Minimax Theory and its Applications
Volume 6 (2021), No. 2, 365–380

Local Boundedness for Weak Solutions to some Quasilinear Elliptic Systems

Salvatore Leonardi
Dept. of Mathematics and Informatics, University of Catania, 95125 Catania, Italy
leonardi@dmi.unict.it

Francesco Leonetti, Cristina Pignotti
DISIM, University of L'Aquila, 67100 L'Aquila, Italy
leonetti@univaq.it, pignotti@univaq.it

Eugenio Rocha, Vasile Staicu
CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
eugenio@ua.pt, vasile@ua.pt

Received: May 15, 2019
Accepted: November 18, 2019

We consider quasilinear elliptic systems in divergence form. In general, we cannot expect that weak solutions are locally bounded because of De Giorgi's counterexample. Here we assume a condition on the support of off-diagonal coefficients that "keeps away" the counterexample and allows us to prove local boundedness of weak solutions.

Keywords: Quasilinear, elliptic, system, weak, solution, regularity.

2010 Mathematics Subject Classification: Primary 35J47; secondary 35B65, 49N60.

1. Introduction

We consider quasilinear elliptic systems in divergence form

$$-div(a(x,u(x))Du(x)) = 0, \quad x \in \Omega, \tag{1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N^2n^2}$ is matrix valued with components $a_{i,j}^{\alpha,\beta}(x,y)$ where $i,j \in \{1,...,n\}$ and $\alpha,\beta \in \{1,...,N\}$. Note that (1) is a system of N equations

$$-\sum_{i=1}^{n} D_i\left(\sum_{j=1}^{n} a_{i,j}^{1,1}(x,u)D_ju^1 + \sum_{j=1}^{n} a_{i,j}^{1,2}(x,u)D_ju^2 + ... + \sum_{j=1}^{n} a_{i,j}^{1,N}(x,u)D_ju^N\right) = 0$$

$$-\sum_{i=1}^{n} D_i\left(\sum_{j=1}^{n} a_{i,j}^{2,1}(x,u)D_ju^1 + \sum_{j=1}^{n} a_{i,j}^{2,2}(x,u)D_ju^2 + ... + \sum_{j=1}^{n} a_{i,j}^{2,N}(x,u)D_ju^N\right) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{2}$$

$$-\sum_{i=1}^{n} D_i\left(\sum_{j=1}^{n} a_{i,j}^{N,1}(x,u)D_ju^1 + \sum_{j=1}^{n} a_{i,j}^{N,2}(x,u)D_ju^2 + ... + \sum_{j=1}^{n} a_{i,j}^{N,N}(x,u)D_ju^N\right) = 0$$

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

Let us assume that coefficients $a_{i,j}^{\alpha,\beta}(x,y)$ are measurable with respect to x, continuous with respect to y, bounded and elliptic. When $N=1$, we have only one equation and the celebrated De Giorgi-Nash-Moser theorem forces weak solutions $u \in W^{1,2}(\Omega)$ to be locally bounded and even Hölder continuous, see section 2.1 in [17]. The result is no longer true, in general, for systems: De Giorgi's counterexample shows that $u(x) = x/|x|^\gamma$ is a weak solution to a particular system (2) where Ω is the ball centered at the origin with radius 1 and $\gamma > 1$ is a suitable exponent; it turns out that u cannot be bounded inside Ω near the origin; see [3], section 3 in [17] and the recent paper [18]; see also [21] and [11]. Now the effort is finding additional restrictions on the coefficients $a_{i,j}^{\alpha,\beta}$ that keep away De Giorgi's counterexample and allow for local boundedness of weak solutions u. The easiest case happens when off-diagonal coefficients vanish, that is

$$a_{i,j}^{\alpha,\beta} = 0 \quad \text{for} \quad \beta \neq \alpha. \tag{3}$$

In such a case, the α row of the system is

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} a_{i,j}^{\alpha,\alpha}(x,u(x)) D_j u^\alpha(x) \right) = 0, \tag{4}$$

so, if $b_{i,j}(x) = a_{i,j}^{\alpha,\alpha}(x,u(x))$ and $v(x) = u^\alpha(x)$, then we are in the linear scalar case

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} b_{i,j}(x) D_j v(x) \right) = 0 \tag{5}$$

and v turns out to be locally bounded and Hölder continuous. A further step has been made in [22]: the system (2) is assumed to be tridiagonal, that is

$$a_{i,j}^{\alpha,\beta} = 0 \quad \text{for} \quad \beta > \alpha. \tag{6}$$

In such a case the system (2) becomes

$$\begin{aligned}
&-\sum_{i=1}^{n} D_i \Bigg(\sum_{j=1}^{n} a_{i,j}^{1,1}(x,u) D_j u^1 \Bigg) = 0 \\
&-\sum_{i=1}^{n} D_i \Bigg(\sum_{j=1}^{n} a_{i,j}^{2,1}(x,u) D_j u^1 + \sum_{j=1}^{n} a_{i,j}^{2,2}(x,u) D_j u^2 \Bigg) = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (7) \\
&-\sum_{i=1}^{n} D_i \Bigg(\sum_{j=1}^{n} a_{i,j}^{N,1}(x,u) D_j u^1 + \sum_{j=1}^{n} a_{i,j}^{N,2}(x,u) D_j u^2 + \ldots + \sum_{j=1}^{n} a_{i,j}^{N,N}(x,u) D_j u^N \Bigg) = 0
\end{aligned}$$

Then we can apply to the first equation the regularity for scalar case: Hölder continuity for u^1 and suitable decay on balls for Du^1. Now the second row can be written as follow

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} a_{i,j}^{2,2}(x,u) D_j u^2 \right) = \sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} a_{i,j}^{2,1}(x,u) D_j u^1 \right); \tag{8}$$

the good behaviour of Du^1 on the right hand side can be transferred to the left hand side so that u^2 inherits Hölder continuity and Du^2 gets a suitable decay on balls.

The procedure can be iterated until we arrive at u^N. Another step has been made in [16] where the local boundedness is obtained under the following structure assumption: there exist numbers $\lambda > 0$, $L \geq 0$ and two nonnegative functions $d(x), g(x)$, such that

$$\sum_{\alpha=1}^{N}\sum_{\gamma=1}^{N}\frac{y^\alpha y^\gamma}{|y|^2}\left(\sum_{i=1}^{n}p_i^\gamma\sum_{\beta=1}^{N}\sum_{j=1}^{n}a_{i,j}^{\alpha,\beta}(x,y)p_j^\beta\right) \geq -\left\{\delta|p|^2+\left(\frac{1}{\delta}\right)^\lambda[d(x)|y|^2+g(x)]\right\} \tag{9}$$

is fulfilled for all $\delta \in (0,1)$ and all (x,y,p), with $|y| > L$. In the present work we assume a condition on the support of off-diagonal coefficients: there exists $L_0 \in (0,+\infty)$ such that $\forall\ L \geq L_0$, when $\alpha \neq \beta$,

$$\begin{aligned}&(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } y^\alpha > L) \Rightarrow y^\beta > L,\\ &(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } y^\alpha < -L) \Rightarrow y^\beta < -L\end{aligned} \tag{10}$$

(see Figure 2.1). Under such a restriction we are able to prove local boundedness of weak solutions. All the necessary assumptions and the result will be listed in section 2 while proofs will be performed in section 3. Let us mention that off-diagonal coefficients with a particular support have been successfully used when proving maximum principles in [12] and when obtaining existence for measure data problems in [14], [15]. It is worth mentioning that, when the ratio between the largest and the smallest eigenvalues of $a_{i,j}^{\alpha,\beta}$ is close to 1, then regularity of u is studied at page 183 of [7]; see also [20], [10], [9], [13]. Let us also say that proving boundedness for weak solutions could be an important tool for getting fractional differentiability, see the estimate after (4.15) in [4]; sometimes, a gain in fractional differentiability can be iterated as in Theorem 3.III of [1] and in Theorem 3.3 of [5].

2. Assumptions and result

Assume Ω is an open bounded subset of $\mathbb{R}^n$, with $n \geq 2$. Consider the system of $N \geq 2$ equations

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u)\frac{\partial}{\partial x_j}u^\beta\right) = 0 \text{ in } \Omega, \text{ for } \alpha = 1,...,N. \tag{11}$$

Note that u^β is the β component of $u = (u^1, u^2, ..., u^N)$. We list our structural conditions.

($\mathcal{A}$) *For all $i,j \in \{1,...,n\}$ and all $\alpha,\beta \in \{1,...,N\}$, we require that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:*

($\mathcal{A}_0$) *$x \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is continuous;*

($\mathcal{A}_1$) *(boundedness of all the coefficients) for some positive constant $c > 0$, we have*

$$|a_{i,j}^{\alpha,\beta}(x,y)| \leq c$$

for almost all $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

($\mathcal{A}_2$) *(ellipticity of all the coefficients) for some positive constant $\nu > 0$, we have*

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y)\,\xi_i^\alpha \xi_j^\beta \geq \nu|\xi|^2$$

for almost all $x \in \Omega$, for all $y \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^{N\times n}$;

($\mathcal{A}_3$) *(support of off-diagonal coefficients) there exists $L_0 \in (0,+\infty)$ such that for all $L \geq L_0$, when $\alpha \neq \beta$,*

$$(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } y^\alpha > L) \Rightarrow y^\beta > L\,, \qquad (\mathcal{A}_3')$$
$$(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } y^\alpha < -L) \Rightarrow y^\beta < -L\,. \qquad (\mathcal{A}_3'')$$

(*see Figure* 2.1).

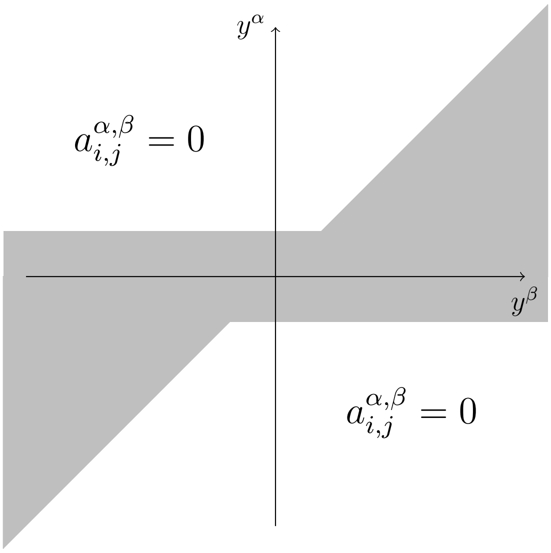

Figure 2.1: Assumption ($\mathcal{A}_3$): off-diagonal entries $a_{i,j}^{\alpha,\beta}$ vanish on the white part of the picture; they might be non zero only on the grey part.

We say that a function $u : \Omega \to \mathbb{R}^N$ is a *weak solution* of the system (11), if $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ and

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx = 0, \tag{12}$$

for all $\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$.

Theorem 2.1. *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution of system (11) under the set ($\mathcal{A}$) of assumptions. Then $u \in L^\infty_{loc}(\Omega,\mathbb{R}^N)$.*

3. Proof of the result

The proof of the Theorem 2.1 will be performed in several steps.

Step 1. Caccioppoli inequality

Theorem 3.1. *(Caccioppoli inequality on superlevel sets) Let* $u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right)$ *be a weak solution of system* (11) *under the assumptions* ($\mathcal{A}_0$), ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{A}'_3$). *For* $0 < s < t$, *let* $B(x_0, s)$ *and* $B(x_0, t)$ *be concentric open balls centered at* x_0 *with radii* s *and* t *respectively. Assume that* $B(x_0, t) \subset \Omega$ *and* $L \geq L_0$. *Then*

$$\sum_{\alpha=1}^{N} \int\limits_{\{u^\alpha > L\} \cap B(x_0,s)} |D\, u^\alpha|^2 \, dx \leq \frac{16 c^2 n^4 N^4}{\nu^2} \sum_{\alpha=1}^{N} \int\limits_{\{u^\alpha > L\} \cap B(x_0,t)} \left(\frac{u^\alpha - L}{t-s} \right)^2 dx, \qquad (13)$$

where c *is the constant involved in assumption (*$\mathcal{A}_1$*),* ν *is given in (*$\mathcal{A}_2$*) and* L_0 *appears in (*$\mathcal{A}_3$*).*

Proof of Theorem 3.1. Let $u \in W^{1,2}\left(\Omega, \mathbb{R}^N\right)$ be a weak solution of system (11). Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be the standard cut-off function such that $0 \leq \eta \leq 1$, $\eta \in C_0^1(B(x_0, t))$, with $B(x_0, t) \subset \Omega$ and $\eta = 1$ in $B(x_0, s)$. Moreover, $|D\eta| \leq 2/(t-s)$ in $\mathbb{R}^n$. For every level $L \geq L_0$, consider the test function $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ with $\varphi = (\varphi^1, ..., \varphi^N)$,

where $\qquad \varphi^\alpha(x) := \eta^2(x) \max\{0, u^\alpha(x) - L\}, \quad \text{for all } \alpha \in \{1, ..., N\}.$

Then $D_i\, \varphi^\alpha = \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i\, u^\alpha + 2\eta (D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)$ for all $i \in \{1, ..., n\}$ and $\alpha \in \{1, ..., N\}$.
Using this test function in the weak formulation (12) of system (11), we obtain

$$0 = \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta D_i \varphi^\alpha \, dx = \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i\, u^\alpha \, dx$$
$$+ \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta 2\eta (D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) dx.$$

Now, assumption ($\mathcal{A}'_3$) guarantees that

$$a_{i,j}^{\alpha,\beta}(x, u(x)) \mathbf{1}_{\{u^\alpha > L\}}(x) = a_{i,j}^{\alpha,\beta}(x, u(x)) \mathbf{1}_{\{u^\beta > L\}}(x) \mathbf{1}_{\{u^\alpha > L\}}(x) \qquad (14)$$

when $\beta \neq \alpha$ and $L \geq L_0$. It is worthwhile to note that (14) holds true when $\alpha = \beta$ as well; then

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i\, u^\alpha \, dx$$
$$= - \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta 2\eta (D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) \, dx. \qquad (15)$$

Now we can use the ellipticity assumption ($\mathcal{A}_2$) with $\xi_i^\alpha = \mathbf{1}_{\{u^\alpha > L\}} D_i\, u^\alpha$ and we get

$$\nu \int_\Omega \eta^2 \sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} |D\, u^\alpha|^2 \, dx \leq \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i\, u^\alpha \, dx. \quad (16)$$

Moreover

$$
\begin{aligned}
&- \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta 2\eta (D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)\, dx \\
&\le \int_\Omega c \sum_{\beta=1}^{N} \sum_{j=1}^{n} \mathbf{1}_{\{u^\beta > L\}} |D_j u^\beta| \sum_{\alpha=1}^{N} \sum_{i=1}^{n} 2\eta |D_i \eta| \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)\, dx \\
&\le \int_\Omega c \sum_{\beta=1}^{N} n \mathbf{1}_{\{u^\beta > L\}} |D u^\beta| \sum_{\alpha=1}^{N} n 2\eta |D\eta| \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)\, dx \\
&\le \int_\Omega c n^2 \epsilon \eta^2 \left(\sum_{\beta=1}^{N} \mathbf{1}_{\{u^\beta > L\}} |D u^\beta| \right)^2 + \int_\Omega \frac{c n^2}{\epsilon} |D\eta|^2 \left(\sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) \right)^2 dx \\
&\le \int_\Omega c n^2 N^2 \epsilon \eta^2 \sum_{\beta=1}^{N} \mathbf{1}_{\{u^\beta > L\}} |D u^\beta|^2 + \int_\Omega \frac{c n^2 N^2}{\epsilon} |D\eta|^2 \sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)^2\, dx, \qquad (17)
\end{aligned}
$$

where we used the inequality $2ab \le \epsilon a^2 + b^2/\epsilon$, provided $\epsilon > 0$. Merging (16) and (17) into (15) we get

$$
\begin{aligned}
&\nu \int_\Omega \eta^2 \sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} |D\, u^\alpha|^2\, dx \\
&\le \int_\Omega c n^2 N^2 \epsilon \eta^2 \sum_{\beta=1}^{N} \mathbf{1}_{\{u^\beta > L\}} |D u^\beta|^2 + \int_\Omega \frac{c n^2 N^2}{\epsilon} |D\eta|^2 \sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)^2\, dx.
\end{aligned}
$$

We take $\epsilon = \nu/(2cn^2N^2)$ and we obtain

$$
\frac{\nu}{2} \int_\Omega \eta^2 \sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} |D\, u^\alpha|^2\, dx \le \int_\Omega \frac{2c^2 n^4 N^4}{\nu} |D\eta|^2 \sum_{\alpha=1}^{N} \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)^2\, dx.
$$

Using the properties of the cut off function η we get

$$
\sum_{\alpha=1}^{N} \int_{\{u^\alpha > L\} \cap B(x_0,s)} |D\, u^\alpha|^2\, dx \le \frac{16 c^2 n^4 N^4}{\nu^2} \sum_{\alpha=1}^{N} \int_{\{u^\alpha > L\} \cap B(x_0,t)} \left(\frac{u^\alpha - L}{t - s} \right)^2 dx. \qquad (18)
$$

This ends the proof of Theorem 3.1. □

The "excess" on superlevel sets

In the previous Caccioppoli inequality the following sum appears on the right hand side:

$$
\sum_{\alpha=1}^{N} \int_{\{u^\alpha > L\} \cap B(x_0,t)} (u^\alpha - L)^2\, dx. \qquad (19)
$$

Note that the sum (19) is zero if and only if all the superlevel sets have zero measure, that is $|\{u^1 > L\}| = 0$, $|\{u^2 > L\}| = 0$, ... , $|\{u^N > L\}| = 0$, where $|A|$ is the n-dimensional Lebesgue measure of $A \subset \mathbb{R}^n$.

This happens when $L \geq \max\{ \operatorname{esssup} u^1, \operatorname{esssup} u^2, ..., \operatorname{esssup} u^N\}$. On the contrary, if $L < \max\{ \operatorname{esssup} u^1, \operatorname{esssup} u^2, ..., \operatorname{esssup} u^N\}$, then the sum (19) is positive. Moreover,

$$L \mapsto \sum_{\alpha=1}^{N} \int_{\{u^\alpha > L\} \cap B(x_0,t)} (u^\alpha - L)^2 \, dx \qquad \text{decreases.} \tag{20}$$

A sum of the form of (19) measures how much L is far from

$$\max\{ \operatorname{esssup} u^1, \operatorname{esssup} u^2, ..., \operatorname{esssup} u^N\}.$$

Let us call (19) the *excess* of u with respect to the level L, the "excess" for short. We aim to show that the excess is zero for a suitable level L. We first show that the excess at level L_2 can be estimated by means of a power σ of the excess at level L_1, for a suitable pair of levels $L_2 > L_1$. Then we iterate the procedure.

Step 2. Decay of the "excess" on superlevel sets

In general, we consider a vector valued function $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $n \geq 2$, with $v = (v^1, ..., v^N)$ and $v \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$, $p \geq 1$, where Ω is an open set in $\mathbb{R}^n$. We fix $B_{R_0} = B(x_0, R_0) \Subset \Omega$, with $R_0 < 1$ small enough so that

$$|B_{R_0}| < 1 \quad \text{and} \quad \sum_{\alpha=1}^{N} \int_{B_{R_0}} |v^\alpha|^{p^*} \, dx < 1, \tag{21}$$

where $A \Subset \Omega$ means that the closure $\overline{A}$ is a compact set contained in Ω; moreover, $p^* = \frac{np}{n-p}$, if $p < n$, and p^* is any $q > p$, else. For every $R \in (0, R_0]$ we define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2}\left(1 + \frac{1}{2^h}\right), \qquad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2}\left(1 + \frac{3}{4 \cdot 2^h}\right).$$

Fix a positive constant $d \geq 1$ and define the increasing sequence of positive real numbers

$$k_h := d\left(1 - \frac{1}{2^{h+1}}\right), \; h \in \{0, 1, 2, ...\}. \tag{22}$$

Moreover, define the sequence (J_h),

$$J_h := \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_h, \rho_h}} (v^\alpha - k_h)^{p^*} \, dx,$$

where $A^\alpha_{k,\rho} = \{v^\alpha > k\} \cap B_\rho$ and $B_\rho = B(x_0, \rho)$. The following result holds.

Proposition 3.2. (Decay of the excess from step h to step h+1) *Let* $v \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$, $p \geq 1$. *Fix* $B(x_0, R_0) \Subset \Omega$, *with* $R_0 < 1$ *small enough such that* (21) *holds. If there exist* $0 \leq \vartheta \leq 1$ *and* $c_0 > 0$ *such that, for every* $0 < s < t \leq R_0$ *and for every* $h \in \{0, 1, 2, ...\}$,

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{k_h, s}} |Dv^\alpha|^p \, dx \leq c_0 \sum_{\alpha=1}^{N} \left\{ \int_{A^\alpha_{k_h, t}} \left(\frac{v^\alpha - k_h}{t - s}\right)^{p^*} dx + |A^\alpha_{k_h, t}|^\vartheta \right\}, \tag{23}$$

then, for every $R \in (0, R_0]$ and for every $h \in \{0, 1, 2, ...\}$,

$$J_{h+1} \leq c(\vartheta, R) \left(2^{\frac{p^* p^*}{p}}\right)^h (J_h)^{\vartheta \frac{p^*}{p}},$$

with the positive constant $c(\vartheta, R)$ independent of h.

Remark 3.3. We want to stress that the exponent on the right hand side is p^* larger than the exponent p on the left hand side: this situation has been studied in the scalar case $N = 1$ in [2], [6], [19].

Proof. Notice that (J_h) is a decreasing sequence, since the following chain of inequalities holds:

$$\begin{aligned} J_{h+1} &= \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_{h+1}}} (v^\alpha - k_{h+1})^{p^*} \, dx \leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p^*} \, dx \\ &\leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_h)^{p^*} \, dx \leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_h,\rho_h}} (v^\alpha - k_h)^{p^*} \, dx = J_h. \end{aligned} \tag{24}$$

Let us now define a sequence (ζ_h) of cut-off functions in $C^1_0(B(x_0, \bar\rho_h))$, such that $0 \leq \zeta_h \leq 1$, $\zeta_h \equiv 1$ in $B_{\rho_{h+1}}$, $|D\zeta_h| \leq \frac{2^{h+4}}{R}$. If we denote $(v - k_{h+1})_+ = \max\{v - k_{h+1}, 0\}$ we get

$$\begin{aligned} J_{h+1} &= \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_{h+1}}} (v^\alpha - k_{h+1})^{p^*} \zeta_h^{p^*} \, dx \leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\bar\rho_h}} (v^\alpha - k_{h+1})^{p^*} \zeta_h^{p^*} \, dx \\ &= \sum_{\alpha=1}^{N} \int_{B_R} (\zeta_h (v^\alpha - k_{h+1})_+)^{p^*} \, dx. \end{aligned} \tag{25}$$

The Sobolev Embedding Theorem and the properties of ζ_h yield

$$\begin{aligned} &\int_{B_R} (\zeta_h (v^\alpha - k_{h+1})_+)^{p^*} \, dx \leq c \left(\int_{B_R} |D(\zeta_h (v^\alpha - k_{h+1})_+)|^p \, dx \right)^{\frac{p^*}{p}} \\ &\leq c \left\{ \left(\int_{B_R} |Dv^\alpha \zeta_h|^p \, 1_{\{v^\alpha > k_{h+1}\}} \, dx \right)^{\frac{1}{p}} + \left(\int_{B_R} |(v^\alpha - k_{h+1})_+ D\zeta_h|^p \, dx \right)^{\frac{1}{p}} \right\}^{p^*} \\ &\leq c \left\{ \left(\int_{A^\alpha_{k_{h+1},\bar\rho_h}} |Dv^\alpha|^p \, dx \right)^{\frac{1}{p}} + \left(\left(\frac{2^h}{R}\right)^p \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^p \, dx \right)^{\frac{1}{p}} \right\}^{p^*}. \end{aligned} \tag{26}$$

Note that, when $p = n$, we used $|B_R| \leq 1$ (see (21)) in the Sobolev inequality.

Substituting $t = \rho_h$ and $s = \bar\rho_h$ in (23) we deduce

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\bar\rho_h}} |Dv^\alpha|^p \, dx \leq c \sum_{\alpha=1}^{N} \left\{ \left(\frac{2^h}{R}\right)^{p^*} \int_{A^\alpha_{k_{h+1},\rho_h}} |v^\alpha - k_{h+1}|^{p^*} \, dx + |A^\alpha_{k_{h+1},\rho_h}|^\vartheta \right\}. \tag{27}$$

Collecting (25), (26), (27), we obtain

$$J_{h+1} \leq c \left\{ \sum_{\alpha=1}^{N} \left(\frac{2^h}{R} \right)^{p^*} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p^*} \, dx + \sum_{\alpha=1}^{N} |A^\alpha_{k_{h+1},\rho_h}|^\vartheta + \right.$$
$$\left. + \sum_{\alpha=1}^{N} \left(\frac{2^h}{R} \right)^{p} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p} \, dx \right\}^{\frac{p^*}{p}} . \tag{28}$$

Since $z^p \leq z^{p^*} + 1$ for every $z \geq 0$, then

$$\left(\frac{2^h}{R} \right)^{p} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p} \, dx \leq \left(\frac{2^h}{R} \right)^{p^*} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p^*} \, dx + |A^\alpha_{k_{h+1},\rho_h}|,$$

and we obtain

$$J_{h+1} \leq c \left\{ \sum_{\alpha=1}^{N} \left(\frac{2^h}{R} \right)^{p^*} \int\limits_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p^*} \, dx + \sum_{\alpha=1}^{N} |A^\alpha_{k_{h+1},\rho_h}|^\vartheta + \sum_{\alpha=1}^{N} |A^\alpha_{k_{h+1},\rho_h}| \right\}^{\frac{p^*}{p}} . \tag{29}$$

Since $$\sum_{\alpha=1}^{N} |A^\alpha_{k_{h+1},\rho_h}| (k_{h+1} - k_h)^{p^*} \leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_h)^{p^*} \, dx \leq J_h,$$

then $$|A^\beta_{k_{h+1},\rho_h}| \leq \sum_{\alpha=1}^{N} |A^\alpha_{k_{h+1},\rho_h}| \leq \frac{J_h}{(k_{h+1} - k_h)^{p^*}} = \left(\frac{2^{h+2}}{d} \right)^{p^*} J_h.$$

Taking also into account that (see (24))

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_{h+1})^{p^*} \, dx \leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_{h+1},\rho_h}} (v^\alpha - k_h)^{p^*} \, dx \leq J_h,$$

inequality (29) gives

$$J_{h+1} \leq c \left\{ \left(\frac{2^h}{R} \right)^{p^*} J_h + \left(\frac{2^h}{d} \right)^{\vartheta p^*} (J_h)^\vartheta + \left(\frac{2^h}{d} \right)^{p^*} J_h \right\}^{\frac{p^*}{p}} . \tag{30}$$

We keep in mind that J_h is decreasing and $k_0 = d/2 > 0$, so

$$J_h \leq J_0 = \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_0,\rho_0}} (v^\alpha - k_0)^{p^*} \, dx \leq \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_0,\rho_0}} (v^\alpha)^{p^*} \, dx \leq \sum_{\alpha=1}^{N} \int_{B_R} |v^\alpha|^{p^*} \, dx \leq 1,$$

where we used (21). Since $J_h \leq 1$ for every h and recalling that $d \geq 1 > R_0 \geq R$, we get

$$\left(\frac{2^h}{R} \right)^{p^*} J_h + \left(\frac{2^h}{d} \right)^{\vartheta p^*} (J_h)^\vartheta + \left(\frac{2^h}{d} \right)^{p^*} J_h$$
$$\leq \left\{ 2 \frac{2^{hp^*}}{R^{p^*}} + \frac{2^{h\vartheta p^*}}{R^{\vartheta p^*}} \right\} (J_h)^\vartheta \leq \left(\frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}} \right) 2^{hp^*} (J_h)^\vartheta.$$

By (30) it follows

$$J_{h+1} \leq c \left\{ \left(\frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}} \right) 2^{hp^*} (J_h)^\vartheta \right\}^{\frac{p^*}{p}} \leq c(\vartheta, R) \left(2^{\frac{p^* p^*}{p}} \right)^h (J_h)^{\vartheta \frac{p^*}{p}} . \qquad \square$$

Step 3. Iteration

We need the following classical result, see e.g. [8].

Lemma 3.4. *Let $\gamma > 0$ and let $J_h \in [0, +\infty)$ be such that*

$$J_{h+1} \le A\,\lambda^h J_h^{1+\gamma} \qquad \forall h \in \mathbb{N} \cup \{0\}, \tag{31}$$

with $A > 0$ and $\lambda > 1$. If $J_0 \le A^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma^2}}$, then $\lim_{h\to\infty} J_h = 0$.

Step 4. Conclusion

We have got all we need to give the proof of Theorem 2.1. Fix $B_{R_0} = B(x_0, R_0) \Subset \Omega$, with $R_0 < 1$ small enough such that $|B_{R_0}| < 1$ and $\int_{B_{R_0}} |u|^{p^*}\, dx \le 1$. From (13) we have that, for every $0 < s < t \le R_0$ and every h, u satisfies

$$\sum_{\alpha=1}^{N} \int_{A^\alpha_{k_h,s}} |D\, u^\alpha|^2\, dx \le c_0 \sum_{\alpha=1}^{N} \left\{ \int_{A^\alpha_{k_h,t}} \left(\frac{u^\alpha - k_h}{t-s} \right)^{2^*} dx + |A^\alpha_{k_h,t}| \right\}, \tag{32}$$

where $c_0 > 0$ is independent of s, t, h, provided $L_0 \le d/2$. Therefore u satisfies (23) of Proposition 3.2 with $p = 2$ and $\vartheta = 1$. Then Proposition 3.2, applied to u, gives

$$J_{h+1} \le c(R) \left(2^{\frac{2^* 2^*}{2}} \right)^h (J_h)^{\frac{2^*}{2}}, \tag{33}$$

with the positive constant $c(R)$ independent of h. Let us note that

$$J_0 := \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_0,\rho_0}} (u^\alpha - k_0)^{2^*}\, dx = \sum_{\alpha=1}^{N} \int_{A^\alpha_{\frac{d}{2},R}} \left(u^\alpha - \frac{d}{2} \right)^{2^*} dx \le \sum_{\alpha=1}^{N} \int_{A^\alpha_{\frac{d}{2},R}} |u^\alpha|^{2^*}\, dx,$$

and so

$$J_0 \to 0 \quad \text{when } d \to +\infty\,.$$

Therefore, we can choose $d > 0$ large enough such that

$$J_0 < c(R)^{-\frac{1}{\frac{2^*}{2}-1}} \left(2^{\frac{2^* 2^*}{2}} \right)^{-\frac{1}{(\frac{2^*}{2}-1)^2}}.$$

Thus, by Lemma 3.4 we deduce that $\lim_{h\to\infty} J_h = 0$; since

$$\begin{aligned} J_h = \sum_{\alpha=1}^{N} \int_{A^\alpha_{k_h,\rho_h}} (u^\alpha - k_h)^{2^*}\, dx &\ge \sum_{\alpha=1}^{N} \int_{\{u^\alpha > k_h\} \cap B_{R/2}} (u^\alpha - k_h)^{2^*}\, dx \\ &\ge \sum_{\alpha=1}^{N} \int_{\{u^\alpha > d\} \cap B_{R/2}} (u^\alpha - k_h)^{2^*}\, dx \ge \sum_{\alpha=1}^{N} \int_{\{u^\alpha > d\} \cap B_{R/2}} (u^\alpha - d)^{2^*}\, dx, \end{aligned}$$

we deduce that $|\{u^\alpha > d\} \cap B_{R/2}| = 0$, namely $u^\alpha \le d$ a.e. in $B_{\frac{R}{2}}$. We have so proved that u^α is locally bounded from above.

Now, let $\tilde{u} = -u$. Then $\tilde{u} \in W^{1,2}(\Omega; \mathbb{R}^N)$ and, since u satifies (12), $\tilde{u}$ satisfies

$$0 = \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \tilde{a}^{\alpha,\beta}_{i,j}(x, \tilde{u}(x)) D_j \tilde{u}^\beta(x) D_i \varphi^\alpha(x) dx \tag{34}$$

for every $\varphi \in W_0^{1,2}\left(\Omega, \mathbb{R}^N\right)$, where

$$\tilde{a}_{i,j}^{\alpha,\beta}(x,y) := a_{i,j}^{\alpha,\beta}(x,-y). \tag{35}$$

We observe that the new coefficients, defined by (35), readily satisfy conditions $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$. Moreover, if $\alpha \neq \beta$ the coefficients $\tilde{a}_{i,j}^{\alpha,\beta}(x,y)$ satisfy $(\mathcal{A}_3')$ provided the $a_{i,j}^{\alpha,\beta}(x,y)$ satisfy $(\mathcal{A}_3'')$. Therefore, we can argue as above on $\tilde{u}$ obtaining the estimate from below for u^α. This ends the proof of Theorem 2.1. □

4. An example

Let us take $N = 2$ and $n = 3$; we define the matrices $a^{\alpha,\beta} \equiv a^{\alpha,\beta}(y)$ ($\alpha, \beta \in \{1,2\}$ and $y = (y^1, y^2)$) as

$$a^{1,1} := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a^{1,2} := \begin{pmatrix} b(y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$a^{2,1} := \begin{pmatrix} 0 & w(y) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a^{2,2} := \begin{pmatrix} 27 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $b : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that $0 \le b(y^1, y^2) \le 2$, $b(k, k+1) = 2$ for every integer $k \ge 2$, $b(0,0) = 2$ and the support of b is contained in the grey part of Figure 2.1 with $\alpha = 1$ and $\beta = 2$; moreover, $w : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that $-10 \le w(y^1, y^2) \le 0$, $w(k+1, k) = -10$ for every integer $k \ge 2$, $w(0,0) = -10$ and the support of w is contained in the grey part of Figure 2.1 with $\alpha = 2$ and $\beta = 1$. It easy to check that assumptions $(\mathcal{A}_0)$–$(\mathcal{A}_3)$ are satisfied with $c = 27$ and $\nu = 1$. For the convenience of the reader, let us do the calculations for the ellipticity $(\mathcal{A}_2)$.

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y)\, \xi_i^\alpha \xi_j^\beta \\
&= \sum_{i,j=1}^{n} a_{i,j}^{1,1}(x,y)\, \xi_i^1 \xi_j^1 + \sum_{i,j=1}^{n} a_{i,j}^{1,2}(x,y)\, \xi_i^1 \xi_j^2 + \sum_{i,j=1}^{n} a_{i,j}^{2,1}(x,y)\, \xi_i^2 \xi_j^1 + \sum_{i,j=1}^{n} a_{i,j}^{2,2}(x,y)\, \xi_i^2 \xi_j^2 \\
&= a_{1,1}^{1,1}(x,y)\, \xi_1^1 \xi_1^1 + a_{2,2}^{1,1}(x,y)\, \xi_2^1 \xi_2^1 + a_{3,3}^{1,1}(x,y)\, \xi_3^1 \xi_3^1 + a_{1,1}^{1,2}(x,y)\, \xi_1^1 \xi_1^2 \\
&\qquad + a_{1,2}^{2,1}(x,y)\, \xi_1^2 \xi_2^1 + a_{1,1}^{2,2}(x,y)\, \xi_1^2 \xi_1^2 + a_{2,2}^{2,2}(x,y)\, \xi_2^2 \xi_2^2 + a_{3,3}^{2,2}(x,y)\, \xi_3^2 \xi_3^2 \\
&= 2\xi_1^1\xi_1^1 + 2\xi_2^1\xi_2^1 + 1\xi_3^1\xi_3^1 + b(y)\, \xi_1^1\xi_1^2 + w(y)\, \xi_1^2\xi_2^1 + 27\xi_1^2\xi_1^2 + 1\xi_2^2\xi_2^2 + 1\xi_3^2\xi_3^2 \\
&\ge 2\xi_1^1\xi_1^1 + 2\xi_2^1\xi_2^1 + 1\xi_3^1\xi_3^1 - 2|\xi_1^1||\xi_1^2| - 10|\xi_1^2||\xi_2^1| + 27\xi_1^2\xi_1^2 + 1\xi_2^2\xi_2^2 + 1\xi_3^2\xi_3^2 \\
&\ge 2\xi_1^1\xi_1^1 + 2\xi_2^1\xi_2^1 + 1\xi_3^1\xi_3^1 - \xi_1^1\xi_1^1 - \xi_1^2\xi_1^2 - \frac{5}{\epsilon}\xi_1^2\xi_1^2 - 5\epsilon\xi_2^1\xi_2^1 + 27\xi_1^2\xi_1^2 + 1\xi_2^2\xi_2^2 + 1\xi_3^2\xi_3^2 \\
&= 1\xi_1^1\xi_1^1 + 2\xi_2^1\xi_2^1 + 1\xi_3^1\xi_3^1 - \xi_1^2\xi_1^2 - 25\xi_1^2\xi_1^2 - 1\xi_2^1\xi_2^1 + 27\xi_1^2\xi_1^2 + 1\xi_2^2\xi_2^2 + 1\xi_3^2\xi_3^2 \\
&= 1\xi_1^1\xi_1^1 + 1\xi_2^1\xi_2^1 + 1\xi_3^1\xi_3^1 + 1\xi_1^2\xi_1^2 + 1\xi_2^2\xi_2^2 + 1\xi_3^2\xi_3^2 = |\xi|^2,
\end{aligned} \tag{36}$$

where we used the inequality $2AB \le A^2 + B^2$ and $2AB \le \frac{A^2}{\epsilon} + \epsilon B^2$ with $\epsilon = \frac{1}{5}$.

On the other hand, this example satisfies neither assumption (3) nor assumption (6) since the two matrices $a^{1,2}$ and $a^{2,1}$ are not zero. Moreover, (9) does not hold true. Indeed, for every integer $k \geq 2$, let us take

$$y^1 = k+1, \quad y^2 = k, \quad p_1^1 = p_2^1 = t > 0, \quad p_i^2 = 0 = p_3^\alpha \tag{37}$$

and let us compute the left hand side of (9); we have

$$\begin{aligned}
I &= \sum_{\alpha=1}^{N} \sum_{\gamma=1}^{N} \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^{n} p_i^\gamma \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y) p_j^\beta \right) \\
&= \sum_{\alpha=1}^{2} \sum_{\gamma=1}^{2} \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^{3} p_i^\gamma \sum_{\beta=1}^{2} \left(a_{i,1}^{\alpha,\beta}(y) p_1^\beta + a_{i,2}^{\alpha,\beta}(y) p_2^\beta \right) \right) \\
&= \sum_{\alpha=1}^{2} \sum_{\gamma=1}^{2} \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^{3} p_i^\gamma \left(a_{i,1}^{\alpha,1}(y) p_1^1 + a_{i,2}^{\alpha,1}(y) p_2^1 \right) \right) \\
&= \sum_{\alpha=1}^{2} \sum_{\gamma=1}^{2} \frac{y^\alpha y^\gamma}{|y|^2} \left(p_1^\gamma \left(a_{1,1}^{\alpha,1}(y) p_1^1 + a_{1,2}^{\alpha,1}(y) p_2^1 \right) + p_2^\gamma \left(a_{2,1}^{\alpha,1}(y) p_1^1 + a_{2,2}^{\alpha,1}(y) p_2^1 \right) \right) \\
&= \sum_{\alpha=1}^{2} \frac{y^\alpha y^1}{|y|^2} \left(p_1^1 \left(a_{1,1}^{\alpha,1}(y) p_1^1 + a_{1,2}^{\alpha,1}(y) p_2^1 \right) + p_2^1 \left(a_{2,1}^{\alpha,1}(y) p_1^1 + a_{2,2}^{\alpha,1}(y) p_2^1 \right) \right) \\
&= \frac{y^1 y^1}{|y|^2} \left(p_1^1 \left(a_{1,1}^{1,1}(y) p_1^1 + a_{1,2}^{1,1}(y) p_2^1 \right) + p_2^1 \left(a_{2,1}^{1,1}(y) p_1^1 + a_{2,2}^{1,1}(y) p_2^1 \right) \right) \\
&\qquad + \frac{y^2 y^1}{|y|^2} \left(p_1^1 \left(a_{1,1}^{2,1}(y) p_1^1 + a_{1,2}^{2,1}(y) p_2^1 \right) + p_2^1 \left(a_{2,1}^{2,1}(y) p_1^1 + a_{2,2}^{2,1}(y) p_2^1 \right) \right) \\
&= \frac{y^1 y^1 |t|^2}{|y|^2} \left(\left(a_{1,1}^{1,1}(y) + a_{1,2}^{1,1}(y) \right) + \left(a_{2,1}^{1,1}(y) + a_{2,2}^{1,1}(y) \right) \right) \\
&\qquad + \frac{y^2 y^1 |t|^2}{|y|^2} \left(\left(a_{1,1}^{2,1}(y) + a_{1,2}^{2,1}(y) \right) + \left(a_{2,1}^{2,1}(y) + a_{2,2}^{2,1}(y) \right) \right) \\
&= \frac{(k+1)(k+1)|t|^2}{(k+1)^2 + k^2} 4 + \frac{k(k+1)|t|^2}{(k+1)^2 + k^2} (-10) \\
&= \frac{(-6k+4)(k+1)|t|^2}{(k+1)^2 + k^2} = \frac{(-6k^2 - 2k + 4)|t|^2}{2k^2 + 2k + 1}.
\end{aligned}$$

Now we compute the right hand side of (9); we have

$$\begin{aligned}
Q &= -\left\{ \delta |p|^2 + \left(\frac{1}{\delta} \right)^\lambda [d(x)|y|^2 + g(x)] \right\} \\
&= -\left\{ \delta 2|t|^2 + \left(\frac{1}{\delta} \right)^\lambda [d(x)[(k+1)^2 + k^2] + g(x)] \right\} \\
&= -\delta 2|t|^2 \left\{ 1 + \frac{1}{2|t|^2 \delta^{1+\lambda}} [d(x)[(k+1)^2 + k^2] + g(x)] \right\}.
\end{aligned}$$

Let us take $|t|^2 = \frac{5}{2\delta^{1+\lambda}}[(d(x)+1)[(k+1)^2+k^2]+g(x)]$ so that, since $\delta \in (0,1)$,

$$\delta 2|t|^2 \left\{1 + \frac{1}{2|t|^2\delta^{1+\lambda}}[d(x)[(k+1)^2+k^2]+g(x)]\right\} \leq 2|t|^2\left(1+\frac{1}{5}\right) = \frac{12}{5}|t|^2$$

and $$\frac{-12}{5}|t|^2 \leq -\delta 2|t|^2 \left\{1 + \frac{1}{2|t|^2\delta^{1+\lambda}}[d(x)[(k+1)^2+k^2]+g(x)]\right\} = Q.$$

For every $L > 0$, we take k so large that $|y|^2 = (k+1)^2 + k^2 > L^2$ and

$$\frac{-6k^2-2k+4}{2k^2+2k+1} < \frac{-12}{5}.$$

Then $$I = \frac{(-6k^2-2k+4)|t|^2}{2k^2+2k+1} < \frac{-12}{5}|t|^2 \leq Q$$

and this shows that the example does not satisfy (9). □

Acknowledgement. Leonardi acknowledges support from the Programma Ricerca di Ateneo UNICT 2020-22 linea 2. Leonardi, Leonetti, Pignotti have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Leonetti and Pignotti acknowledge the support from RIA-UNIVAQ. Rocha and Staicu acknowledge the partial support by the Portuguese Foundation for Science and Technology (FCT), through CIDMA – Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019(CIDMA).

References

[1] S. Campanato, P. Cannarsa: *Differentiability and partial Hölder continuity of the solutions of non-linear elliptic systems of order 2m with quadratic growth,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981) 285–309.

[2] G. Cupini, F. Leonetti, E. Mascolo: *Local boundedness for minimizers of some polyconvex integrals*, Arch. Ration. Mech. Analysis 224 (2017) 269–289.

[3] E. De Giorgi: *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Un. Mat. Ital. (4) 1 (1968) 135–137.

[4] L. Esposito, F. Leonetti, G. Mingione: *Regularity for minimizers of functionals with p-q growth*, NoDEA Nonlinear Differential Equations Appl. 6 (1999) 133–148.

[5] G. Floridia, M. A. Ragusa: *Differentiability and partial Hölder continuity of solutions of nonlinear elliptic systems*, J. Convex Analysis 19 (2012) 63–90.

[6] N. Fusco, C. Sbordone: *Some remarks on the regularity of minima of anisotropic integrals*, Comm. Partial Diff. Equations 18 (1993) 153–167.

[7] M. Giaquinta: *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, Princeton (1983).

[8] E. Giusti: *Direct Methods in the Calculus of Variations*, World Scientific Publishing, River Edge (2003).

[9] A. I. Koshelev: *Regularity Problem for Quasilinear Elliptic and Parabolic Systems*, Springer, Berlin (1995).

[10] J. Kottas: *Interior regularity for the quasilinear elliptic systems with nonsmooth coefficients*, Comm. Math. Univ. Carolinae 28 (1997) 95–102.

[11] S. Leonardi: *On constants of some regularity theorems. De Giorgi's type counter-example*, Math. Nachrichten 192 (1998) 191–204.

[12] S. Leonardi, F. Leonetti, C. Pignotti, E. Rocha, V. Staicu: *Maximum principles for some quasilinear elliptic systems*, Nonlinear Analysis 194 (2020) doi.org/10.1016/j.na.2018.11.004.

[13] S. Leonardi, J. Kottas, J. Stara: *Hölder regularity of the solutions of some classes of elliptic systems in convex nonsmooth domains*, Nonlinear Analysis 60 (2005) 925–944.

[14] F. Leonetti, E. Rocha, V. Staicu: *Quasilinear elliptic systems with measure data*, Nonlinear Analysis 154 (2017) 210–224.

[15] F. Leonetti, E. Rocha, V. Staicu: *Smallness and cancellation in some elliptic systems with measure data*, J. Math. Analysis Appl. 465 (2018) 885–902.

[16] M. Meier: *Boundedness and integrability properties of weak solutions of quasilinear elliptic systems*, J. Reine Angew. Math. 333 (1982) 191–220.

[17] G. Mingione: *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math. 51 (2006) 355–426.

[18] C. Mooney, O. Savin: *Some singular minimizers in low dimensions in the calculus of variations*, Arch. Rational Mech. Analysis 221 (2016) 1–22.

[19] G. Moscariello, L. Nania: *Hölder continuity of minimizers of functionals with non standard growth conditions*, Ricerche Mat. 40 (1991) 259–273.

[20] J. Necas: *On the Regularity of Weak Solutions to Nonlinear Elliptic Systems of Partial Differential Equations*, Lectures at the Scuola Normale Superiore, Pisa (1979).

[21] P. Podio Guidugli: *De Giorgi's counterexample in elasticity*, Quart. Appl. Math. 34 (1977) 411–419.

[22] Z. Q. Yan: *Everywhere regularity for solutions to quasilinear elliptic systems of triangular form*, in: *Partial Differential Equations*, Tianjin 1986, Lecture Notes in Mathematics 1306, Springer, Berlin (1988) 255–261.