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ORIGINAL PAPER

Existence and multiplicity of positive solutions for parametric nonlinear nonhomogeneous singular Robin problems

S. Leonardi[1] · Nikolaos S. Papageorgiou[2]

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Abstract
We consider nonlinear Robin problems driven by a nonhomogeneous differential operator and with a reaction that has a singular term and a parametric $(p-1)$-superlinear perturbation which need not satisfy the Ambrosetti–Rabinowitz condition. We are looking for positive solutions. Using variational arguments and a suitable truncation and comparison techniques, we prove a bifurcation-type theorem which describes the set of positive solutions as the parameter $\lambda > 0$ varies. Also we show the for every admissible value of the parameter $\lambda > 0$, the problem has a smallest solution $\bar{u}_\lambda$ and we determine the monotonicity and continuity properties of the map $\lambda \to \bar{u}_\lambda$.

Keywords Nonhomogeneous differential operator · Singular term · $(p-1)$-superlinear parametric perturbation · Nonlinear regularity · Bifurcation-type theorem · Minimal positive solutions · Robin boundary condition

Mathematics Subject Classification 35J92 · 35P30

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous parametric Robin problem with singular terms in the reaction:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma} + \lambda f(z, u(z)), \quad u(z) > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_\alpha} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with $1 < p < +\infty$, $\lambda > 0$ and $0 < \gamma < 1$.

✉ S. Leonardi
leonardi@dmi.unict.it

Nikolaos S. Papageorgiou
npapg@math.ntua.gr

1 Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria, 6, 95125 Catania, Italy

2 Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

The map $a : \mathbb{R}^n \to \mathbb{R}^N$, involved in the definition of the differential operator, is in general nonhomogeneous, monotone, continuous (hence maximal monotone too) and satisfies certain regularity and growth conditions listed below in the hypotheses (i)–(v) (see Sect. 2). The conditions on $a(\cdot)$ are not restrictive and incorporate in our framework many differential operators of interest, such as the p-Laplacean $(1 < p < +\infty)$ and the (p, q)-Laplacean (that is the sum of a p-Laplacean and a q-Laplacean, $1 < q < p < +\infty$). The potential function $\xi(\cdot)$ is in $L^\infty(\Omega)$, $\xi \geq 0$ for a.a. $z \in \Omega$ and $\xi \neq 0$. In the reaction (i.e. the right-hand side of (1.1)), $u^{-\gamma}$ is the singular term while the perturbation term $\lambda f(z, x)$ is parametric, with $\lambda > 0$ being the parameter and $f(z, x)$ being a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$, but it does not satisfy the usual in such cases Ambrosetti–Rabinowitz condition (the AR-condition for short).

In the boundary condition, $\frac{\partial u}{\partial n_\alpha}$ denotes the conormal derivative of u defined by extension of the map

$$C^1(\bar{\Omega}) \ni u \to (a(Du), n)_{\mathbb{R}^n},$$

with n being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(z) \in C^{0,\mu}(\partial\Omega)$ for some $\mu \in]0, 1[$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we recover the Neumann problem.

We are looking for positive solutions and our goal is to describe the set of positive solutions as the parameter $\lambda > 0$ varies. So we prove a bifurcation-type result describing the dependence of the set of positive solutions on the parameter $\lambda > 0$. We show that there exists a critical parameter value $\lambda^* \in]0, +\infty[$ such that

- for all $\lambda \in]0, \lambda^*[$ problem (1.1) has at least two positive solutions;
- for $\lambda = \lambda^*$ problem (1.1) has at least one positive solution;
- for all $\lambda > \lambda^*$ problem (1.1) has no solution.

Moreover, we show that for every $\lambda \in]0, \lambda^*]$ problem (1.1) has a smallest positive solution $\bar{u}_\lambda$ and we examine the monotonicity and the continuity properties of the map $\lambda \to \bar{u}_\lambda$.

Such an analysis of parametric singular problems was conducted by Papageorgiou–Smyrlis [30] for equations driven by the Dirichlet p-Laplacean. In [30] the perturbation $f(z, \cdot)$ satisfies a different superlinearity condition. In general, singular problems were studied primarily in the context of Dirichlet equations driven by the Laplacean or p-Laplacean. We mention the works of Giacomoni–Schindler–Takač [12], Hirano–Saccon–Shioji [13], Kyritsi–Papageorgiou [15], Lair–Shaker [16], Papageorgiou–Radulescu–Repovs [27], Perera–Zhang [33], Sun–Wu–Lang [36]. For semi linear Neumann problem driven by the Laplacean we mention the work of Chabrowski [2].

To have a complete overview on the literature one can refer also to the relevant papers [29,31,32,35].

For other kind of operators with singular lower order terms see also [3–8,17–20].

2 Mathematical background, hypotheses

Suppose X is a Banach space and let X^* be its topological dual. Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the C-condition for short) if the following property holds:

"Every sequence $\{u_n\} \subset X$ such that

$$\{\varphi(u_n)\} \subset \mathbb{R} \quad \text{is bounded}$$

and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^*, \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ which leads to the "mountain pass theorem".

Theorem 2.1 *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$$

and $c = \inf_{\psi\in\Gamma} \max_{0\le t\le 1} \varphi(\psi(t))$ where $\Gamma = \{\psi \in C([0, 1], X) : \psi(0) = u_0, \psi(1) = u_1\}$, then $c \ge m_\rho$ and c is a critical value of φ.

By $\|\cdot\|$ we denote the usual norm of the Sobolev space $W^{1,p}(\Omega)$ and we recall that $W^{1,p}(\Omega)$ is a separable and reflexive Banach space.

The space $C^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \ge 0 \ \forall z \in \bar{\Omega}\}.$$

This cone has nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \ \forall z \in \bar{\Omega}\}.$$

Also we will use another open cone in $C^1(\bar{\Omega})$, namely the cone

$$\hat{D}_+ = \left\{ u \in W^{1,p}(\Omega) : u(z) > 0 \text{ for all } z \in \Omega, \ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega\cap u^{-1}(0)} < 0 \right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue space $L^q(\partial\Omega)$, $1 \le q \le +\infty$. We know that there exists a continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u_{|\partial\Omega} \ \ \forall u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}).$$

So, the trace map defines boundary values for every Sobolev function.

The trace map is not surjective and we have

$$im\,\gamma_0 \in W^{\frac{1}{p'},p}(\partial\Omega) \ \left(1/p + 1/p' = 1\right) \quad \text{and} \quad ker\,\gamma_0 = W_0^{1,p}(\Omega).$$

We know that

- $\gamma_0(\cdot)$ is compact onto $L^q(\partial\Omega)$ for all $q \in \left[1, \frac{(N-1)p}{N-p}\right[$, if $p < N$;
- $\gamma_0(\cdot)$ is compact onto $L^q(\partial\Omega)$ for all $q \in [1, +\infty[$, if $p \ge N$.

In what follows, for notational economy, we drop the use of trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Suppose $\theta \in C(0, +\infty)$ with $\theta(t) > 0$ for all $t > 0$ and assume that

$$0 < \tau \le \frac{\theta'(t)t}{\theta(t)} \le c_0 \quad \text{and} \quad c_1 t^{p-1} \le \theta(t) \le c_2(t^{q-1} + t^{p-1}) \quad \text{for all } t > 0, \tag{2.1}$$

with $0 < c_1, c_2$ constants and $1 \le q < p$.

The hypotheses on the map $a(\cdot)$ are the following:

(i) $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$;
(ii) $a_0 \in C^1(0, +\infty)$, $t \to a_0(t)t$ is strictly increasing on $]0, +\infty[$, $a_0(t)t \to 0$ as $t \to 0^+$ and

$$\lim_{t \to 0^+} \frac{a_0(t)t}{a_0(t)} > -1;$$

(iii) there exists a constant $c_3 > 0$ such that

$$|\nabla a(y)| \le c_3 \frac{\theta(|y|)}{|y|} \quad \forall y \in \mathbb{R}^n \backslash \{0\};$$

(iv) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \dfrac{\theta(|y|)}{|y|} |\xi|^2 \;\; \forall y \in \mathbb{R}^N \backslash \{0\}, \;\; \forall \xi \in \mathbb{R}^N$;
(v) if $G_0(t) = \displaystyle\int_0^t a_0(s)s\, ds$, then $p\, G_0(t) - a_0(t)t^2 \ge 0$ for all $t > 0$.

Remark 2.2 Hypotheses (i)–(iv) are dictated by the nonlinear regularity theory of Lieberman [21] and the nonlinear maximum principle of Pucci–Serrin [34]. Hypothesis (v) serves the needs of our problem, but it is very mild. Similar hypotheses on the differential operator can be found in [1]. Evidently $G_0(\cdot)$ is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then $G(y)$ is convex and $G(0) = 0$. Also

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \;\; \forall y \in \mathbb{R}^n \backslash \{0\}, \quad \nabla G(0) = 0.$$

So, $G(\cdot)$ is the primitive of $a(y)$ and the convexity go $G(\cdot)$ implies

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \forall y \in \mathbb{R}^N. \tag{2.2}$$

An easy consequence of hypotheses (i)–(iv) and of (2.1) is the following

Lemma 2.3 *If hypotheses (i)–(iv) hold then*

(a) $y \to a(y)$ *is strictly monotone and continuous (hence maximal monotone too);*
(b) $|a(y)| \le c_4[|y|^{q-1} + |y|^{p-1}]$ *for some constant* $c_4 > 0$ *and* $\forall y \in \mathbb{R}^N$*;*
(c) $(a(y), y)_{\mathbb{R}^N} \ge \dfrac{c_1}{p-1}|y|^p \quad \forall y \in \mathbb{R}^N$.

From this lemma and (2.2), we deduce the following growth properties for $G(\cdot)$.

Corollary 2.4 *If hypotheses (i)–(iv) hold then*

$$\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_5(|y|^q + |y|^p) \;\; \textit{for some constant } c_5 > 0 \textit{ and } \forall y \in \mathbb{R}^N.$$

Example The following maps $a(\cdot)$ satisfy hypotheses (i)–(iv).

(a)

$$a(y) = |y|^{p-2}y, \;\; 1 < p < +\infty.$$

This map corresponds to the p-Laplacean operator define by

$$\Delta_p u = \text{div}\,(|Du|^{p-2}Du) \quad \forall u \in W^{1,p}(\Omega).$$

(b)

$$a(y) = |y|^{p-2}y + |y|^{q-2}y, \quad 1 < q < p < +\infty.$$

This map corresponds to the (p, q)-Laplacean defined by

$$\Delta_p u + \Delta_q u, \quad \forall u \in W^{1,p}(\Omega).$$

(c)

$$a(y) = [1 + |y|^2]^{\frac{p-2}{2}} y, \quad 1 < p < +\infty.$$

This map corresponds to the generalized p-mean curvature differential operator defined by

$$\mathrm{div}\left[(1 + |Du|^2)^{\frac{p-2}{2}} Du\right] \quad \forall u \in W^{1,p}(\Omega).$$

(d)

$$a(y) = |y|^{p-2}\left[1 + \frac{1}{1 + |y|^p}\right] y, \quad 1 < p < +\infty.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Gasinski–Papageorgiou [11].

Proposition 2.5 *If hypotheses (i)–(iv) hold, then*

$$A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$$

is monotone, continuous (hence maximal monotone too) and of type $(S)_+$*, that is, if*

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega)$$

and

$$\limsup_{n\to+\infty} < A(u_n), \quad u_n - u > \leq 0$$

then

$$u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

The hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$ are the following:

($\boldsymbol{\xi}$) $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\xi \neq 0$.
($\boldsymbol{\beta}$) $\beta \in C^{0,\mu}(\partial\Omega)$, with $0 < \mu < 1$ and $\beta(z) \geq 0$, $\forall z \in \partial\Omega$.

Remark 2.6 The case $\beta \equiv 0$ corresponds to the Neumann problem.

From Mugnai–Papageorgiou [23] (Lemma 4.11), we have:

Proposition 2.7 *If hypotheses* (ξ) *hold then*

$$\frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz \geq c_6\|u\|^p$$

for some constant $c_6 > 0$ *and all* $u \in W^{1,p}(\Omega)$.

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_0(z,x)| \leq \alpha_0(z)[1+|x|^{r-1}] \quad \text{for a.a. } z \in \Omega \text{ and } \forall x \in \mathbb{R},$$

with $\alpha_0 \in L^\infty(\Omega)$, $1 < r \leq p^*$ ([1]).

We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1–functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma$$
$$- \int_\Omega F_0(z,u)\,dz, \quad \forall u \in W^{1,p}(\Omega).$$

From Papageorgiou–Radulescu [25] (Proposition 8), we have

Proposition 2.8 *If hypotheses (i)–(v),* (ξ), (β) *hold and* $u_0 \in W^{1,p}(\Omega)$ *is a local* $C^1(\bar{\Omega})$-*minimizer of* φ_0, *that is, there exists* $\rho_0 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi(u_0+h) \quad \text{for all } h \in C^1(\bar{\Omega}) \text{ and } \|h\|_{C^1(\bar{\Omega})} \leq \rho_0$$

then $u_0 \in C^{1,\eta}(\bar{\Omega})$ *for some* $0 < \eta < 1$ *and* u_0 *is a local* $W^{1,p}(\Omega)$-*minimizer of* φ_0, *that is, there exists* $\rho_1 > 0$ *such that*

$$\varphi(u_0) \leq \varphi(u_0+h) \quad \text{for all } h \in W^{1,p}(\Omega),\ \|h\| \leq \rho_1.$$

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given $u, v \in W^{1,p}(\Omega)$ with $u(z) \leq v(z)$ for a.a. $z \in \Omega$, we define

$$[u,v] = \{y \in W^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \text{ for a.a. } z \in \Omega\},$$
$$int_{C^1(\bar{\Omega})}[u,v] = \text{the interior in } C^1(\bar{\Omega}) \text{ of } [u,v] \cap C^1(\bar{\Omega}).$$

Finally, given $\varphi \in C^1(X,\mathbb{R})$, by K_φ we denote the critical set of φ, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

3 A purely singular problem

In this section we deal with the following nonlinear purely singular Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma}, \quad u(z) > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_\alpha} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

[1] $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}$

with $0 < \gamma < 1$.

We shall use the hypotheses of Sect. 2.

Proposition 3.1 *If hypotheses (i)–(v), (ξ), (β) hold then problem (3.1) admits a unique solution $\tilde{u} \in D_+$.*

Proof For every $\varepsilon > 0$ we consider the following approximation of problem (3.1)

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = (u(z)+\varepsilon)^{-\gamma}, \quad u(z) > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_\alpha} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

First we solve (3.2). So, let $v \in L^p(\ddot{})$ and consider the following nonlinear Robin problem

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = (|v(z)|+\varepsilon)^{-\gamma}, \quad u(z) > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_\alpha} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

Since the map $u \to A(u) + \xi(z)|u(z)|^{p-2}u$ is maximal monotone and coercive (see Propositions 2.5 and 2.7) it is surjective and so problem (3.3) admits a solution $\tilde{u} \in W^{1,p}(\Omega)$, $\tilde{u} \geq 0$, $\tilde{u} \neq 0$ which is unique. This can be established as in [28], using the Diaz–Saa inequality (see Diaz–Saa [9]).

Moreover, since $(|v(z)|+\varepsilon)^{-\gamma} \in L^\infty(\Omega)$, from Proposition 7 of Papageorgiou–Radulescu [25] we have that $\tilde{u} \in C_+$. Finally the nonlinear maximum principle of Pucci–Serrin [34] (pp. 111, 120) implies that $\tilde{u} \in D_+$.

We consider the map $S_\varepsilon : L^p(\Omega) \to L^p(\Omega)$ defined by

$$S_\varepsilon(v) = \tilde{u}.$$

We have

$$\langle A(\tilde{u}), h\rangle + \int_\Omega \xi(z)\tilde{u}^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1}h\,d\sigma = \int_\Omega \frac{h}{(|v|+\varepsilon)^\gamma}\,dz, \tag{3.4}$$

for all $\in W^{1,p}(\Omega)$.

In (3.4) we choose $h = S_\varepsilon(v) \in W^{1,p}(\Omega)$. Using Lemma 2.3, we obtain

$$\begin{aligned} &\frac{c_1}{p-1}\|DS_\varepsilon(v)\|_p^p + \int_\Omega \xi(z)S_\varepsilon(v)^p\,dz + \int_{\partial\Omega} \beta(z)S_\varepsilon(v)^p\,d\sigma \\ &= \int_\Omega \frac{S_\varepsilon(v)}{(|v|+\varepsilon)^\gamma}\,dz \leq \frac{c_7}{\varepsilon^\gamma}\|S_\varepsilon(v)\|, \end{aligned}$$

for some constant $c_7 > 0$, and this implies

$$\|S_\varepsilon(v)\|^{p-1} \leq \frac{c_8}{\varepsilon^\gamma}, \tag{3.5}$$

for some constant $c_8 > 0$ and all $v \in W^{1,p}(\Omega)$ (see Proposition 2.7).

We now claim that $S_\varepsilon(\cdot)$ is compact.

First we show the continuity of $S_\varepsilon(\cdot)$. To this aim, let $v_n \to v$ in $L^p(\Omega)$. From (3.5) it follows that $\tilde{u}_n = S_\varepsilon(v_n) \subset W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$\tilde{u}_n \xrightarrow{w} \tilde{u} \ \text{ in } W^{1,p}(\Omega) \text{ and } \ \tilde{u}_n \to \tilde{u} \ \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.6}$$

We have

$$\langle A(\tilde{u}_n), h\rangle + \int_\Omega \xi(z)\tilde{u}_n^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_n^{p-1}h\,d\sigma = \int_\Omega \frac{h}{(|v_n|+\varepsilon)^\gamma}\,dz, \tag{3.7}$$

for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$.

In (3.5) we choose $h = \tilde{u}_n - \tilde{u} \in W^{1,p}(\Omega)$. Then using (3.6) we see that

$$\lim_{n\to+\infty} \langle A(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle = 0$$

and this implies

$$\tilde{u}_n \to \tilde{u} \ \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.5).} \tag{3.8}$$

If in (3.7) we pass to the limit as $n \to +\infty$ and use (3.8), then

$$\langle A(\tilde{u}), h\rangle + \int_\Omega \xi(z)\tilde{u}^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1} h\, d\sigma = \int_\Omega \frac{h}{(|v| + \varepsilon)^\gamma}\, dz,$$

for all $h \in W^{1,p}(\Omega)$ and this implies

$$\begin{cases} -\text{div}\ a(D\tilde{u}(z)) + \xi(z)\tilde{u}(z)^{p-1} = (|v(z)| + \varepsilon)^{-\gamma}, \ \ \tilde{u}(z) \geq 0, \ \ \tilde{u} \neq 0 & \text{in } \Omega \\ \dfrac{\partial \tilde{u}}{\partial n_\alpha} + \beta(z)\tilde{u}^{p-1} = 0 & \text{on } \partial\Omega, \end{cases}$$

(see Papageorgiou–Radulescu [24]), so that $\tilde{u} = S_\varepsilon(v) \in D_+$ and $S_\varepsilon(\cdot)$ is continuous.

Moreover, the compact embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ implies that $S_\varepsilon(\cdot)$ is compact (see (3.5)). Invoking the Schauder–Tichonov fixed point theorem (see e.g. Gasinski–Papageorgiou [10], p. 827), we can find $\tilde{u}_\varepsilon \in W^{1,p}(\Omega)$ such that

$$\tilde{u}_\varepsilon = S_\varepsilon(\tilde{u}_\varepsilon).$$

From the first part of the proof we know that $\tilde{u}_\varepsilon \in D_+$.

We prove now that the map $\varepsilon \to \tilde{u}_\varepsilon$ from $]0, +\infty[$ into C_+ is non increasing, that is

$$\varepsilon' < \varepsilon \ \Rightarrow \ \tilde{u}_{\varepsilon'} - \tilde{u}_\varepsilon \in C_+ \backslash \{0\}.$$

Let $0 < \varepsilon' < \varepsilon$ then we have

$$\begin{aligned} -\operatorname{div} a(D\tilde{u}_{\varepsilon'}(z)) + \xi(z)\tilde{u}_{\varepsilon'}(z)^{p-1} &= (\tilde{u}_{\varepsilon'}(z) + \varepsilon')^{-\gamma} \\ &\geq (\tilde{u}_{\varepsilon'}(z) + \varepsilon)^{-\gamma} \ \text{ for a.a. } z \in \Omega. \end{aligned} \tag{3.9}$$

We introduce the Carathéodory function $k_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_\varepsilon(z, x) = \begin{cases} \dfrac{1}{(x^+ + \varepsilon)^\gamma} & \text{if } x \leq \tilde{u}_{\varepsilon'}(z) \\ \dfrac{1}{(\tilde{u}_{\varepsilon'} + \varepsilon)^\gamma} & \text{if } x > \tilde{u}_{\varepsilon'}(z). \end{cases} \tag{3.10}$$

We set

$$K_\varepsilon(z, x) = \int_0^x k_\varepsilon(z, s)\, ds$$

and consider the C^1-functional $\psi_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \psi_\varepsilon(u) = \int_\Omega G(Du(z))\, dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\, d\sigma \\ - \int_\Omega K_\varepsilon(z, u)\, dz, \ \ \forall u \in W^{1,p}(\Omega). \end{aligned}$$

From (3.10), Lemma 2.3 and Proposition 2.7 it follows that

$$\psi_\varepsilon(\cdot) \ \text{ is coercive.}$$

Also the Sobolev embedding theorem and the compactness of the trace map imply that

$$\psi_\varepsilon(\cdot) \text{ is sequentially weakly lower semicontinuos.}$$

So, by the Weierstrass–Tonelli theorem, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\psi_\varepsilon(\hat{u}_\varepsilon) = \inf\{\psi_\varepsilon(u) \,:\, u \in W^{1,p}(\Omega)\}. \tag{3.11}$$

Recall that $\tilde{u}_{\varepsilon'} \in D_+$. So $\min \tilde{u}_{\varepsilon'} = m_{\varepsilon'} > 0$.
Let now $\tau \in]0, \min\{1, m_{\varepsilon'}\}[$, then

$$\psi_\varepsilon(\tau) = \frac{\tau^p}{p}(\|\xi\|_1 + \|\beta\|_{L^1(\partial\Omega)}) - \frac{\tau^{1-\gamma}}{1-\gamma}[(\tau+\varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}]\,\mathrm{meas}_{\mathbb{R}^N}(\Omega).$$

Since $\tau < 1$ and $0 < 1-\gamma < 1 < p$, by choosing τ small, we will have

$$\begin{aligned} \psi_\varepsilon(\tau) < 0 \;&\Rightarrow\; \psi_\varepsilon(\hat{u}_\varepsilon) < 0 = \psi_\varepsilon(0) \;\; \text{(see (3.11))} \\ &\Rightarrow \hat{u}_\varepsilon \neq 0. \end{aligned}$$

From (3.11) we have

$$\begin{aligned} \psi'_\varepsilon(\hat{u}_\varepsilon) \;&=\; 0 \\ &\Rightarrow \langle A(\hat{u}_\varepsilon), h\rangle + \int_\Omega \xi(z)|\hat{u}_\varepsilon|^{p-2}\hat{u}_\varepsilon h\, dz + \int_{\partial\Omega} \beta(z)|\hat{u}_\varepsilon|^{p-2}\hat{u}_\varepsilon h\, d\sigma \\ &= \int_\Omega k_\varepsilon(z, \hat{u}_\varepsilon) h\, dz \end{aligned} \tag{3.12}$$

for all $h \in W^{1,p}(\Omega)$.
In (3.12) first we choose $h = -\hat{u}_\varepsilon^- \in W^{1,p}(\Omega)$. Then using Lemma 2.3 we obtain

$$\begin{aligned} &\frac{c_1}{p-1}\|D\hat{u}_\varepsilon^-\|_p^p + \int_\Omega \xi(z)(\hat{u}_\varepsilon^-)^p\, dz + \int_{\partial\Omega} \beta(z)(\hat{u}_\varepsilon^-)^p\, d\sigma = \int_\Omega \frac{-\hat{u}_\varepsilon^-}{\varepsilon^\gamma}\, dz \le 0 \;\; \text{(see (3.10))} \\ &\Rightarrow c_9\|\hat{u}_\varepsilon^-\|^p \le 0 \qquad \text{for some constant } c_9 > 0 \text{ (see Proposition 2.7)} \\ &\Rightarrow \hat{u}_\varepsilon \ge 0 \;\; \hat{u}_\varepsilon \neq 0. \end{aligned}$$

Next in (3.12) we choose $(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(\hat{u}_\varepsilon), (\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+)\rangle + \int_\Omega \xi(z)\hat{u}_\varepsilon^{p-1}(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+\, dz + \int_{\partial\Omega} \beta(z)\hat{u}_\varepsilon^{p-1}(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+\, d\sigma \\ &= \int_\Omega \frac{(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+}{(\tilde{u}_{\varepsilon'}+\varepsilon)^\gamma}\, dz \qquad \text{(see (3.10))} \\ &\le \int_\Omega \frac{(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+}{(\tilde{u}_{\varepsilon'}+\varepsilon')^\gamma}\, dz \qquad (\text{since } \varepsilon' < \varepsilon) \\ &= \langle A(\tilde{u}_{\varepsilon'}), (\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+)\rangle + \int_\Omega \xi(z)\tilde{u}_{\varepsilon'}^{p-1}(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}_{\varepsilon'}^{p-1}(\hat{u}_\varepsilon - \tilde{u}_{\varepsilon'})^+\, d\sigma \\ &\Rightarrow \;\hat{u}_\varepsilon \le \tilde{u}_{\varepsilon'}. \end{aligned}$$

So, we have proved that

$$\hat{u}_\varepsilon \in [0, \tilde{u}_{\varepsilon'}], \quad \hat{u}_\varepsilon \neq 0. \tag{3.13}$$

From (3.10), (3.12) and (3.13) it follows that

$$\hat{u}_\varepsilon \quad \text{is a positive solution of (3.2)}$$

$$\Rightarrow \hat{u}_\varepsilon = \tilde{u}_\varepsilon$$
$$\Rightarrow \hat{u}_\varepsilon \le \tilde{u}_{\varepsilon'}.$$

Therefore the map $\varepsilon \to \tilde{u}_\varepsilon$ is nonincreasing from $]0, +\infty[$ into $C^1(\bar{\Omega})$ and this proves the claim.

Now let $\varepsilon_n \to 0$ and let $\tilde{u}_n = \tilde{u}_{\varepsilon_n} \in D_+$ fro all $n \in \mathbb{N}$. We have

$$\langle A(\tilde{u}_n), h\rangle + \int_{\partial\Omega} \xi(z)\tilde{u}_n^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}_n^{p-1} h\, d\sigma = \int_\Omega \frac{h}{(\tilde{u}_n + \varepsilon_n)^\gamma}\, dz \tag{3.14}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.14) we choose $h = \tilde{u}_n \in W^{1,p}(\Omega)$. Then using Lemma 2.3 and Proposition 2.7 we obtain

$$\begin{aligned} c_{10}\|\tilde{u}_n\|^p &\le \int_\Omega \frac{\tilde{u}_n}{(\tilde{u}_n + \varepsilon_n)^\gamma} \le \int_\Omega \frac{\tilde{u}_n}{\tilde{u}_1^\gamma}\, dz \quad (\text{since by the claim } \tilde{u}_1 \le \tilde{u}_n\ \forall n \in \mathbb{N}) \\ &\le \frac{c_{11}}{\tilde{m}_1^\gamma} \quad \text{for some constants } c_{10}, c_{11} > 0,\ \text{with } \tilde{m}_1 = \min_{\bar{\Omega}} \tilde{u}_1, \forall n \in \mathbb{N} \\ &\Rightarrow \{\tilde{u}_n\} \subset W^{1,p}(\Omega) \quad \text{is bounded.} \end{aligned}$$

So we may assume that

$$\tilde{u}_n \xrightarrow{w} \tilde{u} \quad \text{in } W^{1,p}(\Omega) \text{ and } \tilde{u}_n \to \tilde{u} \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega),\ \tilde{u} \ne 0 \tag{3.15}$$

(recall that $\tilde{u}_n \ge \tilde{u}_1\ \forall n \in \mathbb{N}$).

In (3.14) we choose $h = \tilde{u}_n - \tilde{u} \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.15). Then

$$\begin{aligned} &\lim_{n\to+\infty} \langle A(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle = 0 \\ &\Rightarrow \tilde{u}_n \to \tilde{u} \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.5).} \end{aligned} \tag{3.16}$$

So, if in 3.14 we pass to the limit as $n \to +\infty$ and use (3.16), then

$$\begin{aligned} &\langle A(\tilde{u}), h)\rangle + \int_\Omega \xi(z)\tilde{u}^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1} h\, d\sigma = \int_\Omega \frac{h}{\tilde{u}^\gamma}\, dz, \quad \forall h \in W^{1,p}(\Omega) \\ &\Rightarrow \begin{cases} -\mathrm{div}\, a(D\tilde{u}(z)) + \xi(z)\tilde{u}(z)^{p-1} = \tilde{u}(z)^{-\gamma} & \text{in } \Omega \\ \dfrac{\partial\tilde{u}}{\partial n_\alpha} + \beta(z)\tilde{u}^{p-1} = 0 & \text{on } \partial\Omega \quad (\text{see [24]}) \end{cases} \\ &\Rightarrow \tilde{u} \in D_+ \quad \text{solves problem (3.1).} \end{aligned}$$

Suppose $\tilde{v} \in W^{1,p}(\Omega)$ is another positive solution of (3.1). Again we show that $\tilde{v} \in D_+$. Then

$$\langle A(\tilde{u}) - A(\tilde{v}), h\rangle + \int_\Omega \xi(z)(\tilde{u}^{p-1} - \tilde{v}^{p-1})h\, dz + \int_{\partial\Omega} \beta(z)(\tilde{u}^{p-1} - \tilde{v}^{p-1})h\, d\sigma = \int_\Omega (\tilde{u}^{-\gamma} - \tilde{v}^{-\gamma})h\, dz, \quad \forall h \in W^{1,p}(\Omega) \tag{3.17}$$

Choosing $h = \tilde{u} - \tilde{v} \in W^{1,p}(\Omega)$, since $x \to x^{-\gamma}$ is strictly decreasing on $]0, +\infty[$, from (3.17) we conclude that

$$\tilde{u} = \tilde{v} \in D_+$$

$$\Rightarrow \tilde{u} \in D_+ \quad \text{is the unique solution of (3.1).} \qquad \square$$

4 Positive solutions of problem (1.1)

In this section we prove a bifurcation-type theorem describing the dependence of the positive solutions on the parameter $\lambda > 0$.

To this end, besides the structural hypotheses of the Sect. 2, we impose the following conditions on the perturbation $f(z, x)$:

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(f1)

$$f(z, 0) = 0;$$

(f2) there exists $\alpha \in L^\infty(\Omega)$ and $r \in]p, p^*[$ such that

$$0 \leq f(z, x) \leq \alpha(z)(1 + x^{r-1})$$

for a.a. $z \in \Omega$ and all $x \geq 0$;

(f3) if $F(z, x) = \int_0^x f(z, s)\, ds$ then

$$\lim_{x \to +\infty} \frac{F(z, x)}{x^p} = +\infty$$

uniformly for a.a. $z \in \Omega$;

(f4) if $\hat{\mu}_\lambda(z, x) = \left(1 - \dfrac{p}{1-\gamma}\right) x^{1-\gamma} + \lambda[f(z, x)x - pF(z, x)]$, $\lambda > 0$, then there exists $g_\lambda \in L^1(\Omega)$ such that

$$\hat{\mu}_\lambda(z, x) \leq \hat{\mu}_\lambda(z, y) + g_\lambda(z)$$

for a.a $z \in \Omega$ and $\forall 0 \leq x \leq y$;

(f5) for every $\rho > 0$ there exists $\hat{\theta}_\rho > 0$ such that, for a.a. $z \in \Omega$, the map $x \to f(z, x) + \hat{\theta}_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$ and, for every $\tau > 0$, there exists $\hat{\eta}_\tau > 0$ such that $f(z, x) \geq \hat{\eta}_\tau$ for a.a. $z \in \Omega$ and all $x \geq \tau$.

Remark 4.1 Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}^+ =]0, +\infty[$, without any loss of generality, we may assume that

$$f(z, x) = 0 \text{ for a.a. } z \in \Omega \text{ and } \forall x \leq 0. \tag{4.1}$$

Hypotheses (f1)–(f4) imply that

$$\lim_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \tag{4.2}$$

uniformly for a.a. $z \in \Omega$.

Therefore the perturbation term $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. For superlinear problems usually we employ the AR-condition, which says that there exist $q>p$ and $M>0$ such that

$$0<qF(z,x)\leq f(z,x)x \tag{4.3}$$

for a.a. $z\in\Omega$ and all $x\geq M$;

$$\operatorname{ess\,inf}_{\Omega} F(\cdot,M)>0. \tag{4.4}$$

Integrating (4.3) and using (4.4), we obtain the weaker condition

$$c_{12}x^q\leq F(z,x) \tag{4.5}$$

for a.a. $z\in\Omega$, all $x\geq M$ and some constant $c_{12}>0$.

We also see that (4.3) and (4.5) imply that (4.2) holds. The AR-condition facilitates the verification of the C-condition for the corresponding energy functional. However from (4.5) we see that it is rather restrictive and excludes from consideration superlinear functions with slower growth near $+\infty$ (see the examples below).

Here we employ the quasimonotonicity condition from hypothesis (f4). This conditions a slight generalization of a condition used by Li–Yang [22]. We mention that (f4) is satisfied if there exists $M>0$ such that for a.a. $z\in\Omega$ the map $x\to\dfrac{x^{-\gamma}+\lambda f(z,x)}{x^{p-1}}$ is nondecreasing on $[M,+\infty[$. The first part of hypothesis (f5) is satisfied if, for example, for a.a. $z\in\Omega$ the function $f(z,\cdot)$ is differentiable and, for every $\rho>0$, there exists c_ρ such that $f'_x(z,x)x\geq -c_\rho x^{p-1}$ for a.a. $z\in\Omega$ and all $0\leq x\leq\rho$.

Example The following functions satisfy hypotheses (f1)–(f5) (for the sake of simplicity we drop the z-dependence):

$$f_1(x)=(x^+)^{q-1}\quad 1<p<q<+\infty,\qquad f_2(x)=(x^+)^{p-1}[\log(x^+)+1]\quad 2\leq p<+\infty.$$

We note that f_1 satisfies the AR-condition but f_2 does not.

We introduce the following sets

$$\mathcal{L}=\{\lambda>0\,:\,\text{problem (1.1) has a positive solution}\},$$
$$S_\lambda=\{\text{positive solutions of problem (1.1)}\}.$$

Let $\tilde u\in D_+$ be the unique solution of problem (3.1) established in Proposition 3.1 and consider the following truncation of the reaction in problem (1.1):

$$\hat f_\lambda(z,x)=\begin{cases}\tilde u(z)^{-\gamma}+\lambda f(z,\tilde u(z)) & \text{if } x\leq\tilde u(z)\\ x^{-\gamma}+\lambda f(z,x) & \text{if } x>\tilde u(z).\end{cases} \tag{4.6}$$

Evidently $\hat f_\lambda(z,x)$ is a Carathéodory function. We set

$$\hat F_\lambda(z,x)=\int_0^x \hat f_\lambda(z,s)\,ds$$

and consider the C^1–functional $\hat\varphi_\lambda : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat\varphi_\lambda(u)=\int_\Omega G(Du(z))\,dz+\frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz+\frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma-\int_\Omega \hat F_\lambda(z,u)\,dz,$$

for all $u\in W^{1,p}(\Omega)$.

Proposition 4.2 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold and $\lambda > 0$, then the functional $\hat{\varphi}_\lambda$ satisfies the C-condition.*

Proof Let $\{u_n\} \subset W^{1,p}(\Omega)$ be a sequence such that

$$|\hat{\varphi}_\lambda(u_n)| \le M_1 \tag{4.7}$$

for some constant $M_1 > 0$ and all $n \in \mathbb{N}$;

$$(1 + \|u_n\|)\hat{\varphi}'_\lambda(u_n) \to 0, \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to 0. \tag{4.8}$$

From (4.8), we have

$$\left| \langle A(u_n), h)\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\,dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\,d\sigma - \int_\Omega \hat{f}_\lambda(z, u_n)h\,dz \right| \le \frac{\varepsilon_n\|h\|}{1 + \|u_n\|}. \tag{4.9}$$

In (4.9) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Using lemma 2.3, Proposition 2.7 and (4.6), we obtain

$$\begin{aligned} &\|u_n^-\|^p \le M_2 \quad \text{for some constant } M_2 > 0 \text{ and all } n \in \mathbb{N} \\ &\Rightarrow \{u_n^-\} \subset W^{1,p}(\Omega) \quad \text{is bounded.} \end{aligned} \tag{4.10}$$

Next in (4.9) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}\,dz - \int_\Omega \xi(z)(u_n^+)^p\,dz - \int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma \\ &+ \int_\Omega \hat{f}_\lambda(z, u_n^+)u_n^+\,dz \le \varepsilon_n, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{4.11}$$

Also from (4.7) and (4.10) we have

$$\begin{aligned} &p\int_\Omega G(Du_n^+)\,dz + \int_\Omega \xi(z)(u_n^+)^p\,dz + \int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma \\ &-p\int_\Omega \hat{F}_\lambda(z, u_n^+)\,dz \le M_3, \end{aligned} \tag{4.12}$$

for some constant $M_3 > 0$ and all $n \in \mathbb{N}$.

Gathering together (4.11), (4.12) and using hypothesis (v) and (4.6), we obtain

$$\int_\Omega \hat{\mu}_\lambda(z, u_n^+)\,dz \le M_4 \quad \text{for some constant } M_4 > 0 \text{ and } \forall n \in \mathbb{N}. \tag{4.13}$$

We use (4.13) to show that $\{u_n^+\} \subset W^{1,p}(\Omega)$ is bounded. Arguing by contradiction, suppose that by passing to a subsequence if necessary, we have $\|u_n^+\| \to +\infty$ as $n \to +\infty$. Let $y_n = \dfrac{u_n^+}{\|u_n^+\|}$ for all $n \in \mathbb{N}$. We have $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega),\ y \ge 0. \tag{4.14}$$

First we suppose that $y \neq 0$. We set $\Omega_+ = \{z \in \Omega : y(z) > 0\}$ and we have $meas_{\mathbb{R}^N}(\Omega_+) > 0$, $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega_+$. Then hypothesis (f3) implies that

$$\begin{aligned}\frac{F(z,u_n^+(z))}{\|u_n^+\|^p} &= \frac{F(z,u_n^+(z))}{u_n^+(z)^p}y_n(z)^p \to +\infty \quad \text{for a.a. } z\in\Omega_+\\ &\Rightarrow \int_{\Omega_+}\frac{F(z,u_n^+(z))}{\|u_n^+\|^p}\,dz \to +\infty \quad \text{(by Fatou's Lemma)}\\ &\Rightarrow \int_{\Omega}\frac{F(z,u_n^+(z))}{\|u_n^+\|^p}\,dz \to +\infty. \end{aligned} \tag{4.15}$$

On the other hand, from (4.7), (4.10) and Corollary 2.4 we have

$$\begin{aligned}&-p\int_\Omega G(Du_n^+)\,dz - \int_\Omega \xi(z)(u_n^+)^p\,dz - \int_{\partial\Omega}\beta(z)(u_n^+)^p\,d\sigma\\ &+p\int_\Omega \hat{F}_\lambda(z,u_n^+)\,dz \le M_5, \quad \text{for some constant } M_5>0 \text{ and } \forall n\in\mathbb{N}\\ &\Rightarrow \int_\Omega \frac{F(z,u_n^+(z))}{\|u_n^+\|^p}\,dz \le M_6 \end{aligned} \tag{4.16}$$

for some constant $M_6>0$ and $\forall n\in\mathbb{N}$ (see (4.6)). Comparing (4.15) and (4.16) we have a contradiction.

Now we assume that $y\equiv 0$. Consider the C^1-functional $\tilde{\varphi}_\lambda :\to \mathbb{R}$ defined by

$$\begin{aligned}\tilde{\varphi}_\lambda(u) = \frac{1}{p}\left[\frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega}\beta(z)|u|^p\,d\sigma\right]\\ - \int_\Omega \hat{F}_\lambda(z,u)\,dz \quad \forall u\in W^{1,p}(\Omega).\end{aligned}$$

Evidently $\tilde{\varphi}_\lambda \le \hat{\varphi}_\lambda$ (see Corollary 2.4) and $\tilde{\varphi}_\lambda$ is p-homogeneous.

We set

$$\eta_n(t) = \tilde{\varphi}_\lambda(tu_n^+), \quad \forall t\in[0,1] \text{ and } \forall n\in\mathbb{N}.$$

We can find $t_n\in[0,1]$ such that

$$\eta_n(t_n) = \max_{0\le t\le 1}\eta_n(t) = \max_{0\le t\le 1}\tilde{\varphi}_\lambda(tu_n^+) \quad \forall n\in\mathbb{N}. \tag{4.17}$$

For $k>0$ we define $v_n=(2k)^{1/p}y_n\in W^{1,p}(\Omega)$. We have $v_n\to 0$ in $L^r(\Omega)$ (see (4.14) and recall that $y=0$). Then

$$\int_\Omega \hat{F}_\lambda(z,v_n)\,dz \to 0 \quad \text{as } n\to+\infty. \tag{4.18}$$

Recalling that $\|u_n^+\|\to+\infty$, we can find $n_0\in\mathbb{N}$ such that

$$(2k)^{1/p}\frac{1}{\|u_n^+\|} \in]0,1[\quad \forall n\ge n_0. \tag{4.19}$$

From (4.17) and (4.19) we deduce

$$\begin{aligned}\eta_n(t_n) &\ge \eta_n\left(\frac{(2k)^{1/p}}{\|u_n^+\|}\right) \quad \forall n\in\mathbb{N}\\ &\Rightarrow \tilde{\varphi}_\lambda(t_nu_n^+) \ge \tilde{\varphi}_\lambda((2k)^{1/p}y_n) = \tilde{\varphi}_\lambda(v_n) \quad \forall n\in\mathbb{N}\\ &\Rightarrow \tilde{\varphi}_\lambda(t_nu_n^+) \ge \frac{2kc_1}{p(p-1)}\left[\|Dy_n\|_p^p + \frac{p-1}{c_1}\int_\Omega \xi(z)y_n^p\,dz\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{p-1}{c_1}\int_{\partial\Omega}\beta(z)y_n^p\,d\sigma\Big] \\
&-\int_\Omega \hat{F}_\lambda(z,v_n)\,dz \\
\geq\ & \frac{2kc_{13}}{p(p-1)}-\int_\Omega \hat{F}_\lambda(z,v_n)\,dz \quad \text{for some constant } c_{13}>0 \\
\geq\ & \frac{kc_{13}}{p(p-1)} \quad \forall n\geq n_1\geq n_0 \quad (\text{see } (4.18)).
\end{aligned} \tag{4.20}
$$

Being $k>0$ arbitrary, from (4.20) we infer that

$$\tilde{\varphi}_\lambda(t_nu_n^+)\to+\infty \quad \text{as } n\to+\infty. \tag{4.21}$$

We have $0\leq t_nu_n^+\leq u_n^+$ for all $n\in\mathbb{N}$ then hypothesis (f4) implies that

$$\int_\Omega \hat{\mu}_\lambda(z,t_nu_n^+)\,dz\leq\int_\Omega \hat{\mu}_\lambda(z,u_n^+)\,dz+\|g_\lambda\|_{L^1(\Omega)}\leq M_7 \tag{4.22}$$

for some positive constant M_7 and all $n\in\mathbb{N}$ (see (4.13)).

Recall that [see (4.6), (4.7) and recall that $\tilde{\varphi}_\lambda\leq\hat{\varphi}_\lambda$]

$$\tilde{\varphi}_\lambda(0)<0 \quad \text{and} \quad \tilde{\varphi}_\lambda(u_n^+)<M_8 \quad \forall n\in\mathbb{N}. \tag{4.23}$$

Then (4.21) and (4.23) imply that

$$t_n\in]0,1[\quad \forall n\geq n_2. \tag{4.24}$$

Thus from (4.17) and (4.24) it follows that

$$
\begin{aligned}
0\ &=\ t_n\left[\frac{d}{dt}\tilde{\varphi}_\lambda(tu_n^+)\right]_{t=t_n}=\langle\tilde{\varphi}'_\lambda(t_nu_n^+),t_nu_n^+\rangle \quad \forall n\geq n_2 \\
\Rightarrow\ &\frac{c_1}{p(p-1)}\|D(t_nu_n^+)\|_p^p+\frac{1}{p}\int_\Omega\xi(z)(t_nu_n^+)^p\,dz+\frac{1}{p}\int_{\partial\Omega}\beta(z)(t_nu_n^+)^p\,d\sigma \\
=\ &\int_\Omega \hat{f}_\lambda(z,t_nu_n^+)(t_nu_n^+)\,dz \quad \forall n\geq n_2
\end{aligned}
$$

whence

$$p\tilde{\varphi}_\lambda(t_nu_n^+)\leq M_8 \tag{4.25}$$

for some constant $M_8>0$ and all $n\geq n_2$ ((see (4.22) and ((4.6))).

If we compare (4.21) and (4.25) we have a contradiction. Therefore

$$
\begin{aligned}
&\{u_n^+\}\subset W^{1,p}(\Omega) \quad \text{is bounded} \\
&\Rightarrow\{u_n\}\subset W^{1,p}(\Omega) \quad \text{is bounded (see (4.10))}.
\end{aligned}
$$

So we may assume that

$$u_n\xrightarrow{w}u \quad \text{in } W^{1,p}(\Omega) \text{ and } u_n\to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{4.26}$$

In (4.9) we choose $h=u_n-u\in W^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (4.26). Then

$$
\begin{aligned}
&\lim_{n\to+\infty}\langle A(u_n),u_n-u\rangle=0 \\
&\Rightarrow u_n\to u \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.5)}.
\end{aligned}
$$

We conclude that $\hat{\varphi}_\lambda$ satisfies the C-condition. □

If $u \in D_+$ then, on account of hypothesis (f3), we have

$$\tilde{\varphi}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{4.27}$$

Proposition 4.3 *If hypotheses (i)–(v), (ξ), (β) and (f1)–(f5) hold, then* $\mathcal{L} \neq \emptyset$

Proof Consider the Robin problem

$$\begin{cases} -\text{div}\, a(Du(z)) + \xi(z)u(z)^{p-1} = \tilde{u}(z)^{-\gamma} + 1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_\alpha} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.28}$$

where $\tilde{u} \in D_+$ is the unique solution of the problem (3.1).

As before (see problem (3.3)), problem (4.28) has a unique solution $u^* \in D_+$. Note that

$$\begin{aligned} &\langle A(\tilde{u}), (\tilde{u} - u^*)^+ \rangle + \int_\Omega \xi(z)\tilde{u}^{p-1}(\tilde{u} - u^*)^+ \, dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1}(\tilde{u} - u^*)^+ \, d\sigma \\ &= \int_\Omega \tilde{u}^{-\gamma}(\tilde{u} - u^*)^+ \, dz \\ &\leq \int_\Omega (\tilde{u}^{-\gamma} + 1)(\tilde{u} - u^*)^+ \, dz \\ &= \langle A(u^*), (\tilde{u} - u^*)^+ \rangle + \int_\Omega \xi(z)(u^*)^{p-1}(\tilde{u} - u^*)^+ \, dz + \int_{\partial\Omega} \beta(z)(u^*)^{p-1}(\tilde{u} - u^*)^+ \, d\sigma \end{aligned}$$

whence

$$\tilde{u} \leq u^*. \tag{4.29}$$

Hypothesis (f2) implies that $f(\cdot, u^*(\cdot)) \in L^\infty(\Omega)$ and so we can find $\lambda_0 > 0$ small enough such that

$$\lambda f(z, u^*(z)) \leq 1 \tag{4.30}$$

for a.a. $z \in \Omega$ and all $0 < \lambda \leq \lambda_0$.

We now introduce the Carathéodory function $\tau_\lambda(z, x)$ defined by

$$\tau_\lambda(z, x) = \begin{cases} \tilde{u}(z)^{-\gamma} + \lambda f(z, \tilde{u}(z)) & \text{if } x \leq \tilde{u}(z) \\ x^{-\gamma} + \lambda f(z, x) & \text{if } \tilde{u}(z) \leq x \leq u^*(z) \\ u^*(z)^{-\gamma} + \lambda f(z, u^*(z)) & \text{if } x \geq u^*(z) \end{cases} \tag{4.31}$$

for $\lambda \in]0, \lambda_0]$ (see (4.29)).

We set $T_\lambda(z, x) = \int_0^x \tau_\lambda(z, s)\, ds$ and consider the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega G(Du)\, dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p \, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p \, d\sigma - \int_\Omega T_\lambda(z, u)\, dz \quad \forall u \in \mathbb{N}.$$

Proposition 2.7 and (4.31) imply that $\psi_\lambda(\cdot)$ is coercive. Also the Sobolev embedding theorem and the compactness of the trace map imply the $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \psi_\lambda(u_0) &= \inf\{\psi_\lambda(u) : u \in W^{1,p}(\Omega)\} \\ &\Rightarrow \psi_\lambda'(u_0) = 0 \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \langle A(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h\,dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h\,d\sigma \\
&= \int_\Omega \tau_\lambda(z, u_0)h\,dz \quad \forall h \in W^{1,p}(\Omega).
\end{aligned} \tag{4.32}
$$

In (4.32) first we choose $h = (\tilde{u} - u_0)^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\langle A(u_0), (\tilde{u} - u_0)^+)\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0(\tilde{u} - u_0)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0(\tilde{u} - u_0)^+\,d\sigma \\
&= \int_\Omega [\tilde{u}^{-\gamma} + \lambda f(z, \tilde{u})](\tilde{u} - u_0)^+\,dz \quad \text{(see (4.31))} \\
&\geq \int_\Omega \tilde{u}^{-\gamma}(\tilde{u} - u_0)^+\,dz \\
&\langle A(\tilde{u}), (\tilde{u} - u_0)^+)\rangle + \int_\Omega \xi(z)|\tilde{u}|^{p-1}(\tilde{u} - u_0)^+\,dz + \int_{\partial\Omega} \beta(z)|\tilde{u}|^{p-1}(\tilde{u} - u_0)^+\,d\sigma \\
&\Rightarrow \tilde{u} \leq u_0.
\end{aligned}
$$

Next in (4.32) we choose $h = (u_0 - u^*)^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\langle A(u_0), (u_0 - u^*)^+)\rangle + \int_\Omega \xi(z)u_0^{p-1}(u_0 - u^*)^+\,dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}(u_0 - u^*)^+\,d\sigma \\
&= \int_\Omega [(u^*)^{-\gamma} + \lambda f(z, u^*)](u_0 - u^*)^+\,dz \quad \text{(see (4.31))} \\
&\leq \int_\Omega [(u^*)^{-\gamma} + 1](u_0 - u^*)^+\,dz \quad (\text{recall } 0 < \lambda \leq \lambda_0 \text{ and see (4.28)}) \\
&= \langle A(u^*), (u_0 - u^*)^+)\rangle + \int_\Omega \xi(z)(u^*)^{p-1}(u_0 - u^*)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)(u^*)^{p-1}(u_0 - u^*)^+\,d\sigma \\
&\Rightarrow u_0 \leq u^*.
\end{aligned}
$$

Therefore we have proved that

$$
u_0 \in [\tilde{u}, u^*]. \tag{4.33}
$$

From (4.31), (4.32), (4.33) it follows that

$$
\begin{cases}
-\operatorname{div} a(Du_0(z)) + \xi(z)u_0(z)^{p-1} = u_0(z)^{-\gamma} + \lambda f(z, u_0(z)) & \text{for a.a. } z \in \Omega \\
\dfrac{\partial u_0}{\partial n_\alpha} + \beta(z)u_0^{p-1} = 0 & \text{on } \partial\Omega \quad \text{(see [24])}
\end{cases}
$$

and this implies that $]0, \lambda_0] \subseteq \mathcal{L}$ and $u_0 \in D_+$ (from the nonlinear regularity theory). □

A useful consequence of the above proof is the following Corollary.

Corollary 4.4 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold and $\lambda > 0$, then $S_\lambda \subseteq D_+$.*

Next we prove a structural property of $\mathcal{L}$, namely we show that $\mathcal{L}$ is an interval.

Proposition 4.5 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq D_+$ and we have

$$\begin{aligned} -\operatorname{div} a(Du_\lambda(z)) + \xi(z)u_\lambda(z)^{p-1} &= u_\lambda(z)^{-\gamma} + \lambda f(z, u_\lambda(z)) \\ &\geq u_\lambda(z)^{-\gamma} + \mu f(z, u_\lambda(z)) \quad \text{in } \Omega. \end{aligned} \tag{4.34}$$

On the other hand we have

$$\begin{aligned} -\operatorname{div} a(D\tilde{u}(z)) + \xi(z)\tilde{u}(z)^{p-1} &= \tilde{u}(z)^{-\gamma} \\ &\leq \tilde{u}(z)^{-\gamma} + \mu f(z, \tilde{u}(z)) \quad \text{in } \Omega. \end{aligned} \tag{4.35}$$

Moreover we we have

$$\begin{aligned} &\langle A(\tilde{u}), (\tilde{u} - u_\lambda)^+)\rangle + \int_\Omega \xi(z)\tilde{u}^{p-1}(\tilde{u} - u_\lambda)^+ \, dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1}(\tilde{u} - u_\lambda)^+ \, d\sigma \\ &= \int_\Omega \tilde{u}^{-\gamma}(\tilde{u} - u_\lambda)^+ \, dz \\ &\leq \int_\Omega u_\lambda^{-\gamma}(\tilde{u} - u_\lambda)^+ \, dz \\ &\leq \int_\Omega [u_\lambda^{-\gamma} + \lambda f(z, u_\lambda)](\tilde{u} - u_\lambda)^+ \, dz \\ &= \langle A(u_\lambda), (\tilde{u} - u_\lambda)^+)\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(\tilde{u} - u_\lambda)^+ \, dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(\tilde{u} - u_\lambda)^+ \, d\sigma \\ &\Rightarrow \tilde{u} \leq u_\lambda. \end{aligned} \tag{4.36}$$

Truncating the reaction of the problem (1.1) at $\tilde{u}(z)$ and at $u_\lambda(z)$ (see (4.36)), using the direct method of the calculus of variations, as before (see the proof of Proposition 4.3), we produce

$$u_\mu \in S_\mu \subseteq D_+, \quad u_\mu \in [\tilde{u}, u_\lambda]$$

and hence $\mu \in \mathcal{L}$. □

An interesting byproduct of the above proof is given by the following Corollary.

Corollary 4.6 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold, $\lambda \in \mathcal{L}$, $0 < \mu < \lambda$ and $u_\lambda \in S_\lambda \subseteq D_+$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subset D_+$ such that*

$$u_\lambda - u_\mu \in C_+ \backslash \{0\}.$$

In fact we can improve the above monotonicity result.

Proposition 4.7 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold, $\lambda \in \mathcal{L}$, $0 < \mu < \lambda$ and $u_\lambda \in S_\lambda \subseteq D_+$ then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subset D_+$ such that*

$$u_\lambda - u_\mu \in \hat{D}_+.$$

Proof From Corollary 4.6 we already know that $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq D_+$ such that

$$u_\lambda - u_\mu \in C_+ \backslash \{0\}. \tag{4.37}$$

Let $0 < m_\mu = \min_{\overline{\Omega}} u_\mu$ (recall that $u_\mu \in D_+$), $\rho = \|u_\lambda\|_\infty$ and let $\hat{\theta}_\rho > 0$ be as postulated by hypothesis (f5). We thus have

$$
\begin{aligned}
&-\operatorname{div} a(Du_\mu) + [\xi(z) + \lambda\hat{\theta}_\rho]u_\mu^{p-1} - u_\mu^{-\gamma} \\
&= \mu f(z, u_\mu) + \lambda\hat{\theta}_\rho u_\mu^{p-1} \\
&= \lambda f(z, u_\mu) + \lambda\hat{\theta}_\rho u_\mu^{p-1} - (\lambda - \mu) f(z, u_\mu) \\
&\leq \lambda f(z, u_\mu) + \lambda\hat{\theta}_\rho u_\mu^{p-1} - (\lambda - \mu)\hat{\eta}_{m_\mu} \quad \text{(see hypothesis (f5))} \\
&< \lambda f(z, u_\lambda) + \lambda\hat{\theta}_\rho u_\lambda^{p-1} \quad \text{(see (4.37) and hypothesis (f5))} \\
&= -\operatorname{div} a(Du_\lambda) + [\xi(z) + \lambda\hat{\theta}_\rho]u_\lambda^{p-1} - u_\lambda^{-\gamma} \quad \text{for all } z \in \Omega \text{ (since } u_\lambda \in S_\lambda).
\end{aligned} \tag{4.38}
$$

Since the map $x \to x^{-\gamma}$ is uniformly continuous on $\left[\frac{1}{\rho^\gamma}, \frac{1}{m_\mu^\gamma}\right]$, from (4.38) and Proposition 3.1 of Papageorgiou–Smyrlis [30] (see also Papageorgiou–Radulescu–Repovs [28], Proposition 2.10), we infer that $u_\lambda - u_\mu \in \hat{D}_+$. □

Let

$$\lambda^* = \sup \mathcal{L}.$$

Proposition 4.8 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold, then $\lambda^* < +\infty$.*

Proof On account of hypothesis (f3), we can find $\tilde{\lambda} > 0$ big enough such that

$$\tilde{\lambda} f(z, x) \geq [\xi(z) + 1]x^{p-1} \quad \text{for a.a. } z \in \Omega, \forall x \geq 0. \tag{4.39}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq D_+$. As before we set $0 < m_\lambda = \min_{\overline{\Omega}} u_\lambda$, $\rho = \|u_\lambda\|_\infty$ and $\hat{\theta}_\rho > 0$ as postulated by hypothesis (f5). For $\delta > 0$ we define $m_{\lambda,\delta} = m_\lambda + \delta \in D_+$ and we have

$$
\begin{aligned}
&-\operatorname{div} a(Dm_{\lambda,\delta}) + [\xi(z) + \lambda\hat{\theta}_\rho](m_{\lambda,\delta})^{p-1} - (m_{\lambda,\delta})^{-\gamma} \\
&\leq [\xi(z) + \lambda\hat{\theta}_\rho]m_\lambda^{p-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0 \text{ as } \delta \to 0 \\
&\leq [\xi(z) + 1]m_\lambda^{p-1} + \lambda\hat{\theta}_\rho m_\lambda^{p-1} + \chi(\delta) \\
&\leq \tilde{\lambda} f(z, m_\lambda) + \lambda\hat{\theta}_\rho m_\lambda^{p-1} + \chi(\delta) \quad \text{(see (4.39))} \\
&= \lambda f(z, m_\lambda) + \lambda\hat{\theta}_\rho m_\lambda^{p-1} - (\lambda - \tilde{\lambda}) f(z, m_\lambda) + \chi(\delta) \\
&< \lambda f(z, u_\lambda) + \lambda\hat{\theta}_\rho u_\lambda^{p-1} \\
&= -\operatorname{div} a(Du_\lambda) + [\xi(z) + \lambda\hat{\theta}_\rho]u_\lambda^{p-1} - u_\lambda^{-\gamma} \quad \text{for all } z \in \Omega \text{ (since } u_\lambda \in S_\lambda).
\end{aligned} \tag{4.40}
$$

As before (see the proof of proposition 4.7), from (4.40) we infer that

$$u_\lambda - m_{\lambda,\delta} \in \hat{D}_+$$

for some small enough $\delta > 0$, which is a contradiction to the definition of m_λ. Therefore $\lambda \notin \mathcal{L}$ and so

$$\lambda^* = \sup \mathcal{L} \leq \tilde{\lambda} < +\infty.$$

□

Next we show that $\lambda^* \in \mathcal{L}$ (that is the critical parameter λ^* is admissible).

Proposition 4.9 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold, then $\lambda^* \in \mathcal{L}$.*

Proof Let $\{\lambda_n\} \subseteq \mathcal{L}$ such that $\lambda_n \to \lambda^*$ as $n \to +\infty$. We can find $u_n \in S_{\lambda_n} \subseteq D_+$ foe all $n \in \mathbb{N}$. We know that we can assume $\{u_n\}$ is increasing. Moreover, on account of hypothesis (f3), there exists a constant $M_9 > 0$ such that

$$\hat{\varphi}_{\lambda_n}(u_n) \leq M_9 \quad \forall n \in \mathbb{N} \tag{4.41}$$

(see Proposition (4.2)). Moreover we have

$$\langle A(u_n), h)\rangle + \int_\Omega \xi(z) u_n^{p-1} h \, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h \, d\sigma = \int_\Omega \hat{f}_{\lambda_n}(z, u_n) h \, dz \tag{4.42}$$

for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$.

From (4.41) we deduce

$$p \int_\Omega G(Du_n) \, dz + \int_\Omega \xi(z) u_n^p \, dz + \int_{\partial\Omega} \beta(z) u_n^p \, d\sigma - p \int_\Omega \hat{F}_{\lambda_n}(z, u_n) \, dz \leq pM_9 \tag{4.43}$$

for all $n \in \mathbb{N}$.

On the other hand, from (4.42) with $h = u_n \in D_+$, we get

$$-\int_\Omega (a(Du_n), Du_n) \, dz - \int_\Omega \xi(z) u_n^p \, dz - \int_{\partial\Omega} \beta(z) u_n^p \, d\sigma = -\int_\Omega \hat{f}_{\lambda_n}(z, u_n) u_n \, dz \tag{4.44}$$

for all $n \in \mathbb{N}$.

Gathering together (4.43), (4.44) and using hypothesis (v) and (4.6), we infer that

$$\int_\Omega \hat{\mu}_{\lambda_n}(z, u_n) \, dz \leq M_{10} \tag{4.45}$$

for some constant $M_{10} > 0$ and all $n \in \mathbb{N}$.

Using (4.45) and reasoning as in the proof of Proposition 4.2 we show that

$$\{u_n\} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \quad \text{in } W^{1,p}(\Omega) \text{ and } u_n \to u^* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{4.46}$$

If in (4.42) we choose $h = u_n - u^* \in W^{1,p}(\Omega)$, we pass to the limit as $n \to 0$ and use (4.46), then

$$\lim_{n \to +\infty} < A(u_n), u_n - u^* >= 0 \;\Rightarrow\; u_n \to u^* \quad \text{in } W^{1,p}(\Omega). \tag{4.47}$$

Therefore passing to the limit as $n \to +\infty$ in (4.42) and using (4.47), we conclude that $u^* \in S_\lambda$, $\lambda^* \in \mathcal{L}$. □

In this way we have proved that

$$\mathcal{L} =]0, \lambda^*].$$

Next we show that for all $\lambda \in]0, \lambda^*[$ we can have at least two positive solutions.

Proposition 4.10 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold and* $0 < \lambda < \lambda^*$, *then problem* (1.1) *admits at least two positive solutions*

$$u_0, \hat{u} \in D_+, \;\; u_0 \neq \hat{u}.$$

Proof We know that we can find $u_0 \in S_\lambda \subseteq D_+$ such that

$$u_0 \in int_{C^1(\bar{\Omega})}[\tilde{u}, u^*] \quad \text{with } u^* \in S_\lambda \subseteq D_+. \tag{4.48}$$

(recall that by $int_{C^1(\bar{\Omega})}[\tilde{u}, u^*]$ we mean the interior in $C^1(\bar{\Omega})$ of $[\tilde{u}, u^*]$; see Proposition 4.7 and its proof).

Also if $\psi_\lambda \in C^1(W^{1,p}(\Omega), \mathbb{R})$ is as in the proof of Proposition 4.3 (see (4.31)), then u_0 is a minimizer of ψ_λ. From (4.31) and (4.6), we see that

$$\begin{aligned} &\psi_\lambda \big|_{[\tilde{u},u^*]} = \hat{\varphi}_\lambda \big|_{[\tilde{u},u^*]} \\ &\Rightarrow u_0 \quad \text{is a local } C^1(\bar{\Omega}) - \text{minimizer of } \hat{\varphi}_\lambda \text{ (see (4.48))} \\ &\Rightarrow u_0 \quad \text{is a local } W^{1,p}(\Omega) - \text{minimizer of } \hat{\varphi}_\lambda \text{ (see Proposition 2.8).} \end{aligned} \tag{4.49}$$

Using (4.6), we can easily check that

$$K_{\hat{\varphi}_\lambda} \subseteq [\tilde{u}[\cap D_+ := \{u \in D_+ \ : \ \tilde{u}(z) \leq u(z) \ \forall z \in \Omega\}. \tag{4.50}$$

Therefore we may assume that $K_{\hat{\varphi}_\lambda}$ is finite or otherwise we already have an infinity of positive smooth solutions. So, on account of (4.49), we can find $\rho \in]0, 1[$ small enough such that

$$\hat{\varphi}_\lambda(u_0) < \inf\{\hat{\varphi}_\lambda(u) \ : \ \|u - u_0\| = \rho\} = \hat{m}_\lambda. \tag{4.51}$$

From Proposition 4.2 we know that

$$\hat{\varphi}_\lambda \quad \text{satisfies the C-condition.} \tag{4.52}$$

Then (4.27), (4.51) and (4.52) permit the use of Theorem 2.1. So we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\hat{\varphi}_\lambda} \quad \text{and } \hat{m}_\lambda \leq \hat{\varphi}_\lambda(\hat{u}). \tag{4.53}$$

From (4.50), (4.53) and (4.6) we conclude that $\hat{u} \in D_+$ is the second positive solution of (1.1) distinct from u_0. □

So, summarizing the situation for the positive solutions of problem (1.1), as $\lambda > 0$ varies, we can state the following bifurcation-type result.

Theorem 4.11 *If hypotheses (i)–(v), (ξ), (β) and (f1)–(f5) hold then there exists a critical parameter value $\lambda^* > 0$ such that*

(a) *for all $\lambda \in]0, \lambda^*[$ problem (1.1) has at least two positive solutions*

$$u_0, \ \hat{u} \in D_+, \ u_0 \neq \hat{u};$$

(b) *for $\lambda = \lambda^*$ problem (1.1) has at least one positive solution*

$$u^* \in D_+;$$

(c) *for all $\lambda > \lambda^*$ problem (1.1) has no positive solutions.*

5 Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L} =]0, \lambda^*[$, problem (1.1) has a smallest positive solution $\bar{u}_\lambda \in D_+$ and we establish the monotonicity and continuity properties of the map $\lambda \to \bar{u}_\lambda$.

We know that S_λ is downward directed, that is, if $u, y \in S_\lambda$ then we can find $v \in S_\lambda$ such that $v \leq u$ and $v \leq y$ (see Papageorgiou–Radulescu–Repovs [26], proof of Proposition 7).

Proposition 5.1 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold and $\lambda \in \mathcal{L}$, then problem (1.1) has a smallest positive solution $\bar{u}_\lambda \in S_\lambda \subseteq D_+$ (that is $\bar{u}_\lambda \le u$ for all $u \in S_\lambda$).*

Proof According to Lemma 3.10, p. 178 of Hu–Papageorgiou [14], we can find $\{u_n\} \subset S_\lambda$ decreasing such that $\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n$. We thus have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, d\sigma = \int_\Omega [u_n^{-\gamma} + \lambda f(z, u_n)] h\, dz \tag{5.1}$$

for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$,

$$\tilde{u} \le u_n \le u_1 \quad \forall n \in \mathbb{N}. \tag{5.2}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ in (5.1) and using (5.2), we infer that

$$\{u_n\} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{5.3}$$

If in (5.1) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (5.3) then

$$\begin{aligned} &\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0 \\ &\Rightarrow \ u_n \to \bar{u}_\lambda \ \text{ in } W^{1,p}(\Omega) \text{ as } n \to +\infty \text{ (see Proposition 2.5).} \end{aligned} \tag{5.4}$$

Therefore if in (5.1) we pass to the limit as $n \to +\infty$ and use (5.4) then

$$\langle A(\bar{u}_\lambda), h\rangle + \int_\Omega \xi(z) \bar{u}_\lambda^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) \bar{u}_\lambda^{p-1} h\, d\sigma = \int_\Omega [\bar{u}_\lambda^{-\gamma} + \lambda f(z, \bar{u}_\lambda)] h\, dz \tag{5.5}$$

for all $h \in W^{1,p}(\Omega)$,

$$\tilde{u} \le \bar{u}_\lambda \quad \text{see (5.2).} \tag{5.6}$$

From (5.5) and (5.6) we conclude that $\bar{u}_\lambda \in S_\lambda$ and $\bar{u}_\lambda = \inf S_\lambda$. □

Next we examine the monotonicity and continuity properties of the map $\lambda \to \bar{u}_\lambda$ from $\mathcal{L}$ into $C^1(\bar{\Omega})$.

Proposition 5.2 *If hypotheses (i)–(v), (ξ), (β), (f1)–(f5) hold, then*

(a) *the map $\lambda \to \bar{u}_\lambda$ from $\mathcal{L}$ into $C^1(\bar{\Omega})$ is strictly increasing, that is*

$$0 < \mu < \lambda \le \lambda^* \ \Rightarrow \ \bar{u}_\lambda - \bar{u}_\mu \in \hat{D}_+;$$

(b) *the map $\lambda \to \bar{u}_\lambda$ from $\mathcal{L}$ into $C^1(\bar{\Omega})$ is left continuous.*

Proof (a) From Proposition 4.7 we know that we can find $u_\mu \in S_+ \subseteq D_+$ such that

$$\begin{aligned} &\bar{u}_\lambda - u_\mu \in \hat{D}_+ \\ &\Rightarrow \ \bar{u}_\lambda - u_\mu \in \hat{D}_+ \ \ (\text{since } \bar{u}_\mu \in u_\mu) \\ &\Rightarrow \ \lambda \to \bar{u}_\lambda \ \ \text{is strictly increasing.} \end{aligned}$$

(b) Suppose $\{\lambda_n, \lambda\} \subseteq \mathcal{L}$ and $\lambda_n \to \lambda$. We have

$$\tilde{u} \le \bar{u}_{\lambda_n} \le \bar{u}_\lambda \ \forall n \in \mathbb{N}, \ \{\bar{u}_{\lambda_n}\} \ \text{ increasing.} \tag{5.7}$$

Hence, as before, we have that $\{\bar{u}_{\lambda_n}\} \subset W^{1,p}(\Omega)$ is bounded. From (5.7) and the nonlinear regularity theory of Lieberman [21], it follows that we can find $\tau \in]0,1[$ and a constant $M_{11} > 0$ such that

$$\bar{u}_{\lambda_n} \in C^{1,\tau}(\bar{\Omega}) \text{ and } \|\bar{u}_{\lambda_n}\|_{C^{1,\tau}(\bar{\Omega})} \leq M_{11} \quad \forall n \in \mathbb{N}. \tag{5.8}$$

From (5.7), (5.8) and the compact imbedding of $C^{1,\tau}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, it follows that

$$\bar{u}_{\lambda_n} \to \tilde{u}_{\lambda_n} \quad \text{in } C^1(\bar{\Omega}) \text{ as } n \to +\infty. \tag{5.9}$$

We claim that $\bar{u}_\lambda = \tilde{u}_\lambda$. Indeed, if this is not the case then we can find $z_0 \in \bar{\Omega}$ such that

$$\begin{aligned} &\bar{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0) \\ &\Rightarrow \bar{u}_{\lambda_n}(z_0) < \tilde{u}_{\lambda_n}(z_0) \quad \forall n \geq n_0 \quad (\text{see } (5.9)) \end{aligned}$$

but this contradicts (a). Therefore the map $\lambda \to \bar{u}_\lambda$ from $\mathcal{L}$ into $C^1(\bar{\Omega})$ is left continuous. □

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