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# Orthogonality in locally convex spaces: Two nonlinear generalizations of Neumann's lemma

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A B S T R A C T

In this note we prove a symmetric version of the Neumann lemma as well as a symmetric version of the Söderlind-Campanato lemma. We establish in this way two partial generalizations of the well-known Casazza-Christenses lemma. This work is related to the Birkhoff-James orthogonality and to the concept of near operators introduced by S. Campanato.



## 1. Introduction

### 1.1. Definitions and notations

Throughout the paper, $X$ is a real Banach space with norm $\|\cdot\|$; several of our results are valid in the larger setting of real normed spaces, while others need, on the contrary, an additional condition on $X$, such as the existence of a Hermitian decomposition. We denote by $\mathscr{L}(X)$ the space of all bounded linear operators on $X$, endowed with the operator norm $\|\cdot\|$ defined by

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$$\|A\| := \inf\{c \geq 0 \,:\, \|A(x)\| \leq c\,\|x\|,\ \forall x \in X\}, \quad \forall A \in \mathscr{L}(X).$$

We use the notation $Id$ for the identity operator in $\mathscr{L}(X)$ and $\mathscr{F}(X)$ for the set of all functions from $X$ to $X$. A function $f \in \mathscr{F}(X)$ is called Lipschitz continuous whenever

$$\mathrm{Lip}\,(f) := \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|},$$

is a real number; in this case $\mathrm{Lip}\,(f)$ is referred to as the Lipschitz modulus of $f$.

### *1.2. Motivation*

The starting point of this research is a series of articles written by Sergio Campanato at the end of the eighties [4–6]. More precisely, to study existence and regularity results for PDEs and systems in non-divergence form, Campanato introduced and developed the abstract notion of "near operators". Using this concept he was able to unify the theories based on the Schauder method, the Lax-Milgram theorem, the Cordes theorem and the theory of monotone operators.

As we have observed and developed in this note, Campanato's nearness goes back to a standard result in functional analysis which is known as Neumann's lemma. This is a generic name given to several strongly related results, establishing, in a real Banach space $X$, the invertibility of a function $A \in \mathscr{F}(X)$. The original result by Carl Neumann[3] was stated as follows (see, e.g.[12, Lemma 5.1.6]):

**Lemma 1** *(**Neumann**). Let $X$ be a real Banach space, and $A \in \mathscr{L}(X)$ be such that*

$$\exists\ \alpha > 0,\ 0 \leq \kappa < 1: \quad \|Id - \alpha\, A\| \leq\ \kappa\,\|Id\|. \tag{1}$$

*Then $A$ is invertible, and $\|A^{-1}\| \leq \frac{\alpha}{1-\kappa}$.*

Of course, $\|Id\| = 1$, and the right hand side of the inequality (1) amounts to $\kappa$. We choose however to state relation (1) in the present form in order to highlight the common points among Lemma 1, Lemma 2, Lemma 3, and the Symmetric Neumann and Sorderling Campaganto lemmas.

Among the various consequences of this ubiquitous lemma, let us mention two classical theorems:

- the set of all invertible operators from $\mathscr{L}(X)$ is open ([12, Theorem 6.4.4]);
- the function which assigns to each operator $A \in \mathscr{L}(X)$ its spectral radius is upper semi-continuous ([10, Theorem 2.1]).

Neumann's lemma was generalized in Euclidean spaces to not necessarily linear functions by Söderlind ([18, Theorem 2.1 & Corollary 2.3]). The Söderlind result was further extended by Campanato, first in the Hilbert space setting ([4, Section 2]), and then in the case of an arbitrary real Banach space ([6, Theorems 1 & 2]), by proving the following statement:

**Lemma 2** *(**Söderlind-Campanato**). Let $X$ be a real Banach space, and consider a not necessarily linear function $A : X \to X$ such that there exist two real numbers $\alpha > 0$ and $0 \leq \kappa < 1$ satisfying*

$$\|(x_1 - \alpha\, A(x_1)) - (x_2 - \alpha\, A(x_2))\| \leq \kappa\,\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \tag{2}$$

[3] This result is usually stated in the particular case $\alpha = 1$, but the proof holds with no modifications in the general setting.

*or, equivalently there exist two real numbers* $\alpha$ *and* $\kappa$ *such that*

$$\alpha > 0, \quad 0 \le \kappa < 1 \quad \textit{and} \quad \operatorname{Lip}(Id - \alpha A) \le \kappa \operatorname{Lip}(Id).$$

*Then* $A$ *is invertible, and* $\operatorname{Lip}(A^{-1}) \le \frac{\alpha}{1-\kappa}$.

In several instances coming from the setting of implicit methods in numerical analysis, Neumann's, or Söderlind-Campanato's lemmata are of little use, since one needs inequalities of the form (1) or (2) in which the dominating right-hand side member should be expressed in terms of the unknown function $A$, and not in terms of the identity. In other words, we need to study the symmetric counterparts of Neumann and Söderlind-Campanato results.

The first result in this direction was obtained by Casazza and Christenses, in their influential study of the frames in a Hilbert space (see [7, Theorem 2]). More precisely, these authors considered functions $A : X \to X$ such that there exist reals $\alpha > 0$ and $0 \le \kappa < 1$ satisfying

$$\|(A(x_1) - \alpha\, x_1) - (A(x_2) - \alpha\, x_2)\| \le \kappa\, \|A(x_1) - A(x_2)\|, \quad \forall x_1, x_2 \in X. \tag{3}$$

Let us note that, when the function $A$ is linear (but not necessarily continuous), relation (3) reduces to:

$$\|A(x) - \alpha\, x\| \le \kappa\, \|A(x)\|, \quad \forall x \in X. \tag{4}$$

The result by Casazza and Christenses establishes the invertibility of the operator $A$ under the additional assumption that $A$ is linear.

**Lemma 3** ***(Casazza and Christenses)****. Let* $X$ *be a real Banach space and* $A \in \mathscr{L}(X)$ *such that there exist real numbers* $\alpha > 0$ *and* $0 \le \kappa < 1$ *satisfying*

$$\|A(x) - \alpha\, x\| \le \kappa\, \|A(x)\|, \quad \forall x \in X.$$

*Then* $A$ *is invertible, and* $\|A^{-1}\| \le \frac{1+\kappa}{\alpha}$.

The main objective of this article is to prove new invertibility theorems for not necessarily linear functions defined on real Banach spaces, similar to the linear Casazza-Christenses result. The main novelty of our work is the use of a method, which, to the best of our knowledge, has never been employed in addressing operator invertibility. Namely, we investigate, in the series of statements Theorems 8–10, several symmetry-related properties of a relation between two operators on a Banach space.

The starting point of our analysis is the notion of orthogonality in a normed space, going back to Birkhoff (see [2], the reader is also referred to the complete survey [1]). This classical definition can easily be transposed to a locally convex space, that is a vector space $Y$ equipped with a family of semi-norms $\mathscr{P} := \{p_\iota : Y \to \mathbb{R} :\ \iota \in I\}$ generating its topology. This was made for instance in [17, Definition 1.1]: given a locally convex space $(Y, \mathscr{P})$, we say that a vector $u \in Y$ is *Birkhoff-James orthogonal* to $v \in Y$ on $(Y, \mathscr{P})$ iff

$$p_\iota(u) \le p_\iota(u - s\, v), \quad \forall s \in \mathbb{R},\ \iota \in I. \tag{5}$$

Of course, when $Y$ is a normed space, $\mathscr{P}$ contains only one element, namely the norm of $Y$, and definition (5) boils down to Birkhoff's original definition which says that there is no point on $\mathbb{R}\, v$, the extended line through $v$, nearer to $u$ than 0.

Our analysis benefits from the introduction of a new concept, strongly related to the Birkhoff-James orthogonality. Given $u, v \in Y$, we say that $u$ is *Campanato*[4] *near* $v$ on $(Y, \mathscr{P})$ iff

$$\exists \ \alpha > 0, \ 0 \leq \kappa < 1 : \quad p_\iota(u - \alpha\, v) \leq \kappa\, p_\iota(u), \quad \forall \iota \in I. \tag{6}$$

As no confusion risks to occur, we will drop, in the remaining part of this article, the wording "Birkhoff-James" and "Campanato", and simply speak of orthogonality and nearness.

If $X$ is a normed space, and $u$, $v$ are two vectors from $X$, there are only three possibilities: $u$ in near $v$, $u$ is orthogonal to $v$, and $u$ is near $-v$. However, in the setting of locally convex spaces, given two vectors $u$ and $v$, it is possible that none of the above three possibilities are satisfied. This explains (inter alia) why stronger results are achieved in normed spaces.

The Neumann lemma, and respectively the Söderlind-Campanato lemma, may now be reworded in terms of nearness applied to the Banach space $(\mathscr{L}(X), \|\cdot\|)$, and respectively to the locally convex space $(\mathscr{F}(X), \mathscr{P})$, where the set $\mathscr{P}$ contains all semi-norms of the form

$$p_{x_1,x_2} : \mathscr{F}(X) \to \mathbb{R}, \quad p_{x_1,x_2}(A) := \|A(x_1) - A(x_2)\|, \quad \forall A \in \mathscr{F}(X), \tag{7}$$

for all the points $x_1, x_2 \in X$ such that $x_1 \neq x_2$.

More precisely, the Neumann lemma says that *an operator $A \in \mathscr{L}(X)$ is invertible provided that the identity is near $A$ on $(\mathscr{L}(X), \|\cdot\|)$*, while the Söderlind-Campanato lemma states that *a function $A \in \mathscr{F}(X)$ is invertible provided that the identity is near $A$ on $(\mathscr{F}(X), \mathscr{P})$*.

Similarly, the Casazza-Christensen lemma may be rephrased as saying that *if $A \in \mathscr{L}(X)$ is near the identity on $(\mathscr{F}(X), \mathscr{P})$, then it is invertible.*

It is now clear that any qualification condition ensuring the symmetry of nearness either on $(\mathscr{L}(X), \|\cdot\|)$, or on $(\mathscr{F}(X), \mathscr{P})$, may be combined with either the Neumann lemma, or with the Söderlind-Campanato lemma. As a result, one automatically obtains a symmetric version of the Neumann lemma, or respectively of the Söderlind-Campanato lemma.

Establishing the symmetry of the orthogonality in a normed space is a surprisingly hard problem. Maybe the best evidence of the difficulty of the subject is the fact that the celebrated theorem enumerating all the cases when orthogonality is symmetric came as the result of a one-decade-long joint effort by Birkhoff ([2] in 1936), Kakutani ([15] in 1939), Day ([9] in 1947) and James ([14] in 1947), and that a rather technical theorem by Blaschke ([3, page 160]), characterizing the ellipsoids among the three-dimensional convex bodies, was a key in the process. This topic is still a very active research area, namely in studying finely tuned symmetry-related properties of the orthogonality on $(\mathscr{L}(X), \|\cdot\|)$ (see [11] and the references therein), or on generalized Minkowski spaces (see [13, Corollary 3.25]).

On the grounds of the Birkhoff-Kakutani-Day-James theorem, our article establishes two symmetry conditions for the nearness property: Proposition 7, which proves that nearness is symmetric on any locally convex space, provided that $\kappa < \frac{1}{2}$, and Theorem 10, showing that, on the particular locally convex space $(\mathscr{F}(X), \mathscr{P})$, nearness is symmetric iff orthogonality is symmetric on the Banach space $X$.

On this basis, we deduce the following two nonlinear generalizations of the Casazza-Christensen lemma.

**Theorem 4.** *Let $X$ be a real Banach space, and let $A \in \mathscr{L}(X)$ be a bounded linear operator verifying*

$$\exists \ \alpha > 0, \ 0 \leq \kappa < \frac{1}{2} : \quad \|A - \alpha\, Id\| \leq \kappa\|A\|. \tag{8}$$

*Then $A$ is invertible, and $\|A^{-1}\| \leq \frac{1-\kappa}{\alpha(1-2\kappa)}$.*

[4] Campanato (see [5]) introduced this notion in the particular case of the locally convex space $\mathscr{F}(X)$ endowed with the family of semi-norms given by relation (7).

**Theorem 5.** *Let $X$ be a real Banach space, and $A : X \to X$ be a not necessarily linear operator such that*

$$\exists\ \alpha > 0,\, 0 \le \kappa < 1 : \quad \|(A(x_1) - \alpha\, x_1) - (A(x_2) - \alpha\, x_2)\| \le \kappa\, \|A(x_1) - A(x_2)\|, \quad \forall x_1, x_2 \in X. \tag{9}$$

*The following statements hold:*

(a) *if $\kappa < \frac{1}{2}$, then $A$ is invertible, and $Lip(A^{-1}) \le \frac{1-\kappa}{\alpha(1-2\kappa)}$;*
(b) *if $X$ is a Hilbert space, then $A$ is invertible and $\mathrm{Lip}\,(A^{-1}) \le \frac{1+\kappa}{\alpha}$.*

Theorem 4 may be viewed as a symmetric version of the Neumann lemma, while Theorem 5 may be considered to be a symmetric version of the Söderlind-Campanato lemma. These two theorems are the main contribution of this paper to the topic of Neumann lemmata.

We conclude the article by pointing out a series of open problems. Namely, Theorems 12 and 13, respectively, prove that the constants used in the Symmetric Neumann lemma (respectively the Symmetric Söderlind-Campanato lemma) are optimal, if the Banach space $X$ admits a Hermitian decomposition (respectively a Hermitian decomposition of $\ell^1$-type). We thus ask whether the qualification conditions in Theorems 4 and 5 are optimal for any Banach space $X$, or if there are classes of spaces $X$ for which those theorems can be refined.

## 2. Two results about the uniform symmetry of nearness

**Definition 6.** Given a locally convex space $(Y, \mathscr{P})$ and $\lambda \in (0, 1]$, we say that nearness is $\lambda$-*uniformly symmetric* if, for every pair $(\alpha, \kappa) \in (0, +\infty) \times [0, \lambda)$, there exists a pair $(\overline{\alpha}, \overline{\kappa}) \in (0, +\infty) \times [0, 1)$ with the property that:

*if $u \in Y$ is near $v \in Y$ for constants $\alpha$ and $\kappa$, then $v$ is near $u$ for constants $\overline{\alpha}$ and $\overline{\kappa}$.*

When nearness is 1-uniformly symmetric, we simply say that it is *uniformly symmetric*.

The first result about the uniform symmetry of nearness says that, in the general setting of a locally convex space, nearness is $\frac{1}{2}$-uniformly symmetric.

**Proposition 7.** *Let $(Y, \mathscr{P})$ be a locally convex space, and assume that a vector $u \in Y$ is near $v \in Y$ for constants $\alpha > 0$ and $0 \le \kappa < \frac{1}{2}$:*

$$p_\iota(u - \alpha\, v) \le \kappa\, p_\iota(u), \quad \forall \iota \in I. \tag{10}$$

*Then $v$ is near $u$ for constants $\frac{1}{\alpha}$ and $\frac{\kappa}{1-\kappa}$, that is*

$$p_\iota\left(v - \frac{1}{\alpha}\, u\right) \le \frac{\kappa}{1-\kappa}\, p_\iota(v), \quad \forall \iota \in I. \tag{11}$$

**Proof.** Since for any $\iota \in I$, $p_\iota$ is a semi-norm, it results that

$$p_\iota(u) - p_\iota(\alpha\, v) \le p_\iota(u - \alpha\, v), \quad \forall \iota \in I,$$

and that $p_\iota(\alpha\, v) = \alpha\, p_\iota(v)$. Hence, from relation (10) it follows that

$$p_\iota(u) - \alpha\, p_\iota(v) \le \kappa\, p_\iota(u), \quad \forall \iota \in I,$$

and thus

$$p_\iota(u) \leq \frac{\alpha}{1-\kappa}\, p_\iota(v), \quad \forall \iota \in I. \tag{12}$$

Again as a consequence of the fact that $p_\iota$ is a semi-norm, we obtain that

$$p_\iota\left(v - \frac{1}{\alpha}\, u\right) = p_\iota\left(\frac{1}{\alpha}\, (\alpha\, v - u)\right) = \frac{1}{\alpha}\, p_\iota\,(\alpha\, v - u) = \frac{1}{\alpha}\, p_\iota\,(u - \alpha\, v)\,, \quad \forall \iota \in I;$$

so, by using once again relation (10), it follows that

$$p_\iota\left(v - \frac{1}{\alpha}\, u\right) \leq \frac{\kappa}{\alpha}\, p_\iota(u), \quad \forall \iota \in I. \tag{13}$$

The desired relation (11) follows by combining relations (12) and (13). □

A much more powerful result holds true when $X$ is equipped with an inner product. The following theorem proves that nearness is uniformly symmetric on any inner product space.

**Theorem 8.** *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space, and assume that a vector $u \in X$ is near $v \in X$ for constants $\alpha > 0$ and $0 \leq \kappa < 1$:*

$$\|u - \alpha\, v\| \leq \kappa\, \|u\|. \tag{14}$$

*Then $v$ is near $u$ for constants $\frac{1-\kappa^2}{\alpha}$ and $\kappa$, that is*

$$\left\| v - \frac{1-\kappa^2}{\alpha}\, u \right\| \leq \kappa\, \|v\|. \tag{15}$$

**Proof.** Since $X$ is an inner product space, it follows that

$$\|u - \alpha\, v\|^2 = \langle u - \alpha\, v,\, u - \alpha\, v\rangle = \|u\|^2 - 2\,\alpha\,\langle u,\, v\rangle + \alpha^2\,\|v\|^2.$$

So relation (14) is equivalent to

$$\|u\|^2 - 2\,\alpha\,\langle u,\, v\rangle + \alpha^2\,\|v\|^2 \leq \kappa^2\,\|u\|^2. \tag{16}$$

It is easy to see that, in its turn, relation (16) is equivalent to each of the following relations:

$$\|u\|^2(1-\kappa^2) - 2\alpha\,\langle u,\, v\rangle + \alpha^2\,\|v\|^2 \leq 0;$$

$$\frac{\|u\|^2(1-\kappa^2)}{\alpha^2} - 2\frac{\langle u,\, v\rangle}{\alpha} + \|v\|^2 \leq 0;$$

$$\frac{\|u\|^2(1-\kappa^2)^2}{\alpha^2} - 2\frac{\langle u,\, v\rangle\,(1-\kappa^2)}{\alpha} + \|v\|^2(1-\kappa^2) \leq 0;$$

$$\|v\|^2 - 2\frac{\langle v,\, u\rangle\,(1-\kappa^2)}{\alpha} + \frac{\|u\|^2(1-\kappa^2)^2}{\alpha^2} \leq \kappa^2\,\|v\|^2;$$

$$\left\| v - \frac{1-\kappa^2}{\alpha}\, u \right\|^2 \leq \kappa^2\,\|v\|^2. \tag{17}$$

The proof of Theorem 8 is complete, as relations (17) and (15) are equivalent. □

In achieving the result of Theorem 8, the hypothesis of completeness of the inner product space $X$ is not needed. Thus Theorem 8 is valid for general inner product spaces, and not necessarily for Hilbert spaces.

## 3. Symmetry and uniform symmetry in real normed spaces

In the framework of real normed spaces, the interplay between the symmetry, the uniform symmetry of nearness, and the symmetry of orthogonality can be clarified.

**Theorem 9.** *Let $X$ be a normed space. The following three statements are equivalent:*

(a) *nearness is uniformly symmetric on $X$;*
(b) *nearness is symmetric on $X$;*
(c) *orthogonality is symmetric on $X$.*

**Proof.** Our plan is to prove that $(a)$ implies $(b)$, $(b)$ implies $(c)$ and $(c)$ implies $(a)$. The proof of the last implication being rather long, we have decided to split it in two: we first prove that $(c)$ implies $(b)$, and on this basis we manage to show that $(c)$ implies $(a)$.

$(a) \Rightarrow (b)$. Obvious.

$(b) \Rightarrow (c)$. Let nearness be symmetric on $X$, and $u, v \in Y$ be two vectors such that $u$ is orthogonal to $v$. We need to prove that $v$ is orthogonal to $u$.

Let us remark that it is impossible that $v$ is near $u$; indeed, as nearness is symmetric, this would mean that $u$ is near $v$, or $u$ is orthogonal to $v$. In exactly the same way, we see that it is impossible that $v$ is near $-u$.

But for two vectors $v$ and $u$ in the normed space $X$, there are only three possibilities:

- $v$ may be near $u$;
- $v$ may be near $-u$;
- $v$ may be orthogonal to $u$.

As the possibilities $v$ near $u$ and $v$ near $-u$ have been eliminated, it results that $v$ is orthogonal to $u$.

$(c) \Rightarrow (b)$. Let orthogonality be symmetric on $X$, and $u, v \in Y$ be two vectors such that $u$ is near $v$, which means that

$$\exists\, \alpha > 0,\ 0 \leq \kappa < 1: \quad \|u - \alpha\, v\| \leq \kappa\, \|u\|. \tag{18}$$

Our aim is to prove that $v$ is near $u$.

Let us first remark that $v$ cannot be orthogonal to $u$; indeed, as orthogonality is symmetric, $v$ orthogonal to $u$ means that $u$ is orthogonal to $v$, which contradicts the fact that $u$ is near $v$. Moreover, as there are only three possibilities: $v$ near $u$, $v$ near $-u$, and $v$ orthogonal to $u$, and as the possibility $v$ orthogonal to $u$ has been eliminated, all what remains to be shown in order to prove the "if" part, is that $v$ is not near $-u$.

To the end of achieving a contradiction, we assume that $v$ is near $-u$. Thus,

$$\exists\, \alpha_1 > 0,\ 0 \leq \kappa_1 < 1: \quad \|v + \alpha_1\, u\| \leq \kappa_1\, \|v\|.$$

The function $f : \mathbb{R} \to [0,\, \infty)$, defined by the formula

$$f(s) := \|v + s\, u\|, \quad \forall s \in \mathbb{R},$$

is convex, continuous, coercive, and therefore attains its minimum at some $c \in \mathbb{R}$. Let us set $w := v + c\, u$. As

$$f(\alpha_1) = \|v + \alpha_1 u\| \leq \kappa_1 \|v\| < \|v\| = f(0),$$

we deduce that $c > 0$. Moreover, since

$$\|w\| = f(c) \leq f(c+s) = \|v + (c+s)\, u\| = \|w + s\, u\|, \quad \forall s \in \mathbb{R},$$

it follows that the vector $w$ is orthogonal to $u$. But orthogonality on X is symmetric, so $u$ is orthogonal to $w$. Accordingly,

$$\|u + s\, w\| \geq \|u\|, \quad \forall s \in \mathbb{R}. \tag{19}$$

As both $c$ and $\alpha$ are positive real numbers, it results that $1 + \alpha\, c > 1 > 0$, so $\frac{1}{1+\alpha\, c}$ is a positive real number. Thus

$$\begin{aligned} \|u - \alpha\, v\| &= (1 + \alpha\, c) \left\| \frac{1}{1+\alpha\, c} (u - \alpha\, v) \right\| \\ &= (1 + \alpha\, c) \left\| \frac{1}{1+\alpha\, c}\, u - \frac{\alpha}{1+\alpha\, c}\, v \right\| \\ &= (1 + \alpha\, c) \left\| \left(1 - \frac{\alpha\, c}{1+\alpha\, c}\right) u - \frac{\alpha}{1+\alpha\, c}\, v \right\| \\ &= (1 + \alpha\, c) \left\| u - \frac{\alpha}{1+\alpha\, c} (cu + v) \right\| \\ &= (1 + \alpha\, c) \left\| u - \frac{\alpha}{1+\alpha\, c}\, w \right\|. \end{aligned} \tag{20}$$

Combining relations (19) and (20) yields

$$\|u - \alpha\, v\| \geq (1 + \alpha\, c)\|u\| > \|u\|,$$

which is a contradiction with relation (18). Our initial assumption is then false, and the "if" part of the proof is completed.

$(c) \Rightarrow (a)$. Let orthogonality be symmetric on $X$. The celebrated Birkhoff-Kakutani-Day-James theorem ([14, Theorem 1]) says that a normed space $X$ on which orthogonality is symmetric is always an inner product space, except when $X$ is two-dimensional. In this case, there are uncountably many normed spaces with symmetric orthogonality, neither two among them being isomorphically isometric (such a Banach space is usually called a Radon plane, see [16]).

Since Theorem 8 proves that nearness is uniformly symmetric when $X$ is an inner-product space, we only need to address the case of a two-dimensional normed space $X$ on which orthogonality is symmetric.

In order to prove that nearness is uniformly symmetric on such a space $X$, let us pick $\alpha > 0$ and $0 \leq \kappa < 1$, and let us denote by $S_{\alpha,\kappa}$ the (possibly empty) set of all the pairs $(u,\, v) \in X \times X$ such that $\|u\| = 1$ and $u$ is near $v$ for constants $\alpha$ and $\kappa$:

$$\|u - \alpha\, v\| \leq \kappa. \tag{21}$$

As the function $\Theta : X \times X \to [0,\, +\infty)$,

$$\Theta(u, v) := \|u - \alpha\, v\|, \quad \forall u, v \in Y$$

is continuous, it follows that any of its sub-level sets is closed. This holds in particular for the sub-level set at level $\kappa$,

$$[\Theta \leq \kappa] := \{(u, v) \in X \times X : \ \|u - \alpha\, v\| \leq \kappa\}.$$

Since the set $S_{\alpha,\kappa}$ is the intersection of $[\Theta \leq \kappa]$ and the closed set

$$\{u \in X : \ \|u\| = 1\} \times X,$$

it follows that $S_{\alpha,\kappa}$ is a closed set. Moreover, from relation (21) it results that

$$\frac{1-\kappa}{\alpha} \leq \|v\| \leq \frac{1+\kappa}{\alpha}, \quad \forall (u, v) \in S_{\alpha,\kappa}, \tag{22}$$

so the set $S_{\alpha,\kappa}$ is also bounded. Being a closed and bounded subset of the vector space $X \times X$ of dimension 4, $S_{\alpha,\kappa}$ is a compact set.

An important step in achieving the desired conclusion is the study of the function $f : S_{\alpha,\kappa} \to [0, +\infty)$,

$$f(u, v) := \min \left\{ \frac{\|v - s\, u\|}{\|v\|} : \ s \geq 0 \right\} \tag{23}$$

(remark that, as $\|v\| \geq \frac{1-\kappa}{\alpha} > 0$ for any $(u, v) \in S_{\alpha,\kappa}$, it is possible to divide by $\|v\|$ in formula (23)).

Implication $(c) \Rightarrow (b)$ ensures us that nearness is symmetric on $X$. But $u$ is near $v$, so $v$ is near $u$. Accordingly,

$$f(u, v) < 1, \quad \forall (u, v) \in S_{\alpha,\kappa}.$$

For any sequence $(u_n, v_n) \in S_{\alpha,\kappa}$ converging to some element $(u, v)$, and any fixed $s \in (0, +\infty)$, we know that the sequence $\frac{\|v_n - s\, u_n\|}{\|v_n\|}$ converges to $\frac{\|v - s\, u\|}{\|v\|}$. By applying this obvious fact to the exact value of $s$ at which the minimum in formula (23) is attained, we deduce that

$$\limsup_{n\to\infty} f(u_n, v_n) \leq \lim_{n\to\infty} \frac{\|v_n - s\, u_n\|}{\|v_n\|} = \frac{\|v - s\, u\|}{\|v\|} = f(u, v).$$

Consequently, $f$ is an upper semi-continuous function defined on a compact set; therefore, this function attains its maximum over $S_{\alpha,\kappa}$. As all values of $f$ are strictly less than 1, the same holds for $\tilde{\kappa}$, its maximum over $S_{\alpha,\kappa}$.

We deduce that, for any $(u, v) \in S_{\alpha,\kappa}$, there exists $s_{u,v} \in (0, +\infty)$ such that

$$\frac{\|v - s_{u,v}\, u\|}{\|v\|} \leq \tilde{\kappa}, \quad \forall (u, v) \in S_{\alpha,\kappa}. \tag{24}$$

From relation (24), it stems that

$$(1 - \tilde{\kappa})\|v\| \leq s_{u,v}, \quad \forall (u, v) \in S_{\alpha,\kappa}. \tag{25}$$

By combining relations (22) and (25), it results that

$$\frac{(1 - \tilde{\kappa})(1 - \kappa)}{\alpha} \leq s_{u,v}, \quad \forall (u, v) \in S_{\alpha,\kappa}. \tag{26}$$

Let us denote the value $\frac{(1-\tilde{\kappa})(1-\kappa)}{\alpha}$ by $\overline{\alpha}$. Given $(u, v) \in S_{\alpha,\kappa}$, the function

$$g : [0, s_{u,v}] \to \mathbb{R}, \quad g(s) := \frac{\|v - s\, u\|}{\|v\|}$$

is convex, and

$$1 = g(0) > \tilde{\kappa} \geq g(s_{u,v}).$$

Accordingly, the function $g$ is strictly decreasing on the line interval $[0, s_{u,v}]$. It follows that $g(\overline{\alpha}) < 1$, that is

$$\frac{\|v - \overline{\alpha}\, u\|}{\|v\|} < 1, \quad \forall (u, v) \in S_{\alpha,\kappa}.$$

The function

$$h : S_{\alpha,\kappa} \to \mathbb{R}, \quad h(u, v) := \frac{\|v - \overline{\alpha}\, u\|}{\|v\|}, \quad \forall (u, v) \in S_{\alpha,\kappa}$$

is a continuous function defined on a compact set. Moreover, all its values are strictly less than 1. We deduce that its maximum over $S_{\alpha,\kappa}$ is again strictly less than 1:

$$\exists \overline{\kappa} \in [0,\, 1) : \quad \|v - \overline{\alpha}\, u\| \leq \overline{\kappa}\, \|v\|, \quad \forall (u, v) \in S_{\alpha,\kappa}.$$

We have thus proved that, given two vectors $u$ and $v$ such that $\|u\| = 1$ and $u$ is near $v$ for constants $\alpha$ and $\kappa$, then $v$ is near $u$ for constants $\overline{\alpha}$ and $\overline{\kappa}$.

Since, for any positive constant $t > 0$, the vectors $t\, u$ and $t\, v$ are near iff $u$ and $v$ are near, and the only vector to which the null vector is near, is the null vector itself, the proof of the implication $(c) \Rightarrow (a)$ is completed. □

## 4. Symmetry of nearness on the locally convex space $(\mathscr{F}(X),\ \mathscr{P})$

Let $X$ be a real normed space, and consider the locally convex space $\mathscr{F}(X)$ whose topology is given by the family of semi-norms

$$\mathscr{P} := \{p_{x_1,x_2} :\ x_1, x_2 \in X,\ \ x_1 \neq x_2\}$$

where

$$p_{x_1,x_2} : \mathscr{F}(X) \to \mathbb{R}, \quad p_{x_1,x_2}(A) := \|A(x_1) - A(x_2)\|, \quad \forall A \in \mathscr{F}(X).$$

Let us remark that the locally convex space $(\mathscr{F}(X),\ \mathscr{P})$ is not Hausdorff. Indeed, when applied to a constant function $A \in \mathscr{F}(X)$, the value of any semi-norm is zero (in other words, the semi-norms do not separate points in $\mathscr{F}(X)$).

**Theorem 10.** *Let us consider a real normed space $X$, and the locally convex space $(\mathscr{F}(X),\ \mathscr{P})$. The following three statements are equivalent:*

(a) *nearness is uniformly symmetric on $(\mathscr{F}(X),\ \mathscr{P})$;*
(b) *nearness is symmetric on $(\mathscr{F}(X),\ \mathscr{P})$;*
(c) *orthogonality is symmetric on $X$.*

**Proof. (b) ⇒ (c).** Let us assume that nearness is symmetric on $(\mathscr{F}(X),\ \mathscr{P})$. Consider $u, v \in X$ such that $u$ is near $v$, and let us define

$$A : X \to X, \quad A(0) = u,\ A(x) = 0, \quad \forall x \in X, x \neq 0,$$

and

$$B : X \to X, \quad B(0) = v,\ B(x) = 0, \quad \forall x \in X, x \neq 0.$$

It is easy to see that, since $u$ in near $v$ in $X$, $A$ is near $B$ in $(\mathscr{F}(X),\ \mathscr{P})$. As nearness is symmetric on $(\mathscr{F}(X),\ \mathscr{P})$, it results that $B$ is near $A$ in $(\mathscr{F}(X),\ \mathscr{P})$, and it is again easy to see that this means that $v$ is near $u$ in $X$.

Accordingly, symmetry of nearness on $(\mathscr{F}(X),\ \mathscr{P})$ implies symmetry of nearness on $X$. Theorem 9 proves now that orthogonality is symmetric on $X$.

**(c) ⇒ (a).** Let orthogonality be symmetric on $X$, and consider the constants $\alpha$ and $\kappa$ such that $\alpha > 0$ and $0 \leq \kappa < 1$, and $A, B \in \mathscr{F}(X)$ such that $A$ is near $B$ for constants $\alpha$ and $\kappa$.

Orthogonality is symmetric on $X$, so, by virtue of implication $(c) \Rightarrow (a)$ in Theorem 9, nearness is uniformly symmetric on $X$: there are two constants $\overline{\alpha} > 0$ and $0 \leq \overline{\kappa} < 1$ such that $v$ is near $u$ for constants $\overline{\alpha}$ and $\overline{\kappa}$ every time when $u$ is near $v$ for constants $\alpha$ and $\kappa$.

But $A$ is near $B$ for constants $\alpha$ and $\kappa$ on $(\mathscr{F}(X),\ \mathscr{P})$, so, for every $x_1, x_2 \in X$, the vector $A(x_1) - A(x_2)$ is near the vector $B(x_1) - B(x_2)$ in $X$, with the same constants $\alpha$ and $\kappa$. It follows that

$$\|(B(x_1) - \overline{\alpha}\, A(x_1)) - (B(x_2) - \overline{\alpha}\, A(x_2))\| \leq \overline{\kappa}\, \|B(x_1) - B(x_2)\|, \quad \forall x_1, x_2 \in X.$$

In conclusion, the operator $B$ is near $A$ in $(\mathscr{F}(X),\ \mathscr{P})$ for constants $\overline{\alpha}$ and $\overline{\kappa}$, so nearness is uniformly symmetric on $(\mathscr{F}(X),\ \mathscr{P})$.

As relation **(a) ⇒ (b)** obviously holds true, the proof of Theorem 10 is complete. □

## 5. Main results and open problems

By combining the classical Neumann lemma with Proposition 7, we are in a position to prove the first Neumann-like result of our study, that is Theorem 4.

**Proof of Theorem 4.** Relation (8) states that $A$ is near $Id$ for constants $\alpha$ an $\kappa < \frac{1}{2}$ on $(\mathscr{L}(X), \|\cdot\|)$. By applying Proposition 7, we deduce that $Id$ is near $A$ for constants $\overline{\alpha} := \frac{1}{\alpha}$ and $\overline{\kappa} := \frac{\kappa}{1-\kappa}$ (remark that, since $0 \leq \kappa < \frac{1}{2}$ it follows that $0 \leq \overline{\kappa} < 1$).

Neumann's lemma may now be invoked in order to deduce that $A$ is invertible, and that

$$\|A^{-1}\| \leq \frac{\overline{\alpha}}{1 - \overline{\kappa}} = \frac{\frac{1}{\alpha}}{1 - \frac{\kappa}{1-\kappa}} = \frac{1 - \kappa}{\alpha(1 - 2\kappa)}.$$

As a consequence, the claim is achieved. □

In order to determine to what extent the constants $\frac{1-\kappa}{\alpha(1-2\kappa)}$ and $\frac{1}{2}$ from Theorem 4 are optimal, we need to introduce two new notions.

Let us first define the *symmetry modulus* of a locally convex space $(Y,\ \mathscr{P})$ as being the supremum of the set of $\lambda$ such that nearness on $X$ is $\lambda$-uniformly symmetric. Proposition 7 implies that the symmetry modulus always lies in the interval $\left[\frac{1}{2},\ 1\right]$. Accordingly, Theorem 4 uses the constant $\frac{1}{2}$ which is a lower bound of the set of all the symmetry moduli, and it is obvious that this theorem may be refined by replacing it with the actual symmetry modulus of the Banach space $X$.

The second notion of interest is that of a Banach space possessing a *Hermitian decomposition*, that is (see [8, Chapter 7]) a space $X$ with the following property: there exist two closed subspaces $M$ and $N$ of $X$ such that $X = M \oplus N$ and $\|u - v\| = \|u + v\|$ for every $u \in M$ and $v \in N$. The Hermitian decomposition is said to be nontrivial if both $M$ and $N$ are unbounded. As $X$ is the direct sum of $M$ and $N$, any element $x$ of $X$ may be uniquely expressed as the sum of $x_M$, a vector of $M$, and $x_N$, a vector of $N$. It is a standard construction to consider two bounded linear operators:

$$\Pi_M : X \to X, \quad \Pi_M(x) := x_M$$

and

$$\Pi_N : X \to X, \quad \Pi_N(x) := x_N,$$

customary called the projections of $X$ onto $M$ (respectively onto $N$) along $N$ (respectively along $M$).

The following result, although presumably known, gathers several basic properties of a Hermitian decomposition.

**Lemma 11.** *Let $X = M \oplus N$ be a nontrivial Hermitian decomposition of $X$. Then the following relations hold true:*

$$\Pi_M \Pi_M = \Pi_M, \quad \Pi_N \Pi_N = \Pi_N, \quad \Pi_M \Pi_N = \Pi_N \Pi_M = 0, \quad \Pi_M + \Pi_N = Id, \tag{27}$$

$$\|a\,\Pi_M + b\,\Pi_N\| = \max(|a|, |b|), \quad \forall a, b \in \mathbb{R}. \tag{28}$$

**Proof.** As relation (27) is easy-to-prove, let us focus on relation (28). Clearly, it results that

$$\|a\,\Pi_M(x) + b\,\Pi_N(x)\| = \|a\,x + b\,0\| = |a|\,\|x\| = |a|, \quad \forall x \in M, \ \|x\| = 1,$$

and

$$\|a\,\Pi_M(x) + b\,\Pi_N(x)\| = \|a\,0 + b\,x\| = |b|\,\|x\| = |b|, \quad \forall x \in N, \ \|x\| = 1.$$

Since $M$ and $N$ both contain unit vectors ($M$ and $N$ are unbounded linear subspaces of $X$), we deduce that

$$\|a\,\Pi_M + b\,\Pi_N\| \geq \max(|a|, |b|), \quad \forall a, b \in \mathbb{R}. \tag{29}$$

Let us first determine the operator norm of the projection $\Pi_M$ (the case of $\Pi_N$ is evidently similar). From the obvious relation

$$\Pi_M(x) = \frac{1}{2}(\Pi_M(x) - \Pi_N(x)) + \frac{1}{2}(\Pi_M(x) + \Pi_N(x)), \quad \forall x \in X,$$

and the fact that $X = M \oplus N$ is a Hermitian decomposition, it follows that

$$\begin{aligned}
\|\Pi_M(x)\| &\leq \frac{1}{2}\|\Pi_M(x) - \Pi_N(x)\| + \frac{1}{2}\|\Pi_M(x) + \Pi_N(x)\| \\
&= \frac{1}{2}\|\Pi_M(x) + \Pi_N(x)\| + \frac{1}{2}\|\Pi_M(x) + \Pi_N(x)\| \\
&= \|\Pi_M(x) + \Pi_N(x)\| \\
&= \|x\|, \quad \forall x \in X.
\end{aligned}$$

Thus, we obtain

$$\|\Pi_M\| \le 1. \tag{30}$$

Combining relations (29) applied for $a = 1$ and $b = 0$, and (30), we deduce that

$$\|\Pi_M\| = 1. \tag{31}$$

A similar result clearly holds true also for $\Pi_N$.

Relation (30) proves the reverse inequality to (29) in the case $a = 1$ and $b = 0$; our aim is to prove this reverse inequality in the general case $a, b \in \mathbb{R}$. We assume without restraining the generality that $a \ge |b|$ (indeed, if $|a| < |b|$ we switch $M$ and $N$, and if $a < 0$, we address the operator $-a\,\Pi_M - b\,\Pi_N$, which obviously has the same operator norm as $a\,\Pi_M + b\,\Pi_N$).

If $a = 0$, then $|b| \le 0$, so $b = 0$, and relation (28) is trivially satisfied. Let us now address the only remaining case: $a \ge |b|$ and $a > 0$. We distinguish two possibilities:

(i) $|b| \neq 0$;
(ii) $b = 0$,

which will be examined one by one.

*(i)* $|b| \neq 0$, that is $a > 0 > |b| \ge -a$. In order to simplify the notation, let us set $c := \frac{|b|}{a}$, and $d = \frac{b}{|b|}$; let us remark that $0 < c \le 1$, and that $d$ amounts to $-1$ or $1$, depending upon the sign of $b$. Then, for every $x \in X$ it follows that

$$\|a\,\Pi_M(x) + b\,\Pi_N(x)\| = \|a\,x_M + b\,x_N\| = a\left\|x_M + \frac{b}{a}\,x_N\right\| = a\,\|x_M + d\,c\,x_N\|.$$

But we have

$$x_M + d\,c\,x_N = (1-c)\,x_M + c\,(x_M + d\,x_N),$$

hence, since $0 < c \le 1$, and $\|x_M + d\,x_N\| = \|x_M + x_N\|$ we get

$$\begin{aligned}\|x_M + d\,c\,x_N\| &\le (1-c)\|x_M\| + c\,\|x_M + d\,x_N\| \\ &= (1-c)\|x_M\| + c\,\|x_M + x_N\| \\ &= (1-c)\|x_M\| + c\,\|x\|.\end{aligned}$$

Accordingly,

$$\begin{aligned}\|a\,\Pi_M(x) + b\,\Pi_N(x)\| &\le a\,\left((1-c)\|x_M\| + c\|x\|\right) \\ &\le a\,\left((1-c)\|x\| + c\|x\|\right) \\ &= a\,\|x\|, \quad \forall x \in X.\end{aligned}$$

Therefore, it results

$$\|a\,\Pi_M + b\,\Pi_N\| \le a = \max(a,\, |b|). \tag{32}$$

Relation (28) follows by combining relations (29) and (32).

$(ii)$ $b = 0$. Thus, by virtue of relation (31), we have that

$$\|a\,\Pi_M + b\,\Pi_N\| = \|a\,\Pi_M\| = a\,\|\Pi_M\| = a = \max(|a|,\,|b|). \quad \square$$

The next theorem establishes the value of the symmetry modulus for the space $(\mathscr{L}(X),\,\|\cdot\|)$, when $X$ is a Banach space with a Hermitian decomposition.

**Theorem 12.** *Let $X$ be a Banach space admitting a nontrivial Hermitian decomposition.*

(a) *For any real values $\alpha$ and $\kappa$ such that $\alpha > 0$ and $0 \le \kappa < \frac{1}{2}$, there exists $A \in \mathscr{L}(X)$ an invertible operator which is near $Id$ for constants $\alpha$ and $\kappa$ such that the operator norm of $A^{-1}$ amounts to $\frac{1-\kappa}{\alpha(1-2\kappa)}$.*
(b) *The symmetry modulus of $(\mathscr{L}(X),\,\|\cdot\|)$ is $\frac{1}{2}$.*

**Proof.** Let $X = M \oplus N$ be a nontrivial Hermitian decomposition of $X$. To address item $(a)$, we define $A \in \mathscr{L}(X)$ by the formula

$$A := \frac{\alpha}{1-\kappa}\,\Pi_M + \frac{\alpha(1-2\,\kappa)}{1-\kappa}\,\Pi_N.$$

Obviously, as $\alpha \ne 0$ and $\kappa \ne \frac{1}{2}$, it follows that $\frac{\alpha}{1-\kappa}$ and $\frac{\alpha(1-2\,\kappa)}{1-\kappa}$ are positive real numbers, hence, direct calculations prove that

$$\left(\frac{\alpha}{1-\kappa}\,\Pi_M + \frac{\alpha(1-2\,\kappa)}{1-\kappa}\,\Pi_N\right)\left(\frac{1-\kappa}{\alpha}\,\Pi_M + \frac{1-\kappa}{\alpha(1-2\,\kappa)}\,\Pi_N\right)$$
$$= \left(\frac{1-\kappa}{\alpha}\,\Pi_M + \frac{1-\kappa}{\alpha(1-2\,\kappa)}\,\Pi_N\right)\left(\frac{\alpha}{1-\kappa}\,\Pi_M + \frac{\alpha(1-2\,\kappa)}{1-\kappa}\,\Pi_N\right) = Id.$$

The inverse of $A$ is thus given by the formula

$$A^{-1} = \frac{1-\kappa}{\alpha}\,\Pi_M + \frac{1-\kappa}{\alpha(1-2\,\kappa)}\,\Pi_N.$$

Also, we obtain

$$A - \alpha\,Id = \left(\frac{\alpha}{1-\kappa} - \alpha\right)\Pi_M + \left(\frac{\alpha(1-2\,\kappa)}{1-\kappa} - \alpha\right)\Pi_N$$
$$= \frac{\alpha\,\kappa}{1-\kappa}\,\Pi_M - \frac{\alpha\,\kappa}{1-\kappa}\,\Pi_N.$$

By using Lemma 11, it results that

$$\|A\| = \max\left(\left|\frac{\alpha}{1-\kappa}\right|,\,\left|\frac{\alpha(1-2\,\kappa)}{1-\kappa}\right|\right),$$
$$\|A^{-1}\| = \max\left(\left|\frac{1-\kappa}{\alpha}\right|,\,\left|\frac{1-\kappa}{\alpha(1-2\,\kappa)}\right|\right),$$
$$\|A - \alpha\,Id\| = \max\left(\left|\frac{\alpha\,\kappa}{1-\kappa}\right|,\,\left|-\frac{\alpha\,\kappa}{1-\kappa}\right|\right) = \left|\frac{\alpha\,\kappa}{1-\kappa}\right|.$$

Since $\alpha > 0$ and $0 \le \kappa < \frac{1}{2}$, it holds that

$$\frac{\alpha}{1-\kappa} \ge \frac{\alpha(1-2\,\kappa)}{1-\kappa} > 0, \quad \frac{1-\kappa}{\alpha(1-2\,\kappa)} \ge \frac{1-\kappa}{\alpha} > 0, \quad \frac{\alpha\,\kappa}{1-\kappa} > 0,$$

so

$$\|A\| = \frac{\alpha}{1-\kappa}, \quad \|A^{-1}\| = \frac{1-\kappa}{\alpha(1-2\,\kappa)}, \quad \|A - \alpha\, Id\| = \frac{\alpha\,\kappa}{1-\kappa}.$$

Obviously, we deduce

$$\|A - \alpha\, Id\| = \frac{\alpha\,\kappa}{1-\kappa} = \kappa\,\frac{\alpha}{1-\kappa} = \kappa\,\|A\|.$$

Therefore $A$ is near $Id$ for constants $\alpha$ and $\kappa$, while the operator norm of $A^{-1}$ is $\frac{1-\kappa}{\alpha(1-2\,\kappa)}$.

Let us now consider item $(b)$. We define the operator $A := 2\,\Pi_M$; of course, $A - Id = 2\,\Pi_M - (\Pi_M + \Pi_N) = \Pi_M - \Pi_N$.

By using once more Lemma 11 we infer that

$$\|A\| = \max(|2|, 0) = 2, \quad \|A - Id\| = \max(|1|, |-1|) = 1,$$

and it yields

$$\|A - Id\| = 1 = \frac{1}{2}\,2 = \frac{1}{2}\,\|A\|,$$

which means that $A$ is near $Id$ for constants $\alpha = 1$ and $\kappa = \frac{1}{2}$. Yet, $A$ is not invertible (as its kernel is equal to the unbounded linear space $N$), so $Id$ is not near $A$ (as from the original Neumann lemma this would imply that $A$ is invertible).

We have thus proved that the symmetry modulus of $(\mathscr{L}(X), \|\cdot\|)$ is less than or equal to $\frac{1}{2}$, and the proof of Theorem 12 is complete. □

Numerous Banach spaces admit easy-to-find non-trivial Hermitian decompositions: this is the case of the Hilbert spaces, of $\mathbb{R}^n$, $n \geq 2$, equipped with one of the $\ell^p$ norms, of spaces $\ell^p(\mathbb{N})$, with $1 < p < \infty$, or of the space $C(K_1 \cup K_2)$, where $K_1$ and $K_2$ are two disjoint compacts.

Consequently, the values $\frac{1}{2}$ and $\frac{1-\kappa}{\alpha(1-2\,\kappa)}$ in Theorem 4 are optimal for each and every of the previously mentioned classes of Banach spaces.

There are, however, Banach spaces for which no non-trivial Hermitian decomposition exists, for instance $C(K)$ when $K$ is a connected compact (see [8, Theorem 7.7]). We are thus led to ask the following question:

*Does the symmetry modulus of* $(\mathscr{L}(X), \|\cdot\|)$ *amount to* $\frac{1}{2}$ *for any Banach space* $X$*? In particular, what is the symmetry modulus of the space* $(\mathscr{L}(X), \|\cdot\|)$ *when* $X := C([0, 1])$*?*

Our second nonlinear generalization of the Casazza-Christensen, Theorem 5, may be proved as follows.

**Proof of Theorem 5.** As the proof of item $(a)$ is obtained from the proof of Theorem 4, by simply using Söderlind-Campanato's lemma instead of Neumann's lemma, we shall focus on item $(b)$.

Since $X$ is a Hilbert space, by combining relation (9) and Theorem 8, we obtain that, on the locally convex space $(\mathscr{F}(X), P)$, $Id$ is near $A$ for constants $\overline{\alpha} := \frac{1-\kappa^2}{\alpha}$ and $\overline{\kappa} := \kappa$.

Söderlind-Campanato's lemma proves now that $A$ is invertible, and that

$$\mathrm{Lip}\,(A^{-1}) \leq \frac{\overline{\alpha}}{1-\overline{\kappa}} = \frac{\frac{1-\kappa^2}{\alpha}}{1-\kappa} = \frac{1+\kappa}{\alpha},$$

which completes the proof. □

Let us note that item $(b)$ in Theorem 5 proves that the symmetry modulus of the locally convex space $(\mathscr{F}(X), \mathscr{P})$ is 1 for every Hilbert space $X$, that is, it achieves its maximal theoretical value. Theorem 13 describes a class of Banach spaces $X$ for which the symmetry modulus of $(\mathscr{F}(X), \mathscr{P})$ is equal to $\frac{1}{2}$, that is to its minimal possible value.

In order to state Theorem 13, let us first introduce a particular class of Hermitian decompositions. Given $X = M \oplus N$, a non-trivial Hermitian decomposition of a Banach space $X$, we call it *of $\ell^1$-type* if

$$\|x\| = \|\Pi_M(x)\| + \|\Pi_N(x)\|, \quad \forall x \in X,$$

and if there is a linear operator $H : M \to N$ such that

$$\|H(x)\| = \|x\|, \quad \forall x \in M.$$

**Theorem 13.** *Let $X$ be a real Banach space admitting a Hermitian decomposition of $\ell^1$-type. Then, the symmetry modulus of the locally convex space $(\mathscr{F}(X), \mathscr{P})$ is equal to $\frac{1}{2}$.*

**Proof.** Our aim is to find an operator $A \in \mathscr{F}(X)$ such that $A$ is near $Id$ with $\kappa = \frac{1}{2}$, but that $Id$ is not near $A$.

Let us define the desired operator $A : X \to X$ via the relation

$$A := Id + H\,\Pi_M + \Pi_N.$$

We claim that $A$ is near $Id$. In order to calculate the norm of $A(x)$, let us remark that

$$\Pi_M\, A = \Pi_M\, Id + \Pi_M\, H\,\Pi_M + \Pi_M\,\Pi_N = \Pi_M + 0 + 0 = \Pi_M,$$

and

$$\Pi_N\, A = \Pi_N\, Id + \Pi_N\, H\,\Pi_M + \Pi_N\,\Pi_N = \Pi_N + H\,\Pi_M + \Pi_N = H\,\Pi_M + 2\,\Pi_N.$$

Since the Hermitian decomposition $X = M \oplus N$ is of $\ell^1$-type, it follows that

$$\|A(x)\| = \|\Pi_M(A(x))\| + \|\Pi_N(A(x))\| = \|\Pi_M(x)\| + \|H(\Pi_M(x)) + 2\,\Pi_N(x)\|.$$

But, it results

$$\|H(\Pi_M(x))\| = \|\Pi_M(x)\|, \quad \forall x \in X,$$

and it yields

$$\|A(x)\| = \|H(\Pi_M(x))\| + \|H(\Pi_M(x)) + 2\,\Pi_N(x)\|.$$

The triangle inequality reads that

$$\begin{aligned} \|A(x)\| &= \|H(\Pi_M(x))\| + \|H(\Pi_M(x)) + 2\,\Pi_N(x)\| \\ &\geq \|H(\Pi_M(x)) + (H(\Pi_M(x)) + 2\,\Pi_N(x))\| \\ &= 2\,\|H(\Pi_M(x)) + \Pi_N(x)\| \\ &= 2\,\|(A - Id)(x)\|, \quad \forall x \in X. \end{aligned} \tag{33}$$

Since $A$ is linear, relation (33) means that $A$ is near $Id$ on $(\mathscr{F}(X), P)$ for constants $\alpha = 1$ and $\kappa = \frac{1}{2}$.

Let us prove now that $Id$ is not near $A$. In this respect, let us pick a non-zero vector $x$ from $M$, and let us calculate $\|(Id - \alpha A)(x)\|$, for some non-negative constant $\alpha$. We have

$$A(x) = Id(x) + H(\Pi_M(x)) + \Pi_N(x) = x + H(x) + 0 = x + H(x),$$

then

$$(Id - \alpha A)(x) = (1-\alpha)x - \alpha H(x).$$

Obviously, it follows

$$\Pi_M((Id - \alpha A)(x)) = (1-\alpha)x, \quad \Pi_N((Id - \alpha A)(x)) = -\alpha H(x),$$

and since the Hermitian decomposition $X = M \oplus N$ is of $\ell^1$-type, it follows that

$$\begin{aligned}\|(Id - \alpha A)(x)\| &= \|\Pi_M((Id - \alpha A)(x))\| + \|\Pi_N((Id - \alpha A)(x))\| \\ &= |1-\alpha|\|x\| + |\alpha| \, \|H(x)\| \\ &= (|1-\alpha| + |\alpha|) \, \|x\| \\ &\geq |1-\alpha+\alpha| \, \|x\| \\ &= \|x\|, \quad \forall \alpha \geq 0.\end{aligned}$$

Accordingly, on $X$, the vector $Id(x)$ is not near the vector $A(x)$ for all non-zero vectors $x$ from $M$; hence $Id$ is not near $A$ on $(\mathscr{F}(X), P)$. $\square$

The spaces $\mathbb{R}^n$, $n \geq 2$, endowed with the $\ell^1$ norm, $\ell^1(\mathbb{N})$, $L^1(\mathbb{R}^n)$, are only a few of the Banach spaces possessing a Hermitian decomposition of $\ell^1$-type, generating thus locally convex spaces of the form $(\mathscr{F}(X), \mathscr{P})$ with a symmetry modulus equal to $\frac{1}{2}$. Hence, the value $\frac{1}{2}$ in item $(a)$ of Theorem 5 cannot be refined for this kind of Banach spaces.

Several open questions are in order:
*Is it true that the symmetry modulus of* $(\mathscr{F}(X), \mathscr{P})$ *is equal to* $\frac{1}{2}$ *for every space* $X$ *of the form* $L^1(S)$, *where* $S$ *is some measurable space? Is it true that all locally convex spaces of the form* $(\mathscr{F}(X), \mathscr{P})$, *when* $X = L^p(S)$ *share a common symmetry modulus, and if yes, may we find its value as a function of* $p$*? What is the value of the symmetry modulus* of $(\mathscr{F}(X), \mathscr{P})$, when $X = C(K)$, with $K$ a compact space?

In conclusion, we observe that Theorems 4 and 5 are optimal within the class of Banach spaces $X$. Precise calculations of the symmetry modulus for the space of linear operators $(\mathscr{L}(X), \|\cdot\|)$, as well as for the locally convex space $(\mathscr{F}(X), \mathscr{P})$, achieved for various families of Banach spaces $X$, should allow us to formulate more accurate versions of our main results.

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