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# Existence results for the mixed Cauchy–Dirichlet problem for a class of hyperbolic operators

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**Abstract**
The paper concerns the study of the Cauchy–Dirichlet problem for a class of hyperbolic second-order operators with double characteristics in presence of transition in a domain of $\mathbb{R}^3$. Firstly, we establish some a priori local and global estimates. Then, we obtain some existence results.

**Keywords** Cauchy–Dirichlet problem · Hyperbolic equations · Pseudodifferential operators · A priori estimates

**Mathematics Subject Classification** 35L10 · 47G30

## 1 Introduction

Let $\Omega = ]0, +\infty[\times\Omega_0$, where $\Omega_0$ is an open set of $\mathbb{R}^2$ with enough smooth boundary (for example $\Omega_0$ is of class $C^m$, with $m \geq 2$). Let us set $S = [0, +\infty[\times\partial\Omega_0$, where $\partial\Omega_0$ is the boundary of $\Omega_0$. Let us consider the following class of hyperbolic second-order operators with double characteristics in presence of transition:

$$P = D^2_{x_0} - D^2_{x_1} - (x_0 - \alpha(x_1, x_2))^2 D^2_{x_2} + \sum_{j=0}^{2} a_j(x) D_{x_j} + b(x), \quad \text{in } \Omega, \tag{1}$$

where $x = (x_0, x_1, x_2)$, $\operatorname{Im} a_2(x) = (x_0 - \alpha(x'))\widetilde{a}_2(x)$, with $\widetilde{a}_2(x)$ real function, $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 0, 1, 2$, the coefficients belong in $C^\infty(\widetilde{\Omega})$, $\widetilde{\Omega} = [0, +\infty[\times\widetilde{\Omega}_0$, with $\widetilde{\Omega}_0$ an open set containing strictly $\Omega_0$, and $\alpha$ is a real function. Let $x' = (x_1, x_2)$, $\xi = (\xi_0, \xi_1, \xi_2) = (\xi_0, \xi')$, where we set $\xi' = (\xi_1, \xi_2)$. Let

$$p(x_0, x', \xi) = -\xi_0^2 + \xi_1^2 + (x_0 - \alpha(x'))^2\xi_2^2 + \frac{1}{i}\sum_{j=0}^{2} a_j(x)\xi_j + b(x)$$

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be the *symbol* of $P$, let

$$\Sigma = \left\{\rho = (x_0, x', \xi) \in T^*\Omega : \ p(\rho) = 0, \ \nabla p(\rho) = 0\right\}$$

be the characteristic set and let

$$F_p(\rho) = \frac{1}{2}\begin{pmatrix} p''_{x\xi}(\rho) & p''_{\xi\xi}(\rho) \\ -p''_{xx}(\rho) & -p''_{\xi x}(\rho) \end{pmatrix}, \quad \forall \rho \in \Sigma$$

be the *fundamental matrix* of $P$ at $\rho$. The spectrum of $F_p(\rho)$, denoted by $\mathrm{Spec}(F_p(\rho))$, has an important rule to study the well-posedness of the Cauchy–Dirichlet problem associated to the operator $P$. In particular, it results (see [10])

$$z \in \mathrm{Spec}(F_p(\rho)) \ \Leftrightarrow \ -z, \bar{z} \in \mathrm{Spec}(F_p(\rho)).$$

The fundamental matrix of $P$ at $\rho$ has only pure imaginary eigenvalues with a possible exception of a pair of nonzero real eigenvalues $\pm\lambda$ (see [9–11]). If $F_p(\rho)$ has a pair of nonzero real eigenvalues, $P$ is called *effectively hyperbolic at* $\rho$. If $F_p(\rho)$ has only pure imaginary eigenvalues and if there are only Jordan blocks of dimension 2 in the Jordan normal form of $F_p(\rho)$ corresponding to the eigenvalue 0, i.e., $\mathrm{Ker}F_p(\rho)^2 \cap \mathrm{Im}F_p(\rho)^2 = \{0\}$, $P$ is called *non-effectively hyperbolic of type 1 at* $\rho$. Instead, if $F_p(\rho)$ has only pure imaginary eigenvalues and if there is only a Jordan block of dimension 4 and no block of dimension 3 in the Jordan normal form of $F_p(\rho)$ corresponding to the eigenvalue 0, i.e., $\mathrm{Ker}F_p(\rho)^2 \cap \mathrm{Im}F_p(\rho)^2$ is 2-dimensional, $P$ is called *non-effectively hyperbolic of type* 2 *at* $\rho$. Furthermore, let

$$\begin{aligned}
\Sigma_+ &= \{\rho \in \Sigma : \ P \text{ is effectively hyperbolic at } \rho\},\\
\Sigma_- &= \{\rho \in \Sigma : \ P \text{ is non-effectively hyperbolic of type 1 at } \rho\},\\
\Sigma_0 &= \{\rho \in \Sigma : \ P \text{ is non-effectively hyperbolic of type 2 at } \rho\},
\end{aligned}$$

(see [9]). It is easy to deduce

$$\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+.$$

We say that *we have a transition* exactly when *at least two among the above sets are nonempty*.

The paper continues the study on the following Cauchy–Dirichlet problem

$$\begin{cases} Pu = f, & \text{in } \Omega = ]0, +\infty[\times\Omega_0 \\ u|_{\partial\Omega} = 0, \ \frac{du}{dn}|_{\Omega_0} = 0, \ u|_S = 0 \end{cases} \tag{2}$$

started in [7]. In fact, in [7], several a priori estimates of local or global nature in Sobolev spaces with general exponent $s \leq 0$ for the class of second-order hyperbolic operators (1) are proved. Here, we establish some existence results for the Cauchy–Dirichlet problem (2). To this aim, we need to obtain other a priori estimates in Sobolev spaces with exponent $s \leq 0$. The proofs of such estimates make use of delicate variational techniques because of the degeneration on the characteristic set and of the transition between $\Sigma_-$, $\Sigma_0$ and $\Sigma_+$. More precisely, the function $\alpha$ in (1) depends on the variables $x_1$ and $x_2$. As a consequence, the coefficient $x_0 - \alpha(x')$ degenerates on the characteristic set with respect to all the variables. Setting $\beta = x_0 - \alpha(x')$, if $|\partial_{x_1}\alpha(x')| < 1$, $\beta = 0$ and $\xi_0 = \xi_1 = 0$, then $F_p(\rho)$ has two distinct nonzero real eigenvalues. If $|\partial_{x_1}\alpha(x')| > 1$, $\beta = 0$ and $\xi_0 = \xi_1 = 0$, $F_p(\rho)$ has two nonzero imaginary eigenvalues. In conclusion, let $\overline{\Sigma}$ be the set of points $\rho = (x_0, x', \xi)$ of $\Sigma$ such that $\beta = 0$ and $\xi_0 = \xi_1 = 0$. We have that $\rho \in \Sigma_+$ if $\rho \in \overline{\Sigma}$ and $|\partial_{x_1}\alpha(x')| < 1$, $\rho \in \Sigma_-$ if $\rho \in \overline{\Sigma}$ and $|\partial_{x_1}\alpha(x')| > 1$, and $\rho \in \Sigma_0$ if $\rho \in \overline{\Sigma}$ and $|\partial_{x_1}\alpha(x')| = 1$. Hence, even if we study the special class of operators (1), the transition from effectively hyperbolic to non-effectively hyperbolic occurs. A class more general of hyperbolic second-order operators with double characteristics is analyzed in [6]. It is worth to underline that the coefficient $x_0 - \alpha(x')$ does not contain the parameter $\lambda$ very helpful to prove global estimates near the boundary of $\Omega$ in [5]. Finally, we remark that the operator (1) contains the first-order terms and the zero-order term, which have an important rule to study the well-posedness of the problem. Instead in [4], the subprincipal term is identically zero; consequently, the Hörmander–Ivrii–Petkov condition is automatically verified.

Several scholars considered the Cauchy problem either for effectively or non-effectively hyperbolic operators with double characteristics (see, for instance, [8, 10–16]). In [9], another class of hyperbolic second-order operators with double characteristics is analyzed. In particular, the $C^\infty$ well-posedness of the Cauchy problem and Carleman estimates for non-effectively hyperbolic operators have been obtained. In [17], some energy estimates for a different class of hyperbolic second-order operators are established. Moreover, the $C^\infty$ well-posedness of the Cauchy problem for non-effectively hyperbolic operators is studied. We underline that in [9, 17] the Cauchy problem for a class of operators in a form more general then (1) is analyzed, but a priori estimates only when $\Sigma = \Sigma_- \sqcup \Sigma_0$ are established. Instead, thanks to variational and pseudodifferential techniques different from the ones used in [9, 17], we are able to examine the mixed Cauchy–Dirichlet problem and we prove a priori estimates when $\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$ or $\Sigma = \Sigma_- \sqcup \Sigma_0$ or $\Sigma = \Sigma_0 \sqcup \Sigma_+$ or $\Sigma = \Sigma_-$ or $\Sigma = \Sigma_+$. Moreover, in the class of operators (1), studied also in [1–3], both the case in which $F_p(\rho)$ has two distinct real eigenvalues and the case in which all the eigenvalues are purely imaginary numbers can occur.

We set $\beta(x) = x_0 - \alpha(x')$, $g(x') = \frac{\alpha(x')}{\partial_{x_1}\alpha(x')}$, $h(x') = 1 - \partial_{x_1} g(x')$, in $\widetilde{\Omega}$,

$$\begin{aligned} \Gamma &= \{x \in \widetilde{\Omega} \,:\; \beta(x) = 0\}, \\ \Gamma' &= \{x \in \Gamma \,:\; \alpha(x') \geq 0\}, \\ \Omega_0' &= \{x' \in \widetilde{\Omega}_0 \,:\; \alpha(x') \geq 0\}. \end{aligned}$$

Moreover, let $B = (b_{hk})_{h,k=0,1}$ be the quadratic matrix-function whose elements are given by:

$$\begin{aligned} b_{00}(x) &= h(x') - 2\alpha(x')\widetilde{a}_0(x), \quad \forall x \in \widetilde{\Omega}, \\ b_{01}(x) &= b_{10}(x) = -g(x')\widetilde{a}_0(x) - \alpha(x')\widetilde{a}_1(x), \quad \forall x \in \widetilde{\Omega}, \\ b_{11}(x) &= h(x') - 2g(x')\widetilde{a}_1(x), \quad \forall x \in \widetilde{\Omega}, \end{aligned}$$

where $\widetilde{a}_0$ and $\widetilde{a}_1$ are the imaginary parts of $a_0$ and $a_1$, respectively.

We suppose

(i) $g, h \in C^\infty(\Omega_0')$, $h(x') \in [h_1, h_2]$, $\forall x' \in \Omega_0'$, with $0 < h_1 < h_2 < 4$;
(ii) the matrix-function $B$ is positive definite in $\Gamma'$, namely there exists $k > 0$ such that $B(x')\eta \cdot \eta \geq k\|\eta\|^2$, $\forall \eta = (\eta_1, \eta_2) \neq (0, 0)$, $\forall x \in \Gamma'$;
(iii) $g(x')n_1|_S \geq 0$, for every $x' \in \Omega_0' \cap \partial\Omega_0$.

We remark that if $\widetilde{a}_0 = \widetilde{a}_1 = 0$, on $\Gamma'$, assumption (ii) is verified.

The main goal of the paper is to prove the following results:

**Theorem 1** *Let (i), (ii) and (iii) be satisfied. If* $f \in L^2_{loc}(\overline{\Omega})$, *there exists* $w \in L^2_{loc}(\overline{\Omega})$ *such that*

$$(w, {}^tPu) = (f, u), \quad \forall u \in C_0^\infty(\overline{\Omega}) : \; u|_S = 0,$$

*where* $\overline{\Omega} = [0, +\infty[\times \overline{\Omega}_0$.

**Theorem 2** *Let (i), (ii) and (iii) be satisfied. Let* $f \in H^r_{loc}(\overline{\Omega})$, *with* $r \geq 2$, *the Cauchy–Dirichlet problem*

$$\begin{cases} Pu = f, & \text{in } \Omega = ]0, +\infty[\times \Omega_0 \\ u|_{\partial\Omega} = 0, \; \frac{du}{dn}|_{\Omega_0} = 0, \; u|_S = 0 \end{cases}$$

*admits a solution* $u \in H^r_{loc}(\overline{\Omega} \setminus \partial\Omega_0)$.

Let us consider some operators which satisfy assumptions (i), (ii) and (iii) and for which we have a transition.

***Example 1*** Let $\alpha(x') = x_1^3 e^{kx_2}$ be functions in an open set $\widetilde{\Omega}_0$ of $\mathbb{R}^2$ contained (0, 0). Let $P = D^{(2)}_{x_0} - D^{(2)}_{x_1} - (x_0 - \alpha(x'))^2 D^{(2)}_{x_2} - ia_0 D_{x_0}$, where $a_0 > 0$. It results $g(x') = \frac{1}{3} x_1$ and $h(x') = \frac{2}{3}$, then assumption (i) is verified for every $\widetilde{\Omega}_0$. Assumption (ii) is satisfied for every $\widetilde{\Omega}_0 \subseteq ] - \infty, \frac{2}{a_0}] \times \mathbb{R}$. Moreover, assumption (iii) is fulfilled if $n_1$ on $\partial\Omega_0 \cap \Omega'_0$ is positive (for example if $\widetilde{\Omega}_0$ is a circle of center in (0, 0)). Then, we can choose $\widetilde{\Omega}_0$ such that $|\partial_{x_1} \alpha(x')|$ admits values either less than or equal than or greater than 1. As a consequence, it follows $\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$, with $\Sigma_-$, $\Sigma_0$ and $\Sigma_+$ nonempty.

***Example 2*** Now, let us consider $\alpha(x') = (ax_1 + bx_2 + c)^2$, with $a, b, c \in \mathbb{R}$, $a, b \neq 0$, in an open set $\widetilde{\Omega}_0$ of $\mathbb{R}^2$ contained (0, 0). Let $P = D^{(2)}_{x_0} - D^{(2)}_{x_1} - (x_0 - \alpha(x'))^2 D^{(2)}_{x_2} + a_0 D_{x_0} - ia_1(x)(x_0 - \alpha(x'))(D_{x_1} + D_{x_2})$, where $a_0 \in \mathbb{R}$ and $a_1 \in C^\infty$. It results $g(x') = \dfrac{ax_1 + bx_2 + c}{2a}$ and $h(x') = \dfrac{1}{2}$. Hence, assumption (i) is always verified. Moreover, we can choose $\widetilde{\Omega}_0$ such that assumption (iii) is fulfilled and both $|\partial_{x_1} \alpha(x')| \leq 1$ and $|\partial_{x_1} \alpha(x')| \geq 1$ hold. Therefore, the existence of a solution is ensured in presence of transition.

The paper is organized as follows. In Sect. 2, some preliminary notations are recalled. In Sect. 3, a priori estimates obtained in [7] are referred. Section 4 is devoted to prove a priori estimates under the assumption $|\partial_{x_1} \alpha(x')| \leq 1$. Instead, Sect. 5 concerns estimates under the assumption $|\partial_{x_1} \alpha(x')| \geq 1$. In Sect. 6, conclusive estimates in $L^2$ are proved. In Sect. 7, estimates in Sobolev spaces with $s < 0$ are established making use of the pseudodifferential operator theory. Section 8 concerns the study of some global estimates. Finally, Sects. 9 and 10 deal with the proofs of Theorems 1 and 2, respectively.

## 2 Notations and preliminaries

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$. We indicate the derivative of order $|\alpha|$ by $\partial^\alpha$, the derivative of order $h$ with respect to $x_j$ by $\partial^h_{x_j}$ and the derivative of order $h$ with respect to $x_j$ and $x_p$ by $\partial^h_{x_j,x_p}$.

We indicate the $L^2$-scalar product, the $L^2$-norm and the $H^r$-norm by $(\cdot,\cdot)$, $\|\cdot\|$ and $\|\cdot\|_{H^r}$ ($r \in \mathbb{N}_0$), respectively. We indicate the external normal versor to the boundary $\partial\Omega$ by $n = (n_0, n_1, n_2)$.

Let $C_0^\infty(\overline{\Omega})$ be the space of restrictions of functions belonging to $C_0^\infty(\mathbb{R}^3)$ on $\overline{\Omega}$. For each $K \subseteq \overline{\Omega}$ compact set, let $C_0^\infty(K)$ be the set of functions $\varphi \in C_0^\infty(\overline{\Omega})$ having support contained in $K$. Set $\Omega_k = [0,k[\times\Omega_0$, let us introduce

$$C_0^\infty(\overline{\Omega}_k) = \left\{ u \in C_0^\infty(\overline{\Omega}) \,:\, \operatorname{supp} u \subseteq [0,k[\times\overline{\Omega}_0 \right\}.$$

Moreover, let $C_0^{*\infty}(\overline{\Omega})$ be the space of functions $u \in C_0^\infty(\overline{\Omega})$ such that $\gamma_1 \partial_{x_0} u(0,x') = \gamma_2 u(0,x')$, where $\gamma_1, \gamma_2 \in \mathbb{R}$. Consequently, we can introduce $C^{*\infty}(\overline{\widetilde{\Omega}})$ and $C^{*\infty}(\overline{\Omega}_k)$. It is worth to remark that if $u \in C_0^\infty(\Omega)$, then $u \in C_0^{*\infty}(\overline{\Omega}_0)$. Furthermore, if $u(x_0,x') = u_1(x')u_2(x_0)$, with $u_1 \in C_0^\infty(\overline{\Omega}_0)$, $u|_{\partial\Omega_0} = 0$ and $u_2 \in C_0^\infty([0,k[)$ then $u \in C_0^{*\infty}(\overline{\Omega}_k)$.

Let $S(\mathbb{R}^3)$ be the space of rapidly decreasing functions. Let $S(\overline{\Omega})$ be the space of restrictions of functions belonging to $S(\mathbb{R}^3)$ on $\overline{\Omega}$.

Let $\Omega = ]0,+\infty[\times\Omega_0$ and let $s \in \mathbb{R}$, the norm in $H^{0,s}$ is given by

$$\|u\|^2_{H^{0,s}} = \frac{1}{(2\pi)^2} \int_0^{+\infty} \mathrm{d}x_0 \int_{\mathbb{R}^2} (1 + |\xi'|^2)^s |\hat{u}(x_0,\xi')|^2 \mathrm{d}\xi',$$
$$\forall u \in C_0^\infty(\overline{\Omega}) \,:\, \operatorname{supp} u \subseteq [0,+\infty[\times\Omega_0,$$

where the Fourier transform is done only with respect to the variable $x'$. Let $A_s : C_0^\infty(\Omega) \to C^\infty(\Omega)$ be the pseudodifferential operator defined by

$$A_s u = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix'\cdot\xi'} (1 + |\xi'|^2)^{\frac{s}{2}} \hat{u}(x_0,\xi') \mathrm{d}\xi',$$
$$\forall u \in C_0^\infty(\overline{\Omega}) \,:\, \operatorname{supp} u \subseteq [0,+\infty[\times\Omega_0.$$

For every $\varphi(x') \in C_0^\infty(\Omega_0)$, the operator $\varphi A_s u$ extends as a linear continuous operator from $H^{0,r}_{comp}(\Omega)$ into $H^{0,r-s}_{loc}(\Omega)$, where $r, s \in \mathbb{R}$. In particular, in $\Omega_k = [0,k[\times\Omega_0$, for $k > 0$, let $\mathring{H}^{0,s}(\Omega_k)$ be the space of $u \in H^{0,s}(\Omega_k)$ such that $\operatorname{supp} u \subseteq \Omega_k$. Moreover, if $\operatorname{supp} \varphi \subseteq \Omega_0 \setminus \operatorname{supp} u$, then $\varphi A_s u$ is a regularizing operator with respect to the variable $x'$. It results

$$\|\varphi A_s u\|_{H^{0,r}} \leq c\|u\|_{H^{0,r'}}, \quad \forall r, r' \in \mathbb{R},\ u \in C^\infty(\overline{\Omega}) \,:\, \operatorname{supp} u \subseteq [0,+\infty[\times\Omega_0.$$

The norms $\|u\|_{H^{0,s}(\Omega)}$ and $\|A_s u\|_{L^2(\Omega)}$ are equivalent for any $s \in \mathbb{R}$.

Let $s \in \mathbb{R}$ and $p \geq 0$. Let $H^{p,s}(\mathbb{R}^3)$ be the space of all the distributions on $\mathbb{R}^3$ such that

$$\|u\|^2_{H^{p,s}(\mathbb{R}^3)} = \frac{1}{(2\pi)^2} \sum_{|h|\leq p} \int_{\mathbb{R}^3} (1 + |\xi'|^2)^s |\partial^h_{x_0} \hat{u}(x_0,\xi')|^2 \mathrm{d}x_0 \mathrm{d}\xi' < +\infty.$$

Let $H^{p,s}(\Omega)$ be the space of restrictions of elements of $H^{p,s}(\mathbb{R}^3)$ on $\Omega$ endowed with the norm

$$\|u\|_{H^{p,s}(\Omega)} = \inf_{\substack{U \in H^{p,s}(\mathbb{R}^3) \\ U|_\Omega = u}} \|U\|_{H^{p,s}(\mathbb{R}^3)}.$$

In the same way, the space $H^{p,s}(\Omega_k)$ can be introduced.

At last, we consider the transposed operator of the operator $P$:

$$\begin{aligned} {}^tP = & -\partial^2_{x_0} + \partial^2_{x_1} + (x_0 - \alpha(x'))^2 \partial^2_{x_2} - 4(x_0 - \alpha(x'))(\partial_{x_2}\alpha)\partial_{x_2} \\ & - \frac{1}{i}\sum_{j=0}^{2} a_j(x)\partial_{x_j} - \frac{1}{i}\sum_{j=0}^{2} \partial_{x_j} a_j(x) - 2(\partial_{x_2}\alpha)^2 + b(x). \end{aligned}$$

## 3 Some known preliminary results

First of all, we recall a priori estimate for the solution to the problem (2) (see [2], Lemma 3.1).

**Lemma 1** *Let $u \in S(\overline{\Omega})$ and let $p, \alpha_0, \alpha_1, \alpha_2 \in \mathbb{N}_0$. Then*

$$\|x_0^{\frac{p}{2}} \partial^{\alpha_0,\alpha_1,\alpha_2} u\| \le \frac{2}{p+1} \|x_0^{\frac{p+2}{2}} \partial^{\alpha_0+1,\alpha_1,\alpha_2} u\|. \tag{3}$$

Moreover, we have the following preliminary result (see [7], Lemma 3.2).

**Lemma 2** *Let $u \in S(\overline{\Omega})$, it results*

$$\int_{\Omega_0} |u(0,x')|^2 \mathrm{d}x' \le 4\|x_0 \partial_{x_0} u\| \|\partial_{x_0} u\|.$$

The next result holds (see [7], Lemma 3.3).

**Lemma 3** *For every $\varepsilon, \delta > 0$ there exists $k > 0$ such that, if*

$$I_{k,\delta} = \left\{ x \in \overline{\Omega} : \; x_0 < k, \; |x_0 - \alpha(x')| > \delta \right\},$$

it results

$$\begin{gathered} \|\partial_{x_0} u\| + \|\partial_{x_1} u\| + \|(x_0 - \alpha(x'))\partial_{x_2} u\| + \|u\| \le \varepsilon \| \, {}^tPu\|, \\ \forall u \in C_0^\infty(\overline{\Omega}) : \; \operatorname{supp} u \subseteq I_{k,\delta}, \; u|_S = 0. \end{gathered} \tag{4}$$

We present a priori estimate (see [7], Theorem 3.4).

**Theorem 3** *Let (i) and (iii) be satisfied. Then, there exist $k > 0$ and $c > 0$ such that*

$$\begin{aligned}&\|\partial_{x_0}u\|+\|\partial_{x_1}u\|+\|(x_0-\alpha(x'))\partial_{x_2}u\|+\|u\|\le c\|\,{}^tPu\|,\\&\qquad\forall u\in C_0^{*\infty}(\overline{\Omega}_k)\,:\;\;u|_S=0.\end{aligned}\tag{5}$$

Moreover, we recall the following result (see [7], Theorem 3.5).

**Theorem 4** *Let (i) and (iii) be satisfied. For every $\varepsilon>0$ there exist $k>0$ and a neighborhood $I_{x'}$ in $\Omega_0\cap\Gamma$ such that*

$$\begin{aligned}&\|\partial_{x_0}u\|+\|\partial_{x_1}u\|+\|(x_0-\alpha(x'))\partial_{x_2}u\|+\|u\|\le \varepsilon\|\,{}^tPu\|,\\&\qquad\forall u\in C_0^{*\infty}(\Omega_k)\,:\;\;\operatorname{supp}u\subseteq[0,k[\times I_{x'},\;u|_S=0.\end{aligned}$$

Let $\overline{x}_0>0$ and let $k>0$, we denote by $\Omega_{\overline{x}_0,k}=]\overline{x}_0,\overline{x}_0+k[\times\overline{\Omega}_0$. Let us show the following preliminary result (see [7], Lemma 4.1).

**Lemma 4** *Let $u\in S(\Omega)$ such that $\partial_{x_0}u|_{\Omega_0}=0$, let $p,\alpha_0,\alpha_1,\alpha_2\in\mathbb{N}$ and $\overline{x}_0>0$. It results*

$$\|(x_0-\overline{x}_0)^{\frac{p}{2}}\partial^{\alpha_0,\alpha_1,\alpha_2}u\|\le\frac{2}{p+1}\|(x_0-\overline{x}_0)^{\frac{p+2}{2}}\partial^{\alpha_0+1,\alpha_1,\alpha_2}u\|.$$

We consider another preliminary lemma (see [7], Lemma 4.2).

**Lemma 5** *For every $\varepsilon,\delta>0$ and $\overline{x}_0>0$, there exists $k>0$ such that, setting*

$$I_{k,\delta}=\left\{x\in\overline{\Omega}\,:\;x_0\in]\overline{x}_0,\overline{x}_0+k[,\;|x_0-\alpha(x')|>\delta\right\},$$

it results

$$\begin{aligned}&\|\partial_{x_0}u\|+\|\partial_{x_1}u\|+\|(x_0-\alpha(x'))\partial_{x_2}u\|+\|u\|\le \varepsilon\|\,{}^tPu\|,\\&\qquad\forall u\in C_0^{\infty}(\overline{\Omega})\,:\;\;\operatorname{supp}u\subseteq I_{k,\delta},\;u|_S=0.\end{aligned}$$

We recall the following preliminary result (see [7], Lemma 4.3).

**Lemma 6** *Let (i), (ii) and (iii) be satisfied. Let $\overline{x}_0>0$, for every $\varepsilon>0$ there exists $k,\delta>0$ such that, setting*

$$J_{k,\delta}=\left\{x\in\overline{\Omega}\,:\;x_0\in]\overline{x}_0,\overline{x}_0+k[,\;|x_0-\alpha(x')|<\delta\right\},$$

*it results*

$$\begin{aligned}&\|(x_0-\alpha(x'))\partial_{x_2}u\|\le\varepsilon\big(\|\,{}^tPu\|+\|\partial_{x_0}u\|+\|\partial_{x_1}u\|+\|u\|\big),\\&\qquad\forall u\in C_0^{\infty}(\overline{\Omega})\,:\;\;\operatorname{supp}u\subseteq J_{k,\delta},\;u|_S=0.\end{aligned}$$

At last, we present the following result (see [7], Theorem 4.4).

**Theorem 5** *Let (i), (ii) and (iii) be satisfied. Let $\overline{x}_0>0$. There exist $k>0$ and $c>0$ such that*

$$\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0-\alpha(x'))\partial_{x_2}u\| + \|u\| \le c\|{}^tPu\|,$$
$$\forall u \in C_0^\infty(\Omega_{\bar{x}_0,k}) : \ u|_S = 0.$$

## 4 Estimates under the assumption $|\partial_{x_1}\alpha(x')| \le 1$

Let $\bar{x}_0 \ge 0$, let us denote by

$$J_{k,\delta,\bar{x}_0} = \left\{ x \in \overline{\Omega} : \ x_0 \in [\bar{x}_0, \bar{x}_0 + k[, \ |x_0 - \alpha(x')| < \delta \right\}.$$

The following result holds.

**Theorem 6** *Let (i), (ii) and (iii) be satisfied. Let us assume that there exist two positive numbers $k'$ and $\delta$ such that $|\partial_{x_1}\alpha(x')| \le 1$ on $\Omega_0 \cap J_{k',\delta,0}$. Then, for every $\varepsilon > 0$ there exists $0 < k \le k'$ such that*

$$\begin{aligned}&\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0-\alpha(x'))\partial_{x_2}u\| + \|u\| \le \varepsilon\|{}^tPu\|,\\ &\forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_k = [0,k[\times\overline{\Omega}_0, \ u|_S = 0.\end{aligned} \tag{6}$$

***Proof*** Let us consider the following inner products

$$({}^tPu, x_0\partial_{x_0}u) + (x_0\partial_{x_0}u, {}^tPu).$$

By means of integrations by parts, for every $u \in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp} u \subseteq \Omega_{k'} \cap J_{k',\delta,0}$ and $u|_S = 0$, we have

$$\begin{aligned}&\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\ &\quad + 2(x_0(x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_2}u) - 4(x_0(x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_0}u)\\ &= ({}^tPu, x_0\partial_{x_0}u) + (x_0\partial_{x_0}u, {}^tPu) - ({}^t(P-P_2)u, x_0\partial_{x_0}u) - (x_0\partial_{x_0}u, {}^t(P-P_2)u).\end{aligned}$$

From which it follows

$$\begin{aligned}&\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \frac{1}{2}\|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\ &\quad + \left((x_0-\alpha(x'))\left(\frac{5}{2}x_0 - \frac{1}{2}\alpha(x')\right)\partial_{x_2}u, \partial_{x_2}u\right)\\ &\quad - 4(x_0(x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_0}u)\\ &= ({}^tPu, x_0\partial_{x_0}u) + (x_0\partial_{x_0}u, {}^tPu) - ({}^t(P-P_2)u, x_0\partial_{x_0}u)\\ &\quad - (x_0\partial_{x_0}u, {}^t(P-P_2)u).\end{aligned} \tag{7}$$

We denote by

$$\Omega_{k',\frac{1}{5}} = \left\{ x \in \Omega_{k'} : \ \frac{1}{5}\alpha(x') \le x_0 \le \alpha(x') \right\}.$$

Since $(x_0-\alpha(x'))\left(\frac{5}{2}x_0 - \frac{1}{2}\alpha(x')\right) > 0$, in $\Omega_{k'} \setminus \Omega_{k',\frac{1}{5}}$, by (7) one has

$$\begin{aligned}
&\|\partial_{x_0}u\|^2+\|\partial_{x_1}u\|^2+\frac{1}{2}\|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\
&\quad\le -\Big((x_0-\alpha(x'))\Big(\frac{5}{2}x_0-\frac{1}{2}\alpha(x')\Big)\partial_{x_2}u,\partial_{x_2}u\Big)_{\Omega_{k',\frac{1}{5}}}\\
&\qquad+4|(x_0(x_0-\alpha(x'))\partial_{x_2}u,\partial_{x_0}u)|\\
&\qquad+2\|x_0\,{}^tPu\|\|\partial_{x_0}u\|+2\|x_0\,{}^t(P-P_2)u\|\|\partial_{x_0}u\|,
\end{aligned}\tag{8}$$

where we denoted by $(\cdot,\cdot)_{\Omega_{k',\frac{1}{5}}}$ the inner product on $\Omega_{k',\frac{1}{5}}$. Furthermore, it results

$$\begin{aligned}
&\|\partial_{x_0}u\|^2+\|\partial_{x_1}u\|^2+\|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\
&\quad\le -2\big(x_0(x_0-\alpha(x'))\partial_{x_2}u,\partial_{x_2}u\big)_{\Omega_{k',\frac{1}{5}}}+4\|x_0(x_0-\alpha(x'))\partial_{x_2}u\|\|\partial_{x_0}u\|\\
&\qquad+2\|x_0\,{}^tPu\|\|\partial_{x_0}u\|+2\|x_0\,{}^t(P-P_2)u\|\|\partial_{x_0}u\|\\
&\quad\le 2\big(\alpha(x')(x_0-\alpha(x'))\partial_{x_2}u,\partial_{x_2}u\big)_{\Omega_{k',\frac{1}{5}}}+4\|x_0(x_0-\alpha(x'))\partial_{x_2}u\|\|\partial_{x_0}u\|\\
&\qquad+2\|x_0\,{}^tPu\|\|\partial_{x_0}u\|+2\|x_0\,{}^t(P-P_2)u\|\|\partial_{x_0}u\|.
\end{aligned}$$

In $\Omega_{k',\frac{1}{5}}$, we consider the following inner products

$$(\partial_{x_0}u,\,{}^tPu)+({}^tPu,\partial_{x_0}u).$$

If $u$ is identically zero on $\Gamma_\eta$, where $\Gamma_\eta$ is the surface $x_0=\eta\alpha(x')$, with $0<\eta\le\frac{1}{5}$, integrating by parts, we have

$$\begin{aligned}
&2\|(\alpha(x')-x_0)^{\frac{1}{2}}\partial_{x_2}u\|^2-4((x_0-\alpha(x'))\alpha(x')\partial_{x_2}u,\partial_{x_0}u)\\
&\quad+\int_\Gamma\big[(\partial_{x_0}u)^2+2\partial_{x_1}\alpha(x')\partial_{x_0}u\,\partial_{x_1}u+(\partial_{x_1}u)^2\big]\mathrm{d}\sigma\\
&\quad=({}^tPu,\partial_{x_0}u)+(\partial_{x_0}u,\,{}^tPu)-({}^t(P-P_2)u,\partial_{x_0}u)-(\partial_{x_0}u,\,{}^t(P-P_2)u).
\end{aligned}\tag{9}$$

By (8) and (9), if $|\partial_{x_1}\alpha(x')|\le 1$, on $\Omega_0\cap J_{k',\delta,0}$, and $k'$ is small enough, the claim follows assuming that $u$ is identically zero on $\Gamma_\eta$.

Let $u\in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp}u\subseteq\Omega_{k'}$ and $u|_S=0$. Let $\chi$ be a function of class $C^\infty$ such that $\chi(t)=1$, for $t\ge\eta$, and $\chi(t)=0$, for $0\le t\le\frac{\eta}{2}$. Rewriting (9) for $u\chi\Big(\frac{x_0}{\alpha(x')}\Big)$ and adding (8), there exists $0<k\le k'$ such that

$$\begin{gathered}
\|\partial_{x_0}u\|+\|\partial_{x_1}u\|+\|(x_0-\alpha(x'))\partial_{x_2}u\|+\|u\|\le\varepsilon\|\,{}^tPu\|,\\
\forall u\in C_0^\infty(\overline{\Omega}):\ \operatorname{supp}u\subseteq\Omega_k\cap J_{k,\delta,0},\ u|_S=0.
\end{gathered}$$

Making use of the previous inequality and Lemma 3 with $k$ small enough, the claim is achieved. □

We set

$$\Omega_{\overline{x}_0,k}=[\overline{x}_0,\overline{x}_0+k[\times\overline{\Omega}_0,$$

with $\overline{x}_0>0$ and $k>0$, and we prove the following result.

**Theorem 7** *Let (i), (ii) and (iii) be satisfied. Let us assume that there exist two positive numbers* $k'$ *an* $\delta$ *such that* $|\partial_{x_1}\alpha(x')| \leq 1$, *on* $\Omega_{\overline{x}_0} \cap J_{k',\delta,\overline{x}_0}$, *where* $\Omega_{\overline{x}_0}$ *is the part of the plane* $x_0 = \overline{x}_0$ *in* $\Omega_{\overline{x}_0,k}$. *Then, for every* $\varepsilon > 0$ *there exists* $0 < k \leq k'$ *such that*

$$\begin{aligned}&\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0-\alpha(x'))\partial_{x_2}u\| + \|u\| \leq \varepsilon\|{}^tPu\|,\\ &\forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k}, \ u|_S = 0.\end{aligned} \tag{10}$$

***Proof*** Let $u \in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k'} \cap J_{k',\delta,\overline{x}_0}$ and $u|_S = 0$, integrating by parts in the following inner products

$$({}^tPu, (x_0-\overline{x}_0)\partial_{x_0}u) + ((x_0-\overline{x}_0)\partial_{x_0}u, {}^tPu),$$

we obtain

$$\begin{aligned}&\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\ &\quad + 2((x_0-\overline{x}_0)(x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_2}u) + 4(x_0(x_0-\alpha(x'))\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u)\\ &\quad = ({}^tPu, (x_0-\overline{x}_0)\partial_{x_0}u) + ((x_0-\overline{x}_0)\partial_{x_0}u, {}^tPu) - ({}^t(P-P_2)u, (x_0-\overline{x}_0)\partial_{x_0}u)\\ &\quad - ((x_0-\overline{x}_0)\partial_{x_0}u, {}^t(P-P_2)u), \quad \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k}, \ u|_S = 0.\end{aligned}$$

Taking into account that $\frac{1}{2}\|(x_0-\alpha(x'))\partial_{x_2}u\|^2 + 2((x_0-\overline{x}_0)(x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_2}u) < 0$, if $x_0 \leq \frac{1}{5}\alpha(x') + \frac{4}{5}\overline{x}_0$ or $x_0 \geq \alpha(x')$, it results

$$\begin{aligned}&\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \frac{1}{2}\|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\ &\quad \leq -\frac{5}{2}\Big((x_0-\alpha(x'))\Big(x_0 - \frac{1}{5}\alpha(x') - \frac{4}{5}\overline{x}_0\Big)\partial_{x_2}u, \partial_{x_2}u\Big)_{\Omega_{k,h,\eta}}\\ &\quad + ({}^tPu, (x_0-\overline{x}_0)\partial_{x_0}u) + ((x_0-\overline{x}_0)\partial_{x_0}u, {}^tPu) - ({}^t(P-P_2)u, (x_0-\overline{x}_0)\partial_{x_0}u)\\ &\quad - ((x_0-\overline{x}_0)\partial_{x_0}u, {}^t(P-P_2)u),\end{aligned} \tag{11}$$

where $\Omega_{\overline{x}_0,k',\eta} = \big\{x \in \Omega_{\overline{x}_0,k'} : \ \eta\alpha(x') + (1-\eta)\overline{x}_0 \leq x_0 \leq \alpha(x')\big\}$, with $0 < \eta \leq \frac{1}{5}$.

In $\Omega_{\overline{x}_0,k',\eta}$, we consider the following inner products

$$({}^tPu, \partial_{x_0}u) + (\partial_{x_0}u, {}^tPu).$$

Proceeding as done above, we obtain

$$\begin{aligned}&2((x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_2}u) - 4((x_0-\alpha(x'))\partial_{x_2}u, \partial_{x_0}u)\\ &\quad = -\int_\Gamma \big[(\partial_{x_0}u)^2 + 2\partial_{x_1}\alpha(x')\partial_{x_0}u\,\partial_{x_1}u + (\partial_{x_1}u)^2\big]\mathrm{d}\sigma\\ &\quad + \int_{\Gamma_{\eta,(1-\eta)\overline{x}_0}} \big[(\partial_{x_0}u)^2 + 2\partial_{x_1}\alpha(x')\partial_{x_0}u\,\partial_{x_1}u + (\partial_{x_1}u)^2\big]\mathrm{d}\sigma\\ &\quad + ({}^tPu, \partial_{x_0}u) + (\partial_{x_0}u, {}^tPu) - ({}^t(P-P_2)u, \partial_{x_0}u) - (\partial_{x_0}u, {}^t(P-P_2)u),\end{aligned} \tag{12}$$

where $\Gamma_{\eta,(1-\eta)\overline{x}_0}$ is the surface $x_0 = \eta\alpha(x') + (1-\eta)\overline{x}_0$, with $0 < \eta \leq \frac{1}{5}$. Making use of (11) and (12), we deduce the claim assuming that the gradient of $u$ with respect to $x_0$ and $x_1$ is zero on $\Gamma_{\eta,(1-\eta)\overline{x}_0}$.

Let $u$ be a function belonging to $C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp} u \subseteq \Omega_{\bar{x}_0}$ and $u|_S = 0$. Let $\chi$ be a function of class $C^\infty$ such that $\chi(t) = 1$, if $|t| \geq \eta$, and $\chi(t) = 0$, if $|t| < \frac{\eta}{2}$. Rewriting (12) for $u\chi\left(\dfrac{x_0}{\alpha(x')}\right)$ and adding (11), there exists $0 < k \leq k'$ such that

$$\|\partial_{x_0} u\| + \|\partial_{x_1} u\| + \|(x_0 - \alpha(x'))\partial_{x_2} u\| + \|u\| \leq \varepsilon \|{}^t P u\|,$$
$$\forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_{\bar{x}_0,k} \cap J_{k,\delta,\bar{x}_0}, \ u|_S = 0.$$

Finally, the claim follows from the previous inequality and by using Lemma 5 for $k$ small enough. □

## 5 Estimates under the assumption $|\partial_{x_1}\alpha(x')| \geq 1$

For every $\bar{x}_0 \geq 0$, we set

$$\Gamma_{\bar{x}_0} = \{x \in \Gamma : \ \alpha(x') = \bar{x}_0\}.$$

The next result holds.

**Theorem 8** *Let (i), (ii) and (iii) be satisfied. Let us assume that* $|\partial_{x_1}\alpha(x')| \geq 1$, *on* $\Gamma_{\bar{x}_0}$. *Then, there exist* $k > 0$ and $c > 0$ *such that*

$$\begin{aligned} &\|\partial_{x_0} u\| + \|\partial_{x_1} u\| + \|(x_0 - \alpha(x'))\partial_{x_2} u\| + \|u\| \leq c\|{}^t P u\|, \\ &\forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_{\bar{x}_0,k} = [\bar{x}_0, \bar{x}_0 + k[\times\Omega_0, \ u|_S = 0, \end{aligned} \tag{13}$$

Moreover, for every $\varepsilon > 0$ there exists $k > 0$ such that

$$\begin{aligned} &\|(x_0 - \alpha(x'))\partial_{x_2} u\| \leq \varepsilon\big(\|{}^t P u\| + \|\partial_{x_0} u\| + \|\partial_{x_1} u\| + \|u\|\big), \\ &\forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_{\bar{x}_0,k}, \ u|_S = 0. \end{aligned} \tag{14}$$

***Proof*** Let $d > 0$ and let us set

$$A_d = (x_0 + d)\partial_{x_0} + g_d(x')\partial_{x_1},$$

where $g_d(x') = \dfrac{\alpha(x') + d}{\partial_{x_1}\alpha(x')}$, and consider the sum of the inner products

$$\begin{aligned} ({}^t P u, A_d u) + (A_d u, {}^t P u) =& ({}^t P_2 u, A_d u) + (A_d u, {}^t P_2 u) + ({}^t P_1 u, A_d u) + (A_d u, {}^t P_1 u) \\ &+ ({}^t P_0 u, A_d u) + (A_d u, {}^t P_0 u). \end{aligned}$$

For every $u \in C_0^\infty(\overline{\Omega})$ such that $u|_S = 0$, it results:

$$\begin{aligned} &({}^t P_2 u, A_d u) + (A_d u, {}^t P_2 u) \\ &\quad = ({}^t P_2 u, (x_0 + d)\partial_{x_0} u) + ((x_0 + d)\partial_{x_0} u, {}^t P_2 u) \\ &\quad\quad + ({}^t P_2 u, g_d(x')\partial_{x_1} u) + (g_d(x')\partial_{x_1} u, {}^t P_2 u). \end{aligned} \tag{15}$$

Let us integrate by parts in the first inner products of the principal part in (15)

$$
\begin{aligned}
&2({}^tP_2u,(x_0+d)\partial_{x_0}u)\\
&\quad=(\partial_{x_0}u,\partial_{x_0}u)+(\partial_{x_1}u,\partial_{x_1}u)+\big((x_0-\alpha(x'))^2\partial_{x_2}u,\partial_{x_2}u\big)\\
&\qquad+2\big((x_0-\alpha(x'))(x_0+d)\partial_{x_2}u,\partial_{x_2}u\big)\\
&\qquad+4\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')\partial_{x_2}u,(x_0+d)\partial_{x_0}u\big)\\
&\qquad+\int_{\Omega_0}(\bar{x}_0+d)\big[(\partial_{x_0}u)^2+(\partial_{x_1}u)^2+(x_0-\alpha(x'))^2(\partial_{x_2}u)^2\big]\mathrm{d}x'.
\end{aligned}
\tag{16}
$$

Moreover, integrating by parts in the second inner products in (15), we have

$$
\begin{aligned}
&2({}^tP_2u,g_d(x')\partial_{x_1}u)\\
&\quad=-(\partial_{x_0}u,\partial_{x_1}g_d(x')\partial_{x_0}u)\\
&\qquad+4((x_0-\alpha(x'))\partial_{x_2}\alpha(x')g_d(x')\partial_{x_2}u,\partial_{x_1}u)\\
&\qquad-2\big((x_0-\alpha(x')^2\partial_{x_2}g_d(x')\partial_{x_2}u,\partial_{x_1}u\big)\\
&\qquad+\big((x_0-\alpha(x'))^2\partial_{x_1}g_d(x')\partial_{x_2}u,\partial_{x_2}u\big)\\
&\qquad-2\big((x_0-\alpha(x'))\partial_{x_1}\alpha(x')g_d(x')\partial_{x_2}u,\partial_{x_2}u\big)\\
&\qquad+\int_S n_1g_d(x')(\partial_{x_0}u)^2d\sigma+\int_S n_1g_d(x')(\partial_{x_1}u)^2\mathrm{d}\sigma\\
&\qquad+2\int_S n_2(x_0-\alpha(x'))^2g_d(x')\partial_{x_0}u\partial_{x_1}u\mathrm{d}\sigma\\
&\qquad-\int_S n_1(x_0-\alpha(x'))^2g_d(x')(\partial_{x_2}u)^2\mathrm{d}\sigma\\
&\qquad+\int_{\Omega_0}2g_d(x')\partial_{x_0}u\partial_{x_1}u\mathrm{d}x'.
\end{aligned}
\tag{17}
$$

Since $u|_S=0$, it results

$$
\int_S n_1g_d(x')(\partial_{x_0}u)^2\mathrm{d}\sigma=0. \tag{18}
$$

Making use of the assumption (iii), it follows

$$
\int_S n_1g_d(x')(\partial_{x_1}u)^2\mathrm{d}\sigma\geq 0. \tag{19}
$$

Denoting the tangential derivative of $u$ along the section of $S$ of the equal height by $\dfrac{\partial u}{\partial\tau}$, we obtain

$$
\begin{aligned}
&2\int_S n_2(x_0-\alpha(x'))^2g_d(x')\partial_{x_2}u\partial_{x_1}ud\sigma-\int_S n_1(x_0-\alpha(x'))^2g_d(x')(\partial_{x_2}u)^2\mathrm{d}\sigma\\
&\quad=-2\int_S(x_0-\alpha(x'))\Big(\frac{\partial u}{\partial\tau}\Big)g_d(x')\partial_{x_2}ud\sigma+\int_S n_1(x_0-\alpha(x'))^2g_d(x')(\partial_{x_2}u)^2\mathrm{d}\sigma\\
&\quad=\int_S n_1(x_0-\alpha(x'))^2g_d(x')(\partial_{x_2}u)^2\mathrm{d}\sigma\geq 0,
\end{aligned}
\tag{20}
$$

where we took into account that $\dfrac{\partial u}{\partial\tau}=0$, since $u=0$ on $S$.

Adding (16) and (17) and making use of (18), (19) and (20), we have

$$
\begin{aligned}
&2({}^tP_2u, A_du) \\
&\geq \|h^{\frac{1}{2}}(x')\partial_{x_0}u\|^2 + \|h^{\frac{1}{2}}(x')\partial_{x_1}u\|^2 + \|(4-h(x'))^{\frac{1}{2}}\big(x_0-\alpha(x')\big)\partial_{x_2}u\|^2 \\
&\quad + 4\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')(x_0+d)\partial_{x_2}u, \partial_{x_0}u\big) \\
&\quad + 4\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')g_d(x')\partial_{x_2}u, \partial_{x_1}u\big) \\
&\quad - 2\big((x_0-\alpha(x'))^2\partial_{x_2}g_d(x')\partial_{x_2}u, \partial_{x_1}u\big) \\
&\quad + \int_{\Omega_0}\big\{(\bar{x}_0+d)\big[(\partial_{x_0}u)^2 + (\partial_{x_1}u)^2 + (x_0-\alpha(x'))^2(\partial_{x_2}u)^2\big] + 2g_d(x')\partial_{x_0}u\partial_{x_1}u\big\}\mathrm{d}x' \\
&= \|h^{\frac{1}{2}}(x')\partial_{x_0}u\|^2 + \|h^{\frac{1}{2}}(x')\partial_{x_1}u\|^2 + \|(4-h(x'))^{\frac{1}{2}}\big(x_0-\alpha(x')\big)\partial_{x_2}u\|^2 \\
&\quad + 4\big((x_0-\alpha(x'))^2\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big) \\
&\quad + 4\big((x_0-\alpha(x'))\alpha(x')\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big) \\
&\quad + 4\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')g_d(x')\partial_{x_2}u, \partial_{x_1}u\big) \\
&\quad - 2\big((x_0-\alpha(x'))^2\partial_{x_2}g_d(x')\partial_{x_2}u, \partial_{x_1}u\big) \\
&\quad + \int_{\Omega_0}\Big\{(\bar{x}_0+d)\big[(\partial_{x_0}u)^2 + (\partial_{x_1}u)^2 + (x_0-\alpha(x'))^2(\partial_{x_2}u)^2\big] + 2g_d(x')\partial_{x_0}u\partial_{x_1}u\Big\}\mathrm{d}x'.
\end{aligned}
\tag{21}
$$

By assumption (i), there exist two positive numbers $k$ and $\delta$ such that, for $d > \frac{1}{h_1}|g(x')|$, where $x' \in \Omega_0 \cap J_{k,\delta,\bar{x}_0}$, it results $(\bar{x}_0+d)^2 - (g_d(x'))^2 \geq 0$ and, hence,

$$
\int_{\Omega_0}\big[(\bar{x}_0+d)(\partial_{x_0}u)^2 + 2g_d(x')\partial_{x_0}u\partial_{x_1}u + (\bar{x}_0+d)(\partial_{x_1}u)^2\big]\mathrm{d}x' \geq 0.
$$

By using (21), we deduce

$$
\begin{aligned}
&({}^tP_2u, A_du) + (A_du, {}^tP_2u) \\
&\quad = 2({}^tP_2u, A_du) \\
&\quad \geq \|h^{\frac{1}{2}}(x')\partial_{x_0}u\|^2 + \|h^{\frac{1}{2}}(x')\partial_{x_1}u\|^2 + \|[4-h(x')]^{\frac{1}{2}}(x_0-\alpha(x'))\partial_{x_2}u\|^2 \\
&\qquad + 4\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')\partial_{x_2}u, g_d(x')\partial_{x_1}u\big) \\
&\qquad - 2\big((x_0-\alpha(x'))^2\partial_{x_2}g_d(x')\partial_{x_2}u, \partial_{x_1}u\big) \\
&\qquad + 4\big((x_0-\alpha(x'))^2\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big) \\
&\qquad + 4\big((x_0-\alpha(x'))\alpha(x')\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big).
\end{aligned}
\tag{22}
$$

Now, we consider the first-order terms. Integrating by parts, it results

$$
\begin{aligned}
&({}^tP_1u, A_du) + (A_du,\ {}^tP_1u)\\
&\quad = -8\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')\partial_{x_2}u, (x_0+d)\partial_{x_0}u + g_d(x')\partial_{x_1}u\big)\\
&\qquad - 2\big(\widetilde{a}_0(x)\partial_{x_0}u + \widetilde{a}_1(x)\partial_{x_1}u + (x_0-\alpha(x'))\widetilde{a}_2(x)\partial_{x_2}u, (x_0+d)\partial_{x_0}u + g_d(x')\partial_{x_1}u\big)\\
&\quad = -8\big((x_0-\alpha(x'))^2\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 8\big((x_0-\alpha(x'))\alpha(x')\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 8\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')\partial_{x_2}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - 2\big(\widetilde{a}_0(x)(x_0-\alpha(x'))\partial_{x_0}u, (x_0+d)\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_0(x)\alpha(x')\partial_{x_0}u, (x_0+d)\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_1(x)(x_0-\alpha(x'))\partial_{x_1}u, (x_0+d)\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_1(x)\alpha(x')\partial_{x_0}u, (x_0+d)\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_2(x)(x_0-\alpha(x'))^2\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_2(x)(x_0-\alpha(x'))\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_0(x)\partial_{x_0}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - 2\big(\widetilde{a}_1(x)\partial_{x_1}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - 2\big(\widetilde{a}_2(x)(x_0-\alpha(x'))\partial_{x_2}u, g_d(x')\partial_{x_1}u\big)
\end{aligned}
\tag{23}
$$

Adding (22) and (23), we have

$$
\begin{aligned}
&({}^tPu, A_du) + (A_du,\ {}^tPu)\\
&\quad \geq h_1\|\partial_{x_0}u\|^2 + h_1\|\partial_{x_1}u\|^2 + (4-h_2)\|(x_0-\alpha(x'))\partial_{x_2}u\|^2\\
&\qquad - 4\big((x_0-\alpha(x'))\partial_{x_2}\alpha(x')\partial_{x_2}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - 4\big((x_0-\alpha(x'))^2\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 4\big((x_0-\alpha(x'))\alpha(x')\partial_{x_2}\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 2\Big|\big((x_0-\alpha(x'))^2\partial_{x_2}g_d(x')\partial_{x_2}u, \partial_{x_1}u\big)\Big|\\
&\qquad - c\|(x_0+d)\partial_{x_0}u\|\|\partial_{x_0}u\|\\
&\qquad - 2\big(\widetilde{a}_0(x)(x_0-\alpha(x'))\partial_{x_0}u, x_0\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_0(x)\alpha(x')\partial_{x_0}u, x_0\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_1(x)(x_0-\alpha(x'))\partial_{x_1}u, (x_0+d)\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_1(x)\alpha(x')\partial_{x_0}u, x_0\partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_2(x)(x_0-\alpha(x'))^2\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_2(x)(x_0-\alpha(x'))\alpha(x')\partial_{x_2}u, \partial_{x_0}u\big)\\
&\qquad - 2\big(\widetilde{a}_0(x)\partial_{x_0}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - 2\big(\widetilde{a}_1(x)\partial_{x_1}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - 2\big(\widetilde{a}_2(x)(x_0-\alpha(x'))\partial_{x_2}u, g_d(x')\partial_{x_1}u\big)\\
&\qquad - \big|({}^tP_0u, A_du)\big| - \big|(A_du,\ {}^tP_0u)\big|, \quad \forall u \in C_0^{*\infty}(\overline{\Omega}_k).
\end{aligned}
\tag{24}
$$

Since $\alpha(x')$, $g_d(x')$ and $\beta(x')$ vanish on $\Omega_0 \cap \Gamma$, for every $\delta > 0$ there exist a neighborhood $I_{x'}$ in $\Omega_0 \cap \Gamma$ and $k > 0$ such that

$$|\alpha(x')| < \delta,\ |g_d(x')| < \delta, \quad \forall x' \in I_{x'},$$
$$|x_0 - \alpha(x')| < \delta, \quad \forall x \in [0, k[\times I_{x'}.$$

Let $\varphi \in C_0^\infty(\overline{\Omega})$ such that $\varphi \equiv 1$, on $[0, k'[\times I'_{x'}$, with $I'_{x'} \subseteq I_{x'}$ and $k' < k$, $0 \le \varphi(x) \le 1$ and $\operatorname{supp}\varphi \subseteq [0, k[\times I_{x'}$. Without lost generality, we can consider $[0, k'[\times I'_{x'}$ such that $|x_0 - \alpha(x')| \ge \frac{\varepsilon}{2}$, for every $x \in \Omega_k \setminus ([0, k'[\times I'_{x'})$. Using (22) and the previous remarks, it follows

$$\begin{aligned}
&h_1\|\partial_{x_0}u\|^2 + h_1\|\partial_{x_1}u\|^2 + (4 - h_2)\|(x_0 - \alpha(x')\partial_{x_2}u\|^2\\
&\quad \le c(\delta + k)(\|\varphi^{\frac12}(x)\partial_{x_0}u\|^2 + \|\varphi^{\frac12}(x)\partial_{x_1}u\|^2 + \|\varphi^{\frac12}(x)(x_0 - \alpha(x'))\partial_{x_2}u\|^2)\\
&\qquad + c(\|(1-\varphi(x))^{\frac12}\partial_{x_0}u\|^2 + \|(1-\varphi(x))^{\frac12}\partial_{x_1}u\|^2 + \|(1-\varphi(x))^{\frac12}(x_0 - \alpha(x'))\partial_{x_2}u\|^2)\\
&\qquad + 2(\|{}^tP_0u\|\|A_du\| + \|{}^tPu\|\|A_du\|).
\end{aligned}$$

Taking into account Lemma 1, we get

$$\begin{aligned}
&h_1\|\partial_{x_0}u\|^2 + h_1\|\partial_{x_1}u\|^2 + (4 - h_2)\|(x_0 - \alpha(x')\partial_{x_2}u\|^2\\
&\quad \le c(\delta + k)(\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \|(x_0 - \alpha(x'))\partial_{x_2}u\|^2) + c(\|(1-\varphi(x))^{\frac12}\partial_{x_0}u\|^2\\
&\qquad + \|(1-\varphi(x))^{\frac12}\partial_{x_1}u\|^2 + \|(1-\varphi(x))^{\frac12}(x_0 - \alpha(x'))\partial_{x_2}u\|^2) + c(\|{}^tPu\| + \|{}^tP_0u\|)\\
&\quad = c(\delta + k)(\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \|(x_0 - \alpha(x'))\partial_{x_2}u\|^2)\\
&\qquad + c(\|\partial_{x_0}(1-\varphi(x))^{\frac12}u + [(1-\varphi)^{\frac12}, \partial_{x_0}]u\|^2 + \|\partial_{x_1}(1-\varphi(x))^{\frac12}u + [(1-\varphi)^{\frac12}, \partial_{x_1}]u\|^2\\
&\qquad + \|(x_0 - \alpha(x'))\partial_{x_2}(1-\varphi(x))^{\frac12}u + (x_0 - \alpha(x'))[(1-\varphi)^{\frac12}, \partial_{x_2}]u\|^2) + c(\|{}^tPu\| + \|{}^tP_0u\|)\\
&\quad \le c(\delta + k)(\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \|(x_0 - \alpha(x'))\partial_{x_2}u\|^2) + c(\|\partial_{x_0}(1-\varphi(x))^{\frac12}u\|^2\\
&\qquad + \|\partial_{x_1}(1-\varphi(x))^{\frac12}u\|^2 + \|(x_0 - \alpha(x'))\partial_{x_2}(1-\varphi(x))^{\frac12}u\|^2) + c(\|{}^tPu\| + \|{}^tP_0u\|).
\end{aligned}$$

Making use of Lemmas 1 and 3, for $\delta$ and $k$ small enough, we obtain

$$\begin{aligned}
&\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0 - \alpha(x')\partial_{x_2}u\| + \|u\| \le c\|{}^tPu\|,\\
&\forall u \in C_0^\infty(I_{\bar{x}_0,k,\delta}) :\ u|_S = 0.
\end{aligned} \tag{25}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(t) = 1$, if $|t| \le \frac12$, and $\chi(t) = 0$, if $|t| > 1$. We rewrite (25) for $u\chi\left(\frac{x_0 - \alpha(x')}{\delta}\right)$ and apply Lemma 5 to $u\left[1 - \chi\left(\frac{x_0 - \alpha(x')}{\delta}\right)\right]$. Adding the obtained estimates, for $\delta$ small enough and $k$ suitable and small, we reach (13).

Instead, in order to get (14), let $\gamma > 0$ and let us consider the operator

$$A_{\bar{x}_0,\gamma} = (x_0 - \alpha(x'))\partial_{x_0} + g_{\bar{x}_0,\gamma}(x')\partial_{x_1},$$

where

$$g_{\bar{x}_0,\gamma}(x') = \frac{\alpha(x') - \bar{x}_0 + \gamma}{\partial_{x_1}\alpha(x')}.$$

Integrating by parts in the inner products $({}^tPu, A_{\overline{x}_0,\gamma}u) + (A_{\overline{x}_0,\gamma}u, {}^tPu)$, using the same arguments as done and since $g_{\overline{x}_0,\gamma}(x')$ has the same sign of $g(x')$ on $S \cap I_{k,\delta}$, we deduce

$$\begin{aligned}&(4-\varepsilon)\|(x_0-\alpha(x'))\partial_{x_2}u\|^2 + \int_{\Omega_0}\left[\gamma(\partial_{x_0}u)^2 + 2g_{\overline{x}_0,\gamma}(x')\partial_{x_0}u\,\partial_{x_1}u + \gamma(\partial_{x_1}u)^2\right]\mathrm{d}x'\\ &\quad\le c\varepsilon(\|\partial_{x_0}u\|^2 + \|\partial_{x_1}u\|^2 + \|(x_0-\alpha(x'))\partial_{x_2}u\|^2 + \|u\|^2 + \|{}^tPu\|^2),\\ &\quad\forall u \in C_0^\infty(\overline{\Omega}) :\ \operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k}\cap I_{k,\delta},\ u|_S = 0.\end{aligned} \tag{26}$$

For $\delta$ small enough and since $|\partial_{x_1}\alpha(x')| > 1$, on $\Gamma_{\overline{x}_0}$, it results

$$\int_{\Omega_0}\left[\gamma(\partial_{x_0}u)^2 + 2g_{\overline{x}_0,\gamma}(x')\partial_{x_0}u\,\partial_{x_1}u + \gamma(\partial_{x_1}u)^2\right]\mathrm{d}x' > 0.$$

As a consequence, for $\varepsilon$ small enough and $k$ suitable and small, we have

$$\begin{aligned}&\|(x_0-\alpha(x'))\partial_{x_2}u\| \le \varepsilon\left(\|{}^tPu\| + \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|u\|\right),\\ &\forall u \in C_0^\infty(\overline{\Omega}) :\ \operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k}\cap I_{\Gamma_{\overline{x}_0}},\ u|_S = 0.\end{aligned}$$

Rewriting the previous inequality for $u\chi\left(\frac{x_0-\alpha(x')}{\delta}\right)$ and applying Lemma 6 to $u\left[1-\chi\left(\frac{x_0-\alpha(x')}{\delta}\right)\right]$, as done above, (14) follows for $\overline{x}_0 > 0$.

On the other hand, if $\overline{x}_0 = 0$, considering the inner products

$$(A_{\overline{x}_0,\gamma}u, {}^tPu) + ({}^tPu, A_{\overline{x}_0,\gamma}u)$$

and proceeding as before, we obtain (13) and, then, (14) for $\gamma$ small enough. □

## 6 Conclusive a priori estimates

Let us assume that $|\partial_{x_1}\alpha(x')| = 1$ in some points of the plane $x_0 = \overline{x}_0$, with $\overline{x}_0 > 0$. Let $\Omega_{\overline{x}_0}$ be the intersection between the plane $x_0 = \overline{x}_0$ and $\Omega$. Let $\Gamma_{\overline{x}_0} = \Gamma \cap \Omega_{\overline{x}_0}$. Let $\Gamma'_{\overline{x}_0}$ be the set of the points of $\Gamma_{\overline{x}_0}$ where $\partial_{x_1}\alpha(x_1) = 1$ and, finally, let $I_{\overline{x}_0}$ be a neighborhood of $\overline{x}_0$ in $\Gamma'_{\overline{x}_0}$ on $\Omega_{\overline{x}_0}$ such that $\partial_{x_1}\alpha(x_1) \lessgtr 1$ outside $I_{\overline{x}_0}$. The following result holds.

**Theorem 9** *Let (i), (ii) and (iii) be satisfied. If on the plane $x_0 = \overline{x}_0 > 0$ there exist points in which $|\partial_{x_1}\alpha(x')| = 1$, then there exist $k > 0$ and $c > 0$ such that*

$$\begin{aligned}&\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0-\alpha(x'))\partial_{x_2}u\| + \|u\| \le c\|{}^tPu\|,\\ &\forall u \in C_0^\infty(\overline{\Omega}) :\ \operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k},\ u|_S = 0.\end{aligned} \tag{27}$$

*Moreover, for every $\varepsilon > 0$ there exists $k > 0$ such that*

$$\begin{aligned}&\|(x_0-\alpha(x'))\partial_{x_2}u\| \le \varepsilon\left(\|{}^tPu\| + \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|u\|\right),\\ &\forall u \in C_0^\infty(\overline{\Omega}) :\ \operatorname{supp} u \subseteq \Omega_{\overline{x}_0,k},\ u|_S = 0.\end{aligned} \tag{28}$$

***Proof*** Let $\Omega_{\overline{x}_0} \cap \Gamma$, let $\overline{x}' \in \Omega_0$ such that $|\partial_{x_1}\alpha(\overline{x}')| = 1$. We set

$$\gamma(\overline{x}') = \begin{cases} \sqrt{\overline{x}_0} - \sqrt{\alpha(\overline{x}')}, & \text{if } \partial_{x_1}\alpha(\overline{x}') = 1, \\ -(\sqrt{\overline{x}_0} - \sqrt{\alpha(\overline{x}')}), & \text{if } \partial_{x_1}\alpha(\overline{x}') = -1. \end{cases}$$

Evidently, it results $|\partial_{x_1}\alpha(x')| \le 1$ on the curve $x_0 - \overline{x}_0 = \gamma(x')$ and $x \in J_{k,\delta,\overline{x}_0}$, with suitable $k$ and $\delta$. Therefore, there exists $\eta$ such that $|\partial_{x_1}\alpha(x')| \le 1$ if $|x_0 - \overline{x}_0| \le \eta\gamma(x')$ and $x \in J_{k,\delta,\overline{x}_0}$. Whereas $|\partial_{x_1}\alpha(x')| \ge 1$ on $\Omega_{\overline{x}_0}$ if $|x_0 - \overline{x}_0| \ge \eta\gamma(x')$. Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 0$ if $t \le \frac{\eta}{2}$ and $\chi(t) = 1$ if $t \ge \eta$. For every $u \in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp} u \subseteq J_{k,\delta,\overline{x}_0}$ and $u|_S = 0$, we rewrite (13) and (14) for $\chi\left(\frac{x_0-\overline{x}_0}{\gamma(x')}\right)u$ and (10) for $\left(1 - \chi\left(\frac{x_0-\overline{x}_0}{\gamma(x')}\right)\right)u$. Adding such inequalities, for $k$ small enough, we have

$$\begin{gathered} \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0 - \alpha(x'))\partial_{x_2}u\| + \|u\| \le c\|\,{}^tPu\|, \\ \forall u \in C_0^\infty(\Omega_{k,\overline{x}_0}) : \ \operatorname{supp} u \subseteq J_{k,\delta,\overline{x}_0}, \ u|_S = 0, \end{gathered} \tag{29}$$

and

$$\begin{gathered} \|(x_0 - \alpha(x'))\partial_{x_2}u\| \le \varepsilon\left(\|\,{}^tPu\| + \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|u\|\right), \\ \forall u \in C_0^\infty(\Omega_{k,\overline{x}_0}) : \ \operatorname{supp} u \subseteq J_{k,\delta,\overline{x}_0}, \ u|_S = 0. \end{gathered} \tag{30}$$

From (29), (30) and Lemma 5, it follows

$$\begin{gathered} \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0 - \alpha(x'))\partial_{x_2}u\| + \|u\| \le c\|\,{}^tPu\|, \\ \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \overline{\Omega}_{\overline{x}_0,k} = [\overline{x}_0, \overline{x}_0 + k[\times\overline{\Omega}_0, \ u|_S = 0, \end{gathered}$$

and

$$\begin{gathered} \|(x_0 - \alpha(x'))\partial_{x_2}u\| \le \varepsilon\left(\|\,{}^tPu\| + \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|u\|\right), \\ \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \overline{\Omega}_{\overline{x}_0,k} = [\overline{x}_0, \overline{x}_0 + k[\times\overline{\Omega}_0, \ u|_S = 0. \end{gathered}$$

□

With the same techniques used in Theorem 6 if $\overline{x}_0 = 0$ and Theorems 8 and 9 if $\overline{x}_0 > 0$, we obtain the next result.

**Theorem 10** *Let (i), (ii) and (iii) be satisfied. If on the plane $x_0 = \overline{x}_0 > 0$ there exist points in which $|\partial_{x_1}\alpha(x')| = 1$, then there exist $k > 0$ and $c > 0$ such that*

$$\begin{gathered} \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0 - \alpha(x'))\partial_{x_2}u\| + \|u\| \le c\|\,{}^tPu\|, \\ \forall u \in C_0^\infty(\widetilde{\Omega}) : \ \operatorname{supp} u \subseteq \widetilde{\Omega}_{\overline{x}_0,k} = [\overline{x}_0, \overline{x}_0 + k[\times\widetilde{\Omega}_0. \end{gathered} \tag{31}$$

*Moreover, for every $\varepsilon > 0$ there exists $k > 0$ such that*

$$\begin{gathered} \|(x_0 - \alpha(x'))\partial_{x_2}u\| \le \varepsilon\left(\|\,{}^tPu\| + \|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|u\|\right), \\ \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \widetilde{\Omega}_{\overline{x}_0,k} = [\overline{x}_0, \overline{x}_0 + k[\times\widetilde{\Omega}_0. \end{gathered} \tag{32}$$

## 7 Estimates in Sobolev spaces with $s < 0$

Let $\Omega'_{\bar{x}_0}$ be the intersection between $\Omega'$ and the plane $x_0 = \bar{x}_0$. Let $\Gamma'_{\bar{x}_0}$ be the set of points belonging into $\Gamma_{\bar{x}_0} = \Gamma \cap \Omega'_{\bar{x}_0}$ such that $|\partial_{x_1}\alpha(x')| = 1$. Moreover, let $J_{\bar{x}_0}$ be the intersection between a neighborhood of $\Gamma'_{\bar{x}_0}$ and $\Omega'_{\bar{x}_0}$. We are able to prove the following estimate in Sobolev spaces with $s < 0$.

**Theorem 11** *Let (i), (ii) and (iii) be satisfied. Then, for every $\bar{x}_0 \geq 0$ and for every $s < 0$ there exist $k > 0$ and $c > 0$ such that*

$$\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0 - \alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}} \leq c\|Pu\|_{H^{0,s}},$$
$$\forall u \in C_0^\infty(\overline{\Omega}_k) : \ \operatorname{supp} u \subseteq \Omega_{\bar{x}_0,k} = [\bar{x}_0, \bar{x}_0 + k[\times\Omega_0. \tag{33}$$

***Proof*** Firstly, let $x_0 > 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}\varphi \subseteq \Omega'_0$, $\varphi \equiv 1$ on $\Omega_0$, with $\Omega_0 \subset \Omega'_0$. For every $u \in C_0^\infty(\overline{\Omega}_k)$ such that $\operatorname{supp} u \subseteq \Omega_{\bar{x}_0,k} = [\bar{x}_0, \bar{x}_0 + k[\times\Omega_0$, we set $v_s = \varphi(x')A_s u$. Making use of Theorem 10, it follows

$$\|\partial_{x_0}v_s\| + \|\partial_{x_1}v_s\| + \|(x_0 - \alpha(x'))\partial_{x_2}v_s\| + \|v_s\| \leq c\|{}^tPv_s\|. \tag{34}$$

We have

$$\begin{aligned} \|\partial_{x_0}v_s\| &= \|\partial_{x_0}\varphi(x')A_s u\| \\ &= \|\varphi(x')A_s\partial_{x_0}u\| \\ &= \|A_s\varphi(x')\partial_{x_0}u + [\varphi, A_s]\partial_{x_0}u\| \\ &\geq \|A_s\partial_{x_0}u\| - \|R\partial_{x_0}u\|, \end{aligned} \tag{35}$$

where $R = [\varphi, A_s]u$ is a regularizing pseudodifferential operator.
By using (35) and Lemma 4, we obtain

$$\begin{aligned} \|\partial_{x_0}v_s\| &\geq \|A_s\partial_{x_0}u\| - c\|R(x_0 - \bar{x}_0)\partial^2_{x_0}u\| \\ &= \|A_s\partial_{x_0}u\| - c\|R(x_0 - \bar{x}_0)(-{}^tPu + {}^tPu + \partial^2_{x_0}u)\| \\ &\geq \|A_s\partial_{x_0}u\| - c\|R(x_0 - \bar{x}_0){}^tPu\| - c\|R(x_0 - \bar{x}_0)({}^tPu + \partial^2_{x_0}u)\| \\ &\geq \|\partial_{x_0}u\|_{H^{0,s}} - c\|(x_0 - \bar{x}_0){}^tPu\|_{H^{0,s}} - c\|(x_0 - \bar{x}_0)\partial_{x_0}u\|_{H^{0,s}} \\ &\quad - c\|(x_0 - \bar{x}_0)u\|_{H^{0,s}}. \end{aligned} \tag{36}$$

Furthermore, it results

$$\begin{aligned} \|\partial_{x_1}v_s\| &= \|\partial_{x_1}\varphi(x')A_s u\| \\ &= \|(\partial_{x_1}\varphi(x'))A_s u + \varphi(x')A_s\partial_{x_1}u\| \\ &= \|(\partial_{x_1}\varphi(x'))A_s u + A_s\partial_{x_1}u + [\varphi, A_s]\partial_{x_1}u\| \\ &\geq \|A_s\partial_{x_1}u\| - \|R_1A_s u\| - \|[\varphi, A_s]\partial_{x_1}u\| \\ &\geq \|\partial_{x_1}u\|_{H^{0,s}} - c\|u\|_{H^{0,s}} - \|R_2\partial_{x_1}u\| \\ &\geq \|\partial_{x_1}u\|_{H^{0,s}} - c\|u\|_{H^{0,s}}, \end{aligned} \tag{37}$$

where $R_1$ and $R_2$ are regularizing pseudodifferential operators.

Finally, we get

$$\begin{aligned}\|(x_0-\alpha(x'))\partial_{x_2}v_s\| =& \|(x_0-\alpha(x'))\partial_{x_2}(\varphi(x')A_s u)\|\\ =& \|(\partial_{x_2}\varphi(x'))(x_0-\alpha(x'))A_s u+(x_0-\alpha(x'))\varphi(x')A_s\partial_{x_2}u\|\\ =& \|(\partial_{x_2}\varphi(x'))(x_0-\alpha(x'))A_s u+(x_0-\alpha(x'))A_s\varphi(x')\partial_{x_2}u\\ &+(x_0-\alpha(x'))[\varphi,A_s]\partial_{x_2}u\|\\ =& \|R_3u+A_s(x_0-\alpha(x'))\varphi(x')\partial_{x_2}u+[x_0-\alpha(x'),A_s]\varphi(x')\partial_{x_2}u\\ &+R_4\partial_{x_2}u\|\\ &\geq \|A_s(x_0-\alpha(x'))\varphi(x')\partial_{x_2}u\|-\|R_3u\|-\|R_4\partial_{x_2}u\|-\|B_{s-1}\partial_{x_2}u\|\\ &\geq \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}-c\|u\|_{H^{0,s}}-\|B'_s u\|\\ &\geq \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}-c\|u\|_{H^{0,s}},\end{aligned} \tag{38}$$

where $R_3$ and $R_4$ are regularizing pseudodifferential operators, $B_{s-1}$ and $B'_s$ are pseudodifferential operators of order $s-1$ and $s$, respectively. Adding (36), (37), (38) and using Lemma 4, it follows

$$\begin{aligned}&\|\partial_{x_0}v_s\|+\|\partial_{x_1}v_s\|+\|(x_0-\alpha(x'))\partial_{x_2}v_s\|\\ &\quad\geq \|\partial_{x_0}u\|_{H^{0,s}}-c\|(x_0-\bar{x}_0)Pu\|_{H^{0,s}}-c\|(x_0-\bar{x}_0)\partial_{x_0}u\|-c\|(x_0-\bar{x}_0)u\|_{H^{0,s}}\\ &\qquad+\|\partial_{x_1}u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}-c\|u\|_{H^{0,s}}\\ &\quad\geq \|\partial_{x_0}u\|_{H^{0,s}}+\|\partial_{x_1}u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}-c\|(x_0-\bar{x}_0)\,{}^tPu\|_{H^{0,s}}\\ &\qquad-c\|(x_0-\bar{x}_0)\partial_{x_0}u\|_{H^{0,s}}-c\|(x_0-\bar{x}_0)u\|_{H^{0,s}}.\end{aligned} \tag{39}$$

Moreover, it results

$$\begin{aligned}\|{}^tPv_s\| =& \|{}^tP(\varphi(x')A_s u)\|\\ =& \|\varphi(x')\,{}^tPA_s u+[\varphi(x'),\,{}^tP]A_s u\|\\ =& \|\varphi(x')A_s\,{}^tPu+\varphi(x')[{}^tP,A_s]u+R_5u\|\\ =& \|A_s\,{}^tPu+[\varphi(x'),A_s]\,{}^tPu+\varphi(x')[{}^tP,A_s]u+R_5u\|\\ =& \|A_s\,{}^tPu+R_6\,{}^tPu+\varphi(x')[{}^tP,A_s]u+R_5u\|,\end{aligned} \tag{40}$$

where $R_5$ and $R_6$ are regularizing operators.

The commutator $[{}^tP,A_s]$ is given by

$$\varphi(x')[{}^tP,A_s]u=\varphi(x')[{}^tP_2,A_s]u+\varphi(x')[{}^tP_1,A_s]u+\varphi(x')[{}^tP_0,A_s]u. \tag{41}$$

We consider the principal part:

$$[{}^tP_2,A_s]u=B_{s+1}u+B_s u,$$

where $B_{s+1}$ and $B_s$ are pseudodifferential operators of order $s+1$ and $s$, respectively. The symbol of $B_{s+1}$ is given by

$$
\begin{aligned}
b(x,\xi') =& -\frac{1}{i}\sum_{h=1}^{2}\partial_{x_h}(\xi_1^2+(x_0-\alpha(x'))^2\xi_2^2)\varphi(x')\partial_{\xi_h}(1+|\xi'|^2)^{\frac{s}{2}}\\
=& -\frac{1}{i}\big(2(x_0-\alpha(x'))(-\partial_{x_1}\alpha(x'))\xi_2^2\big)\varphi(x')\partial_{\xi_1}(1+|\xi'|^2)^{\frac{s}{2}}\\
&-\frac{1}{i}\big(2(x_0-\alpha(x'))(-\partial_{x_2}\alpha(x'))\xi_2^2\big)\varphi(x')\partial_{\xi_2}(1+|\xi'|^2)^{\frac{s}{2}}
\end{aligned}
$$

Then, $B_{s+1}u = (x_0-\alpha(x'))\varphi(x')\partial_{x_2}B'_s u$, where $B'_s$ is a pseudodifferential operator of order $s$. Moreover, taking into account Theorem 10, we deduce

$$
\begin{aligned}
\|B_{s+1}u\| =&\|(x_0-\alpha(x'))\varphi(x')\partial_{x_2}B'_s u\|\\
\le&\varepsilon\big(\|{}^tPB'_s u\| + \|\partial_{x_0}B'_s u\| + \|\partial_{x_1}B'_s u\| + \|B'_s u\|\big)\\
\le&\varepsilon\big(\|B'_s\,{}^tPu\| + \|[{}^tP,B'_s]u\| + \|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|u\|_{H^{0,s}}\big)\\
\le&\varepsilon\big(\|{}^tPu\|_{H^{0,s}} + \|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}}\big),
\end{aligned}
$$

being $[{}^tP,B'_s]$ a pseudodifferential operator of order $s-1$ and its principal symbol $b'(x,\xi)$ of the same type of $b(x,\xi)$. Hence, making use of Lemma 4, it results

$$
\begin{aligned}
\|\varphi(x')[{}^tP_2,A_s]u\| \le&\varepsilon c\big(\|{}^tPu\|_{H^{0,s}} + \|\partial_{x_0}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}\\
&+\|u\|_{H^{0,s}}\big) + \|(x_0-\overline{x}_0)\partial_{x_0}u\|_{H^{0,s}}.
\end{aligned}
\tag{42}
$$

We consider the first-order part of the commutator

$$
\varphi(x')[{}^tP_1,A_s]u = B_{s-1}\partial_{x_0}u + B_s u + B_{s-1}u,
\tag{43}
$$

where $B_{s-1}$ and $B_s$ are pseudodifferential operators of order $s-1$ and $s$, respectively.

By using Lemma 4, we have

$$
\begin{aligned}
\|B_{s-1}\partial_{x_0}u\| \le&c\|(x_0-\overline{x}_0)\partial_{x_0}B_{s-1}\partial_{x_0}u\|\\
=&c\|(x_0-\overline{x}_0)B_{s-1}\partial_{x_0}^2u\|\\
\le&c\big(\|(x_0-\overline{x}_0)B_{s-1}Pu\| + \|(x_0-\overline{x}_0)B'_s\partial_{x_0}u\|\\
&+\|(x_0-\overline{x}_0)B''_s(x_0-\alpha(x'))\partial_{x_1}u\|\\
&+\|(x_0-\overline{x}_0)B'''_s(x_0-\alpha(x'))\partial_{x_2}u\|\\
&+\|(x_0-\overline{x}_0)B_s^{(iv)}u\|\big),
\end{aligned}
$$

where $B_s^{(i)}$ are pseudodifferential operators of order $s$. Hence, it results

$$
\begin{aligned}
\|B_{s-1}\partial_{x_0}u\| \le&c\big(\|(x_0-\overline{x}_0)Pu\|_{H^{0,s}} + \|(x_0-\overline{x}_0)\partial_{x_0}u\|_{H^{0,s}} + \|(x_0-\overline{x}_0)\partial_{x_1}u\|_{H^{0,s}}\\
&+\|(x_0-\overline{x}_0)(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|(x_0-\overline{x}_0)u\|_{H^{0,s}}\big).
\end{aligned}
\tag{44}
$$

Taking into account (43) and (44), it follows

$$
\begin{aligned}
\|\varphi(x')[{}^tP_1,A_s]\| \le&c\big(\|(x_0-\overline{x}_0)Pu\|_{H^{0,s}} + \|(x_0-\overline{x}_0)\partial_{x_0}u\|_{H^{0,s}}\\
&+\|(x_0-\overline{x}_0)(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}\\
&+\|(x_0-\overline{x}_0)u\|_{H^{0,s}} + \|(x_0-\overline{x}_0)\partial_{x_0}u\|_{H^{0,s}}\big).
\end{aligned}
\tag{45}
$$

We estimate the zero-order part:

$$\begin{aligned}\|\varphi(x')[{}^tP_0, A_s]u\| \le & c\|u\|_{H^{0,s}} \\ \le & c\|(x_0-\overline{x}_0)\partial_{x_0}u\|_{H^{0,s}}.\end{aligned} \tag{46}$$

Making use of (42), (45), (46) and for $|x_0-\overline{x}_0| \le k < \varepsilon$, we obtain

$$\begin{aligned}\|\varphi(x')[{}^tP, A_s]u\| \le & c\varepsilon\big(\|{}^tPu\|_{H^{0,s}} + \|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} \\ & + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}}\big).\end{aligned} \tag{47}$$

Taking into account (40), (47) and Lemma 4, denoted the generic regularizing operator by $R$, it follows

$$\begin{aligned}\|{}^tPv_s\| \le & \|A_s{}^tPu\| + \|R{}^tPu\| + \|\varphi(x')[P, A_s]u\| + \|Ru\| \\ \le & c\|{}^tPu\|_{H^{0,s}} + \varepsilon c\big(\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_1}u\|_{H^{0,s}}\big) \\ & + c\|u\|_{H^{0,s}} \\ \le & c\|{}^tPu\|_{H^{0,s}} + \varepsilon c\big(\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}\big) \\ & + c\|(x_0-\overline{x}_0)\partial_{x_0}u\| \\ \le & c\|{}^tPu\|_{H^{0,s}} + \varepsilon c\big(\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}\big).\end{aligned} \tag{48}$$

By using (34), (39), (48) and Lemma 4, it results

$$\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} \le c\|{}^tPu\|_{H^{0,s}} + c\varepsilon\|u\|_{H^{0,s}}.$$

For $\varepsilon$ small enough and making use of Lemma 4, we have

$$\begin{aligned}&\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} \\ &\quad \le c(\|{}^tPu\|_{H^{0,s}} + \|u\|_{H^{0,s}}) \\ &\quad \le c(\|{}^tPu\|_{H^{0,s}} + \|(x_0-\overline{x}_0)\partial_{x_0}u\|_{H^{0,s}}).\end{aligned}$$

For $|x_0-\overline{x}_0|$ small enough and using Lemma 4, we deduce

$$\begin{aligned}&\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}} \le c\|{}^tPu\|_{H^{0,s}}, \\ &\forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq ]\overline{x}_0, \overline{x}_0 + k[\times\Omega_0.\end{aligned} \tag{49}$$

Since the function $\varphi$ is the same for every functions $u$, then $c$ does not depend on $u$ but depends on the distance between $\partial\widetilde{\Omega}'_0$ and $\partial\widetilde{\Omega}_0$ and $k$ is small enough.

Now, if $x_0 = 0$, for every $u \in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp} u \subseteq [0, k[\times\Omega_0$, we set $v_s = \varphi(x')A_su$. Making use of Theorem 10, it results

$$\|\partial_{x_0}v_s\| + \|\partial_{x_1}v_s\| + \|(x_0-\alpha(x'))\partial_{x_2}v_s\| + \|v_s\| \le \varepsilon\|{}^tPv_s\|. \tag{50}$$

Proceeding as done above, we obtain the analogous inequality of (39):

$$\begin{aligned}&\|\partial_{x_0}v_s\| + \|\partial_{x_1}v_s\| + \|(x_0-\alpha(x'))\partial_{x_2}v_s\| \\ &\quad \ge \|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} - c\|x_0{}^tPu\|_{H^{0,s}} \\ &\qquad - c\|x_0\partial_{x_0}u\|_{H^{0,s}} - c\|x_0u\|_{H^{0,s}},\end{aligned} \tag{51}$$

where we used Lemma 1 instead of Lemma 4. Considering $\| {}^tPv_s\|$ and proceeding again as done before and taking into account Theorem 10 and Lemma 1, we have

$$\| {}^tPv_s\| \leq c\big(\| {}^tPu\|_{H^{0,s}} + \|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x')\partial_{x_2}u\|_{H^{0,s}}\big). \tag{52}$$

Moreover, using (50), (51) and (52), we obtain

$$\begin{aligned} &\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}} \leq c\|Pu\|_{H^{0,s}},\\ &\quad \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq [0,k[\times\Omega_0. \end{aligned} \tag{53}$$

□

## 8 Global estimates

In this section, we obtain fundamental global estimates in order to prove the existence of a solution to the Cauchy–Dirichlet problem (2).

**Theorem 12** *Let (i), (ii) and (iii) be satisfied. Then, for every $k > 0$ and $s < 0$ there exists $c > 0$ such that*

$$\begin{aligned} &\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}} \leq c\| {}^tPu\|_{H^{0,s}},\\ &\quad \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \Omega_k = [0,k[\times\Omega_0. \end{aligned} \tag{54}$$

*Moreover, for $s = 0$ and for every $k > 0$ there exists $c > 0$ such that*

$$\begin{aligned} &\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|(x_0-\alpha(x'))\partial_{x_2}u\| + \|u\| \leq c\| {}^tPu\|,\\ &\quad \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \overline{\Omega}_k = [0,k[\times\overline{\Omega}_0, \ u|_S = 0. \end{aligned} \tag{55}$$

*Finally, for every $k > 0$ and $s < 0$ there exists $c > 0$ such that*

$$\begin{aligned} &\|\partial_{x_0}u\|_{H^{0,s}} + \|\partial_{x_1}u\|_{H^{0,s}} + \|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}} + \|u\|_{H^{0,s}} \leq c\| {}^tP[u]\|_{H^{0,s}},\\ &\quad \forall u \in C_0^\infty(\overline{\Omega}) : \ \operatorname{supp} u \subseteq \overline{\Omega}_k = [0,k[\times\overline{\Omega}_0, \ u|_S = 0, \end{aligned} \tag{56}$$

*where* $[u] = \begin{cases} u, & \text{in } \Omega_k = [0,k[\times\Omega_0 \\ 0, & \text{in } \Omega_k = [0,k[\times(\mathbb{R}^2 \setminus \Omega_0) \end{cases}$.

***Proof*** Let $k > 0$, let us set $\Omega_k = [0,k[\times\Omega_0$. For the compactness of $[0,k]\times\overline{\Omega}_0$, there exists a finite number of subsets $\{\Omega_1, \Omega_2, \dots, \Omega_p\}$ of $\Omega_k$, given by

$$\Omega_1 = [0,h_1[\times\Omega_0, \ \Omega_2 = [h'_1, h_2[\times\Omega_0, \dots, \ \Omega_p = [h'_{p-1}, h_p[\times\Omega_0,$$

with $h_0 = 0$, $h_p = h$, $h_{i-1} < h'_i < h_i$, for every $i = 1, \dots, p$, and such that (33) holds in every $\Omega_i$, for $i = 1, \dots, p$.

Let $u \in C_0^\infty(\Omega_k)$, let $\varphi \in C_0^\infty([0,h_1[)$, with $\varphi \equiv 1$ on $[0,h'_1[$ and $0 \leq \varphi \leq 1$ in $[0,h_1[$. Rewriting (33) for $\varphi u$, it results

$$
\begin{aligned}
&\|\partial_{x_0}\varphi u\|_{H^{0,s}}+\|\partial_{x_1}\varphi u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}\varphi u\|_{H^{0,s}}+\|\varphi u\|_{H^{0,s}}\\
&\quad\le c\|P\varphi u\|_{H^{0,s}}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|[P,\varphi]u\|_{H^{0,s}}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|\partial_{x_0}\varphi\partial_{x_0}u\|_{H^{0,s}}+c\|(\partial^2_{x_0}\varphi)u\|_{H^{0,s}}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|\partial_{x_0}u\|_{H^{0,s}([h_1',h_1[\times\Omega_0)}+c\|u\|_{H^{0,s}([h_1',h_1[\times\Omega_0)}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|\partial_{x_0}u\|_{H^{0,s}([h_1',h_2'[\times\Omega_0)}+c\|u\|_{H^{0,s}([h_1',h_2'[\times\Omega_0)}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|\partial_{x_0}\varphi_1 u\|_{H^{0,s}([h_1',h_2[\times\Omega_0)}+c\|\varphi_1 u\|_{H^{0,s}([h_1',h_2[\times\Omega_0)},
\end{aligned}
$$

where $\varphi_1\in C_0^\infty(\Omega_0)$ such that $\operatorname{supp}\varphi_1\subseteq[h_1',h_2[$, $\varphi_1\equiv 1$ in $[h_1',h_2']\times\Omega_0$.

We can deduce that

$$
\begin{aligned}
&\|\partial_{x_0}\varphi_{i-1}u\|_{H^{0,s}}+\|\partial_{x_1}\varphi_{i-1}u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}\varphi_{i-1}u\|_{H^{0,s}}+\|\varphi_{i-1}u\|_{H^{0,s}}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|\partial_{x_0}\varphi_i u\|_{H^{0,s}([h_i',h_{i+1}[\times\Omega_0)}+c\|\varphi_i u\|_{H^{0,s}([h_i',h_{i+1}[\times\Omega_0)},
\end{aligned}
$$

where $\varphi_0=\varphi$ and $\varphi_i\in C_0^\infty([0,k[)$ such that $\operatorname{supp}\varphi_i\subseteq[h_i',h_{i+1}[$, for every $i=1,\dots,p$.

On the other hand, we have

$$
\begin{aligned}
&\|\partial_{x_0}\varphi_{p-1}u\|_{H^{0,s}}+\|\partial_{x_1}\varphi_{p-1}u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}\varphi_{p-1}u\|_{H^{0,s}}+\|\varphi_{p-1}u\|_{H^{0,s}}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\|\partial_{x_0}\varphi_p u\|_{H^{0,s}(\Omega_p)}+c\|\varphi_p u\|_{H^{0,s}(\Omega_p)}\\
&\quad\le c\|Pu\|_{H^{0,s}}+c\left(\|\partial_{x_0}u\|_{H^{0,s}(\Omega_p)}+\|u\|_{H^{0,s}(\Omega_p)}\right)\\
&\quad\le c\|Pu\|_{H^{0,s}}.
\end{aligned}
\tag{57}
$$

Using (33), (57) and proceeding by recurrence on $i$, we easily obtain

$$
\|\partial_{x_0}\varphi_i u\|_{H^{0,s}}+\|\partial_{x_1}\varphi_i u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}\varphi_i u\|_{H^{0,s}}+\|\varphi_i u\|_{H^{0,s}}\le c\|Pu\|_{H^{0,s}},
$$

for $i=1,\dots,p$. Taking into account the previous inequality, we have

$$
\begin{aligned}
&\|\partial_{x_0}u\|_{H^{0,s}}+\|\partial_{x_1}u\|_{H^{0,s}}+\|(x_0-\alpha(x'))\partial_{x_2}u\|_{H^{0,s}}+\|u\|_{H^{0,s}}\le c\|Pu\|_{H^{0,s}},\\
&\quad\forall u\in C_0^\infty(\overline{\Omega}):\ \operatorname{supp}u\subseteq\Omega_k.
\end{aligned}
\tag{58}
$$

For the arbitrariness of $k$, (58) holds for every $u\in C_0^\infty(\overline{\Omega})$. The proof of (54) is thereby completed.

Furthermore, taking into account (31), we obtain (55).

Finally, we prove (56). Let $u\in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp}u\subseteq[0,k[\times\overline{\Omega}_0$ and $u|_S=0$. Let $\{u_n\}$ be a sequence in $C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp}u_n\subseteq[0,k[\times\overline{\Omega}_0$ and $u_n\to u$ in $H^{2,1}$. We have that $u_n\to u$ and $Pu_n\to P[u]$ in $H^{0,s}$, for every $s<0$. Hence, rewriting (54) for $u_n$, for every $n\in\mathbb{N}$, and passing to the limit as $n\to+\infty$, we obtain (56). □

## 9 Proof of Theorem 1

Let $V$ be the subspace of $L^2(\Omega_k)$, where $\underline{\Omega}_k=]0,k[\times\Omega_0$, made up of functions $\psi={}^tPu$, with $u\in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp}u\subseteq[0,k[\times\overline{\Omega}_0$ and $u|_S=0$. Let us consider the functional

$$T(\psi) = T({}^tPu) = (f, u).$$

It results

$$\begin{aligned}|T(\psi)| =&|T({}^tPu)|\\ =&|(f,u)|\\ \leq&\|f\|_{L^2(\Omega_k)}\|u\|_{L^2(\Omega_k)}.\end{aligned}$$

Making use of (55), we have

$$\begin{aligned}|T(\psi)| \leq&c\|f\|_{L^2(\Omega_k)}\|{}^tPu\|_{L^2(\Omega_k)}\\ =&c\|f\|_{L^2(\Omega_k)}\|\psi\|_{L^2(\Omega_k)}\\ =&c'\|\psi\|_{L^2(\Omega_k)},\end{aligned}$$

where $c' = c\|f\|_{L^2(\Omega_k)}$. Therefore, it is possible to extend $T$ as a linear continuous functional into $L^2(\Omega_k)$. Making use of a representation theorem, there exists $w \in L^2(\Omega_k)$ such that

$$T(v) = (w, v), \quad \forall v \in L^2(\Omega_k).$$

In particular, we have

$$T(\psi) = T({}^tPu) = (w, {}^tPu) = (f, u), \quad \forall u \in C_0^\infty(\overline{\Omega}_k) : \; u|_S = 0.$$

Hence, $w$ is a solution in the sense of distributions to the equation

$$Pu = f, \quad \text{in } \Omega_k.$$

For the arbitrariness of $k$ and since $f \in L^2_{loc}(\overline{\Omega})$, Theorem 1 is proved.

## 10 Proof of Theorem 2

Let us denote by $W$ the subspace of $\mathcal{D}'([0,k[\times\Omega_0)$ containing extensions of linear continuous functionals to functions $\varphi \in C_0^\infty(\overline{\Omega}_k)$ such that $\varphi|_S = 0$, where $\overline{\Omega}_k = [0,k[\times\overline{\Omega}_0$. It results that $P[u] \in W$, where $u \in C_0^\infty(\overline{\Omega}_k)$ such that $u|_S = 0$ and $u = 0$ in $[0,k[\times(\mathbb{R}^2 \setminus \overline{\Omega}_0)$. Moreover, we have

$$\langle\varphi, {}^tP[u]\rangle = (\varphi, {}^tP[u]) = (\varphi, {}^tPu), \quad \forall \varphi \in C_0^\infty(\overline{\Omega}_k) : \; \varphi|_S = 0.$$

Therefore, the distributions ${}^tP[u]$ and ${}^tPu$ are equal in $W$. Let $T$ be the functional defined into the subspace of $W$ containing the distributions $\psi = {}^tP[u]$, for every $u \in C^\infty([0,k[\times\overline{\Omega}_0)$ such that $u|_S = 0$, given by

$$T(\psi) = T({}^tP[u]) = (f, u).$$

Making use of (56), it follows

$$\begin{aligned}\|T(\psi)\| =&|T({}^tP[u])|\\ =&|(f,u)|\\ \leq&\|f\|_{H^{0,s}}\|u\|_{H^{0,-s}}\\ \leq&c\|{}^tP[u]\|_{H^{0,-s}}, \quad \forall u\in C_0^\infty(\overline{\Omega}_k)\,:\; u|_S=0.\end{aligned}$$

with $s\leq r$. Then, $T$ can be extended in the subspace $W'$ of $W$ containing the distributions of $W$ with finite $H^{0,-s}(\overline{\Omega}_k)$-norm. As a consequence, there exists $w\in W'^*$, where $W'^*$ is the topological dual of $W'$, such that

$$T(\psi)=T({}^tP[u])=(w,{}^tP[u])=(f,u). \tag{59}$$

On the other hand, it results $w\in H^{0,s}(\Omega_k)$ and since

$$(\varphi,{}^tP[u])=(\varphi,{}^tPu),\quad \forall\varphi,u\in C_0^\infty([0,k[\times\overline{\Omega}_0)\,:\; \varphi|_S=0,\ u|_S=0,$$

it follows for every $\{\varphi_n\}\subseteq C_0^\infty([0,+\infty[\times\overline{\Omega}_0)$ such that $\varphi_n|_S=0$, $\forall n\in\mathbb{N}$, and $\varphi_n\rightharpoonup w$ in $W'^*$,

$$\begin{aligned}(Pw,u)=&(w,{}^tP[u])\\ =&\lim_{n\to+\infty}(\varphi_n,{}^tP[u])\\ =&\lim_{n\to+\infty}(\varphi_n,{}^tPu)\\ =&(w,{}^tPu),\end{aligned} \tag{60}$$

we deduce that $w|_S=0$ (see also below).

Taking into account (61) and (60), we get

$$(w,{}^tPu)=(Pw,u)=(f,u),\quad \forall u\in C_0^\infty(\overline{\Omega})\,:\; \operatorname{supp}u\subseteq[0,k[\times\overline{\Omega}_0. \tag{61}$$

From (61), we have

$$Pw=f,\quad \text{in the sense of distributions.}$$

and

$$w\in H^r(\Omega_k\setminus\partial\Omega_0).$$

Indeed, set $Lw=Pw+\partial_{x_0}^2w-\frac{1}{i}a_0(x)\partial_{x_0}w-b(x)w$, it results

$$-\partial_{x_0}^2w+\frac{1}{i}a_0(x)\partial_{x_0}w+b(x)w=f-Lw, \tag{62}$$

with $w\in\mathcal{D}'(\Omega_k)\cap H^{0,r}(\Omega_k)$ and $f-Lw\in L^2(\Omega_k)$. From (62), it follows that $w$ is a solution to a second-order differential equation with zero-order term belonging to $L^2(\Omega_k)$. Hence, we have $w\in H^{2,0}(\Omega_k)\cap H^{0,r}(\Omega_k)$. On the other hand, (62) implies

$$\partial^{0,\alpha_1,\alpha_2}\left(-\partial_{x_0}^2w+\frac{1}{i}a_0(x)\partial_{x_0}w+b(x)w\right)=\partial^{0,\alpha_1,\alpha_2}(f-Lw),$$

with $\alpha_1+\alpha_2\leq s-r+2$. Therefore, we obtain

$$
\begin{aligned}
&-\partial_{x_0}^2\partial^{0,\alpha_1,\alpha_2}w+\frac{1}{i}a_0(x)\partial_{x_0}\partial^{0,\alpha_1,\alpha_2}w+b(x)\partial^{0,\alpha_1,\alpha_2}w\\
&\quad=\partial^{0,\alpha_1,\alpha_2}(f-Lw)+\left[\partial^{0,\alpha_1,\alpha_2}-\partial_{x_0}^2+\frac{1}{i}a_0(x)\partial_{x_0}+b(x)\right]w.
\end{aligned}
\tag{63}
$$

Proceeding by induction in the previous equality, assuming $u\in H^{2,p-1}$, with $1\le p\le r-2$ and taking into account (63), it results

$$
w\in H^{2,p}(\Omega_k\setminus\partial\Omega_0).
$$

Subsequently, by the equality

$$
\partial^{p-2,\alpha_1,\alpha_2}\left(-\partial_{x_0}^2w+\frac{1}{i}a_0(x)\partial_{x_0}w+b(x)w\right)=\partial^{p-2,\alpha_1,\alpha_2}(f-Lw),
$$

with $0\le p-2+\alpha_1+\alpha_2\le r-2$, and proceeding by induction on $p$, it follows

$$
w\in H^r(\Omega_k\setminus\partial\Omega_0).
$$

From (61), we deduce

$$
\begin{aligned}
\langle Pw,u\rangle=&(Pw,u)\\
=&(w,\,{}^tPu)\\
=&(f,u),\quad \forall u\in C_0^\infty(\Omega_k):\ \operatorname{supp}u\subseteq]0,k[\times\Omega_0.
\end{aligned}
$$

Then, we obtain

$$
Pw=f,\quad \text{a.e. in}\ \operatorname{int}\Omega_k.
$$

Now, making use of (61), we show that the boundary conditions on $\Omega_0$ are satisfied. Let $u(x_0,x')=u_0(x_0)u_1(x')$ such that $u_0\in C_0^\infty([0,k_1[)$, $u_0(0)=1$, $\partial_{x_0}u_0(0)=0$ and $u_1\in C_0^\infty(\Omega_0)$. Integrating by parts in (61), we have

$$
(Pw,u)-\int_{\Omega_0}w(0,x')u_1(x')\mathrm{d}x'=(w,\,{}^tPu).
$$

It follows

$$
\int_{\Omega_0}w(0,x')u_1(x')\mathrm{d}x'=0,\quad \forall u_1\in C_0^\infty(\Omega_0).
$$

It implies

$$
w(0,x')=0,\quad \text{a.e. in}\ \Omega_0.
$$

Instead, if $u(x_0,x')=u_0(x_0)u_1(x')$, with $u_0\in C_0^\infty([0,k_1[)$, $u_0(0)=0$, $\partial_{x_0}u_0(0)=1$ and $u_1\in C_0^\infty(\Omega_0)$, integrating by parts, we obtain

$$
\int_{\Omega_0}\partial_{x_0}w(0,x')u_1(x')\mathrm{d}x'=0,\quad \forall u_1\in C_0^\infty(\Omega_0).
$$

Hence, it results

$$\partial_{x_0} w(0,x') = 0, \quad \text{a.e. in } \Omega_0.$$

Then, we have proved that the Cauchy problem

$$\begin{cases} Pw = f, & \text{in } \Omega_k, \\ w|_{\Omega_0} = 0, & \frac{\mathrm{d}w}{\mathrm{d}n}|_{\Omega_0} = 0, \end{cases}$$

admits a solution $w \in H^r(\overline{\Omega}_k \setminus \partial\Omega_0)$, for every $k > 0$, under assumptions (i), (ii) and (iii) and if $f \in H^r(\overline{\Omega}_k)$. Finally, we justify that $w|_S = 0$, as written above. In fact, integrating by parts in (61), we get

$$(Pw, u) + \int_S wn_1 \partial_{x_1} u \mathrm{d}\sigma + \int_S wn_2 (x_0 - \alpha(x'))^2 \partial_{x_2} u \mathrm{d}\sigma = (w, {}^t Pu).$$

It follows

$$\int_S w(n_1 \partial_{x_1} u + n_2 (x_0 - \alpha(x'))^2 \partial_{x_2} u) \mathrm{d}\sigma = 0.$$

Fixed an arbitrary test function $\phi$ on $S$, it is possible to determine $u$ such that $n_1 \partial_{x_1} u + n_2(x_0 - \alpha(x'))^2 \partial_{x_2} u|_S = \phi(x_0, x')$. Then, we obtain

$$\int_S w\phi \mathrm{d}\sigma = 0, \quad \forall \phi \in C_0^\infty(S), \tag{64}$$

which implies

$$w = 0, \quad \text{a.e. in } S.$$

In the following, a brief proof of the previous claim is given. Parameterizing the surface $S$ in the following way:

$$x_0 = x_0, \quad x_1 = \varphi_1(s), \quad x_2 = \varphi_2(s),$$

with $x_0 \in [0, k[$ and $s \in [0, L(\partial\Omega_0)]$, being $s$ the arc length of $\partial\Omega_0$, we have

$$\begin{aligned} &\int_S w(n_1 \partial_{x_1} u + n_2 (x_0 - \alpha(x'))^2 \partial_{x_2} u) \mathrm{d}\sigma \\ &= \int_{[0,k]\times[0,L(\partial\Omega_0)]} w(x_0, \varphi_1(s), \varphi_2(s)) \varphi_2'(s) \partial_{x_1} u(x_0, \varphi_1(s), \varphi_2(s)) \mathrm{d}x_0 \mathrm{d}s \\ &- \int_{[0,k]\times[0,L(\partial\Omega_0)]} w(x_0, \varphi_1(s), \varphi_2(s)) \varphi_1'(s)(x_0 - \alpha(\varphi_1(s), \varphi_2(s)))^2 \\ &\cdot \partial_{x_2} u(x_0, \varphi_1(s), \varphi_2(s)) \mathrm{d}x_0 \mathrm{d}s \\ &= \int_{[0,k]\times[0,L(\partial\Omega_0)]} w(x_0, s) \frac{du}{dn}(x_0, s)((\varphi_2'(s))^2 + (x_0 - \alpha(s))^2 (\varphi_1'(s))^2) \mathrm{d}x_0 \mathrm{d}s, \end{aligned}$$

where $n$ is the external normal vector to the surface $S$. Hence, in order to obtain (64), we need that $\frac{du}{dn}|_S = \phi(x_0, s)$, where $\phi$ is an arbitrary function belonging to $C_0^\infty([0, k[\times]0, L(\partial\Omega_0)[)$. As a consequence, we have proved the existence of a solution $w \in H^r(\Omega_k \setminus \partial\Omega_0)$ to the following Cauchy–Dirichlet problem

$$\begin{cases} Pw = f, & \text{in } \Omega_k, \\ w|_{\Omega_0} = 0, \ \frac{dw}{dn}|_{\Omega_0} = 0, \ w|_S = 0, \end{cases}$$

where $f \in H^r(\overline{\Omega}_k)$. Since $f \in H^r_{loc}(\overline{\Omega})$ and for the arbitrariness of $k$, Theorem 2 is obtained.

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