpha_2 = &g_0 - \gamma - \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}}{2\varepsilon} - \sigma\frac{h_1}{2\mu_2}.
\end{aligned}$$

Once $\sigma > 0$ and $\varepsilon > 0$ are fixed, if

$$\begin{aligned}
\alpha > &\frac{h_1}{2} + \frac{\bar{u}h_2}{\mu_2} + \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}\varepsilon}{2\sigma} \\
g_0 > &\gamma + \frac{\bar{u}\chi_1\mu_2^{-\frac{1}{2}}}{2\varepsilon} + \frac{\sigma h_1}{2\mu_2},
\end{aligned} \tag{43}$$

then $\alpha_1 > 0$ and $\alpha_2 > 0$. In particular, the inequalities 43 justify the largeness assumptions on α and g_0 claimed in the hypothesis of this theorem.

Using the Schwarz inequality in the third term of 42 we have

$$\|\Delta v\|_2\|\nabla u\|_2 \le \frac{1}{2\sqrt{\alpha_1\alpha_2}}\Big(\alpha_1\|\Delta v\|_2^2 + \alpha_2\|\nabla u\|_2^2\Big),$$

from which we obtain 15 with

$$\alpha_3 = \frac{\chi_2\sqrt{c_1c_2}}{2\sqrt{\sigma\alpha_1\alpha_2}}, \quad \alpha_4 = \frac{\chi_1\sqrt{c_1}}{2\sqrt{2\alpha_1\alpha_2}}.$$

Now, if 43 and 16 hold, then $\Phi(t)$ decays exponentially in time. □

5. **Proof of Theorem 2.3.**

Proof. Differentiating 17 we have

$$\begin{aligned}
\Psi'(t) = &\int_\Omega (u-\bar{u})u_t dx + \sigma\int_\Omega \nabla v\nabla v_t dx + \sigma_1\int_\Omega \Delta v\Delta v_t dx \\
= &J_1 + J_2 + J_3.
\end{aligned} \tag{44}$$

To estimate J_1 we can follow the steps of section 4 up to 29 so that we can write

$$\begin{aligned} J_1(t) = &\int_\Omega (u-\bar u)u_t dx \le -g_0\|\nabla u\|_2^2 + \int_\Omega (u-\bar u)\chi(v)\nabla u\nabla v dx \\ &+ \bar u\int_\Omega \chi(v)\nabla u\nabla v dx + a\int_\Omega u(u-\bar u)dx - b\int_\Omega u^2(u-\bar u)dx \\ \le& - g_0\|\nabla u\|_2^2 + \int_\Omega (u-\bar u)\chi(v)\nabla u\nabla v dx + \bar u\int_\Omega \chi(v)\nabla u\nabla v dx \\ &+ (a+b\bar u)\int_\Omega u^2 dx - a|\Omega|\bar u^2 - b|\Omega|\bar u^3. \end{aligned} \tag{45}$$

To bound the second term of 45 we use 6 and Schwarz's inequality and we have

$$\begin{aligned} \int_\Omega (u-\bar u)\chi(v)\nabla u\nabla v dx \le& \chi_1\Big(\int_\Omega (u-\bar u)^2|\nabla v|^2 dx\Big)^{\frac12}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac12} \\ \le& \chi_1\Big(\int_\Omega (u-\bar u)^4 dx\Big)^{\frac14}\Big(\int_\Omega |\nabla v|^4 dx\Big)^{\frac14}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac12} \\ \le& \frac{\sqrt2\chi_1\bar c_1^{\frac14}\bar c_2^{\frac14}}{\sigma^{\frac18}\sigma_1^{\frac14}}\Psi^{\frac12}\|\nabla u\|_2^{\frac74}\|\Delta v\|_2^{\frac14}, \end{aligned} \tag{46}$$

where in the last inequality we have used 25 and 26 with respectively $w = u$ and $w = v$.

Now, we note that 32 is valid also in a domain of $\mathbb{R}^3$ so that the third term of 45 can be rewritten as

$$\bar u\int_\Omega \chi(v)\nabla u\nabla v dx \le \frac{\bar u\chi_1\mu_2^{-\frac12}}{2\varepsilon}\|\nabla u\|_2^2 + \frac{\bar u\chi_1\mu_2^{-\frac12}\varepsilon}{2}\|\Delta v\|_2^2. \tag{47}$$

As in the estimates 35, for the last term in 45 we have that:

$$(a+b\bar u)\int_\Omega u^2 dx - a|\Omega|\bar u^2 - b|\Omega|\bar u^3 \le \tilde\gamma\|\nabla u\|_2^2, \tag{48}$$

with $\tilde\gamma = C_P(a+bC_{\bar u})$. Combining 45, 46, 47 and 48 we have

$$\begin{aligned} J_1 \le& - g_0\|\nabla u\|_2^2 + \frac{\sqrt2\chi_1\bar c_1^{\frac14}\bar c_2^{\frac14}}{\sigma^{\frac18}\sigma_1^{\frac14}}\Psi^{\frac12}\|\nabla u\|_2^{\frac74}\|\Delta v\|_2^{\frac14} + \frac{\bar u\chi_1\mu_2^{-\frac12}}{2\varepsilon}\|\nabla u\|_2^2 \\ &+ \frac{\bar u\chi_1\mu_2^{-\frac12}\varepsilon}{2}\|\Delta v\|_2^2 + \tilde\gamma\|\nabla u\|_2^2. \end{aligned} \tag{49}$$

To estimate J_2 we observe that 37-41 are also valid here, so that we can write

$$\begin{aligned} J_2 =& \sigma\int_\Omega \nabla v\nabla v_t dx \le -\Big(\sigma\alpha - \sigma\Big[\frac{h_1}{2} + \frac{\bar u h_2}{\mu_2}\Big]\Big)\|\Delta v\|_2^2 \\ &+ \frac{\sigma}{2}h_1\mu_2^{-1}\|\nabla u\|_2^2. \end{aligned} \tag{50}$$

Finally, we now fix our attention on J_3:

$$\begin{aligned} J_3 =& \sigma_1\int_\Omega \Delta v\Delta v_t dx = \sigma_1\alpha\int_\Omega \Delta v\Delta^2 v dx - \sigma_1\int_\Omega \Delta v\Delta\big(k(v)v\big)dx \\ &+ \sigma_1\int_\Omega \Delta v\Delta\big(uh(v)\big)dx. \end{aligned} \tag{51}$$

We recall that since $\frac{\partial v}{\partial n}=0$ and $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$, differentiating the second equation of 4 with respect to n we obtain

$$\frac{\partial \Delta v}{\partial n}=0 \quad \text{on}\, \partial\Omega$$

so that

$$\sigma_1\alpha\int_\Omega \Delta v\Delta^2 v dx=-\sigma_1\alpha\|\nabla\Delta v\|_2^2. \tag{52}$$

Replacing 52 in 51 and using the divergence theorem we arrive at

$$\begin{aligned} J_3=&-\sigma_1\alpha\|\nabla\Delta v\|_2^2-\sigma_1\int_\Omega \Delta v\Delta\big(k(v)v\big)dx\\ &+\sigma_1\int_\Omega \Delta v\Delta\big(uh(v)\big)dx=-\sigma_1\alpha\|\nabla\Delta v\|_2^2\\ &+\sigma_1\int_\Omega \nabla\Delta v\nabla\big(k(v)v\big)dx-\sigma_1\int_\Omega\nabla\Delta v\nabla\big(uh(v)\big)dx. \end{aligned} \tag{53}$$

To estimate the second term in 53, we note that $\nabla\big(k(v)v\big)=\dfrac{m\nabla v}{(1+Kv)^2}$ and we can write

$$\begin{aligned} \sigma_1\int_\Omega \nabla\Delta v\nabla\big(k(v)v\big)dx=&\sigma_1 m\int_\Omega \nabla\Delta v\frac{\nabla v}{(1+Kv)^2}dx\\ \le&\Big(\frac{\sigma_1}{\epsilon_1}\int_\Omega(\nabla\Delta v)^2dx\Big)^{\frac12}\Big(\sigma_1 m^2\epsilon_1\int_\Omega\frac{|\nabla v|^2}{(1+Kv)^4}dx\Big)^{\frac12}\\ \le&\frac{\sigma_1}{2\epsilon_1}\|\nabla\Delta v\|_2^2+\frac{\sigma_1 m^2\epsilon_1}{2}\|\nabla v\|_2^2\\ \le&\frac{\sigma_1}{2\epsilon_1}\|\nabla\Delta v\|_2^2+\frac{\sigma_1 m^2\epsilon_1}{2\mu_2}\|\Delta v\|_2^2, \end{aligned} \tag{54}$$

with $\epsilon_1>0$ a suitable constant to be choose later on. In 54 we have also used ϵ-Schwarz, arithmetic-geometric mean inequalities and 21.

To estimate the third term in 53 we use ϵ-Schwarz and arithmetic-geometric mean inequalities to write

$$\begin{aligned} &-\sigma_1\int_\Omega\nabla\Delta v\nabla\big(uh(v)\big)dx\\ =&-\sigma_1\int_\Omega\nabla\Delta v(u-\bar u)h'(v)\nabla v dx\\ &-\bar u\sigma_1\int_\Omega\nabla\Delta v h'(v)\nabla v dx-\sigma_1\int_\Omega\nabla\Delta v\nabla u h(v)dx\\ \le&\frac{\sigma_1}{2\epsilon_2}\|\nabla\Delta v\|_2^2+\frac{\sigma_1 h_2^2\epsilon_2}{2}\int_\Omega|u-\bar u|^2|\nabla v|^2dx+\frac{\sigma_1}{2\epsilon_3}\|\nabla\Delta v\|_2^2\\ &+\frac{\sigma_1\bar u^2h_2^2\epsilon_3}{2}\int_\Omega|\nabla v|^2dx+\frac{\sigma_1}{2\epsilon_4}\|\nabla\Delta v\|+\frac{\sigma_1 h_1\epsilon_4}{2}\int_\Omega|\nabla u|^2dx, \end{aligned} \tag{55}$$

with $\epsilon_i,\ i=2,..,4$ positive constants to be choose later on. To complete the estimate 55 we employ 21 and (36) in [14].

Following the steps at pag. 346 in [14] we choose $\epsilon_i=\frac{4}{\alpha},\ i=1,..,4$ and we get

$$J_3\le-\gamma_1\|\nabla\Delta v\|_2^2+\gamma_2\|\Delta v\|_2^2+\gamma_3\|\nabla u\|_2^2+\gamma_4\Psi\|\nabla u\|_2^{\frac32}\|\Delta v\|_2^{\frac12}, \tag{56}$$

with

$$\gamma_1 = \sigma_1 \frac{\alpha}{2}, \qquad \gamma_2 = \sigma_1 \Big(\frac{2m^2}{\mu_2 \alpha} + \frac{2h_2^2 \bar{u}^2}{\mu_2 \alpha} \Big)$$

$$\gamma_3 = \frac{2\sigma_1 h_1^2}{\alpha}, \qquad \gamma_4 = \frac{2\sigma_1 \sqrt{\bar{c}_1 \bar{c}_2}}{\sigma^{\frac{1}{4}} \sigma_1^{\frac{1}{2}}}.$$

Combining 44, 49, 50 and 56 provides

$$\begin{aligned} \Psi'(t) \leq & - \xi_1 \|\nabla u\|_2^2 - \xi_2 \|\Delta v\|_2^2 - \xi_3 \|\nabla \Delta v\|_2^2 \\ & + \bar{c} \Psi^{\frac{1}{2}} \|\nabla u\|_2^{\frac{7}{4}} \|\Delta v\|_2^{\frac{1}{4}} + \gamma_4 \Psi \|\nabla u\|_2^{\frac{3}{2}} \|\Delta v\|_2^{\frac{1}{2}} \end{aligned} \tag{57}$$

with

$$\begin{aligned} \xi_1 = & g_0 - \frac{\bar{u} \chi_1 \mu_2^{-\frac{1}{2}}}{2\varepsilon} - \tilde{\gamma} - \frac{\sigma}{2} h_1 \mu_2^{-1} - \gamma_3, \\ \xi_2 = & \sigma \alpha - \sigma \Big[\frac{h_1}{2} + \frac{\bar{u} h_2}{\mu_2} \Big] - \frac{\bar{u} \chi_1 \mu_2^{-\frac{1}{2}} \varepsilon}{2} - \gamma_2, \\ \xi_3 = & \gamma_1 > 0, \\ \bar{c} = & \frac{\sqrt{2} \chi_1 \bar{c}_1^{\frac{1}{4}} \bar{c}_2^{\frac{1}{4}}}{\sigma^{\frac{1}{8}} \sigma_1^{\frac{1}{4}}}. \end{aligned}$$

Now, fixing $\sigma >$, $\sigma_1 >$ and $\varepsilon > 0$, if

$$\begin{aligned} \alpha > & \Big[\frac{h_1}{2} + \frac{\bar{u} h_2}{\mu_2} \Big] + \frac{\bar{u} \chi_1 \mu_2^{-\frac{1}{2\sigma}} \varepsilon}{2} + \gamma_2, \\ g_0 > & \frac{\bar{u} \chi_1 \mu_2^{-\frac{1}{2}}}{2\varepsilon} + \tilde{\gamma} + \frac{\sigma}{2} h_1 \mu_2^{-1} + \gamma_3, \end{aligned} \tag{58}$$

we have $\xi_1 > 0$, $\xi_2 > 0$. Again, inequalities 58 are in line with the largeness assumptions on α and g_0 of this theorem.

In the fourth and fifth terms of 57 we apply the arithmetic-geometric inequality 33 to obtain

$$\begin{aligned} \bar{c} \Psi^{\frac{1}{2}} \|\nabla u\|_2^{\frac{7}{4}} \|\Delta v\|_2^{\frac{1}{4}} \leq & \frac{\bar{c}}{\xi_1^{\frac{7}{8}} \xi_2^{\frac{1}{8}}} \Psi^{\frac{1}{2}} \Big[\xi_1 \|\nabla u\|_2^2 + \xi_2 \|\Delta v\|_2^2 + \xi_3 \|\nabla \Delta v\|_2^2 \Big], \\ \gamma_4 \Psi \|\nabla u\|_2^{\frac{3}{2}} \|\Delta v\|_2^{\frac{1}{2}} \leq & \frac{\gamma_4}{\xi_1^{\frac{3}{4}} \xi_2^{\frac{1}{4}}} \Psi \Big[\xi_1 \|\nabla u\|_2^2 + \xi_2 \|\Delta v\|_2^2 + \xi_3 \|\nabla \Delta v\|_2^2 \Big]. \end{aligned} \tag{59}$$

Finally, we substitute 59 in 57 and we arrive at 18. Since 19 holds, then Ψ decays exponentially fast in time. The Theorem 2.3 is proved. □

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