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Nonlinear Robin problems with indefinite potential

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ABSTRACT

We consider a nonlinear Robin problem driven by the p-Laplacian plus an indefinite potential. The conditions on the source term are minimal. We prove two multiplicity theorems with sign information for all the solutions. In the semilinear case ($p = 2$), we show that we can have multiple nodal solutions. We apply our results to a special class of logistic equations with equidiffusive reaction.

1. Introduction

In this paper we study the existence and multiplicity of nontrivial smooth solutions for the nonlinear Robin problem:

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In this problem $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega$. By Δ_p we denote the p-Laplace differential operator defined by

$$\Delta_p u = div(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < +\infty.$$

The potential function $\xi(z) \in L^\infty(\Omega)$ and is, in general, sign changing. So, the differential operator of the problem cannot be coercive. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$

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the function $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$ the function $x \to f(z,x)$ is continuous). The conditions on $f(z,\cdot)$ are minimal and imply that the energy (Euler) functional of the problem is coercive. Our aim is to prove multiplicity theorems for problem (1.1) and provide sign information for all the solution produced. We prove two such multiplicity theorems, which differ in the geometry of the energy functional near the origin. In the first, we assume the presence of a concave term (that is, of a $(p-1)$-sublinear term) near zero. In the second, we assume that $f(z,\cdot)$ has $(p-1)$-linear growth near zero. In both cases, we prove the existence of at least three nontrivial smooth solutions all with sign information (a positive solution, a negative solution and a nodal (sign changing) solution). When $p = 2$ (semilinear problem), we can improve the second multiplicity theorem and, using the theory of critical groups, we can generate a second nodal solution for a total of four nontrivial smooth solutions with sign information. Moreover, in the semilinear case, we can relax the conditions on potential function $\xi(\cdot)$ and assume that $\xi \in L^s(\Omega)$, with $s > N$, and $\xi^+ \in L^\infty(\Omega)$ (so $\xi(\cdot)$ can be unbounded below). At the end we use our results to a particular parametric logistic equation with reaction of equidiffusive type. We show that for all $\lambda > \lambda^*$ ($\lambda \in \mathbb{R}$ being the parameter), the problem has three nontrivial solutions all with sign information. Also, we identify exactly the critical parameter value $\lambda^* > 0$ by means of the spectrum of the differential operator. Our work here complements that of Papageorgiou–Radulescu–Repovs [23], where the reaction $f(z,\cdot)$ is asymptotically, as $x \to \pm\infty$, at resonance with respect to a non principal eigenvalue of the differential operator $u \to -\Delta_p u + \xi(z)|u|^{p-2}u$ with Robin boundary condition. This makes the energy functional noncoercive. To have a complete overview on the literature one can refer also to the relevant papers [15,19–21].

For other kind of operators with lower order terms see also [7–9]

2. Mathematical background. Hypotheses

In the analysis of problem (1.1), we will use the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\bar{\Omega})$. By $\|\cdot\|_{W^{1,p}(\Omega)}$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\|_{W^{1,p}(\Omega)} := \left[\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)}\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\bar{\Omega})$ is ordered with positive (order) cone given by

$$C_+ = \{u \in C^1(\bar{\Omega}) \,:\, u(z) \geq 0 \ \forall z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$int\, C_+ = \{u \in C_+ \,:\, u(z) > 0 \ \forall z \in \bar{\Omega}\}.$$

Also we will use another open cone in $C^1(\bar{\Omega})$, namely

$$D_+ = \left\{u \in C^1(\bar{\Omega}) \,:\, u(z) > 0 \ \forall z \in \bar{\Omega},\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure we can define, in the usual way, the boundary Lebesgue spaces $L^p(\partial\Omega)$, $1 \leq p \leq +\infty$. There is a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u_{|\partial\Omega} \ \ \forall u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. This map is not surjective. In fact we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \ \ (1/p + 1/p' = 1) \ \text{ and } \ker\gamma_0 = W^{1,p}_0(\Omega).$$

Moreover, $\gamma_0(\cdot) \in \mathcal{L}(W^{1,p}(\Omega), L^p(\partial\Omega))$ is compact.

In the sequel, for the sake of notational simplicity, we drop the use of trace map. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

We will use some facts about the spectrum, of the differential operator $u \to -\Delta_p u + \xi(z)|u|^{p-2}u$ with Robin boundary condition. So, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}\,|u(z)|^{p-2}u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" if (2.1) admits a nontrivial solution $\hat{u} \in W^{1,p}(\Omega)$ which is known as an "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. We impose the following conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

H(ξ): $\xi \in L^\infty(\Omega)$.

Remark 2.1. So the potential function is in general sign changing.

H(β): $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in]0,1[$ and $\beta(z) \geq 0$, $\forall z \in \partial\Omega$.

Remark 2.2. The case $\beta \equiv 0$ is also included and corresponds to the Neumann problem.

Let $K : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$K(u) = \|Du\|^p_{L^p(\Omega)} + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

The eigenvalue problem (2.1) was studied by Papageorgiou–Radulescu [12] and Fragnelli–Mugnai–Papageorgiou [5]. We know that the set $\hat{\sigma}(p) \subset \mathbb{R}$ of eigenvalues is closed and there is a smallest eigenvalue $\hat{\lambda}_1 \in \mathbb{R}$ which has the following properties:

(*a*) $\hat{\lambda}_1$ is isolated, that is, there exists $\varepsilon > 0$ such that

$$]\hat{\lambda}_1, \hat{\lambda}_1 + \varepsilon[\cap \hat{\sigma}(p) = \emptyset,$$

(*b*) $\hat{\lambda}_1$ is simple, that is, if $\hat{u}, \hat{v}$ are eigenfunctions corresponding to $\hat{\lambda}_1$, then $\hat{u} = \eta\hat{v}$ for some $\eta \in \mathbb{R} \setminus \{0\}$,

(*c*)

$$\hat{\lambda}_1 = \inf\left\{ \frac{K(u)}{\|u\|^p_{L^p(\Omega)}} : u \in W^{1,p}(\Omega),\, u \neq 0 \right\}. \tag{2.2}$$

The infimum in (2.2) is realized on the corresponding one dimensional eigenspace (see (b)). From (2.2) it follows that the elements of this eigenspace have fixed sign. In fact $\hat{\lambda}_1$ is the only eigenvalue with eigenfunctions of fixed sign. All the other eigenvalues have eigenfunctions which are nodal (sign changing).

By $\hat{u}_1$ we denote the positive, L^p-normalized (that is, $\|\hat{u}_1\|_{L^p(\Omega)} = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. Nonlinear regularity theory (see Lieberman [10]) and the nonlinear maximum principle (see Pucci–Serrin [22], pp. 111, 120) imply that $\hat{u}_1 \in intC_+$. Since $\hat{\lambda}_1$ is isolated and $\hat{\sigma}(p)$ is closed, the second eigenvalue $\hat{\lambda}_2^*$ is well defined by

$$\hat{\lambda}_2^* = \inf\left\{ \hat{\lambda} : \hat{\lambda} \in \hat{\sigma}(p),\, \hat{\lambda} > \hat{\lambda}_1 \right\}.$$

Moreover, using Ljusternik–Schnirelmann minimax scheme, we can generate a whole sequence $\left\{\hat{\lambda}_k\right\} k \geq 1$ of distinct eigenvalues (LS-eigenvalues for short). We do not know if the LS-eigenvalues exhaust $\hat{\sigma}(p)$. When $p = 2$, then $\hat{\sigma}(p) = \left\{\hat{\lambda}_k\right\}_{k\geq 1}$. Also $\hat{\lambda}_2^* = \hat{\lambda}_2$, that is, the second eigenvalue coincides with the second LS-eigenvalue. The Ljusternik–Schnirelmann minimax scheme provides a variational (minimax) characterization of $\hat{\lambda}_2$. But for our purposes this characterization is not convenient. Instead we will use an alternative one which can be found in Papageorgiou–Radulescu [12].

So, let $\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ and $M = W^{1,p}(\Omega) \cap \partial B_1^{L^p}$. We consider the following set of continuous paths

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M) : \hat{\gamma}(-1) = -\hat{u}_1, \hat{\gamma}(1) = \hat{u}_1\}.$$

Proposition 2.3. $\hat{\lambda}_2 = \inf_{\hat{\gamma}\in\hat{\Gamma}} \max_{t\in[-1,1]} K(\hat{\gamma}(t))$.

By $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ we denote the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

The next proposition presents the main properties of this map (see Papageorgiou–Radulescu–Repovs [18], pp. 114, 158).

Proposition 2.4. *The map $A(\cdot)$ is bounded, continuous, monotone, and of type $(S)_+$, that is, the following implication holds:*

"if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0 \Rightarrow u_n \to u$ in $W^{1,p}(\Omega)$" (see [18], Definition 2.10.11(a)).

If $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$ and if $u \in W^{1,p}(\Omega)$, then we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that

$$u^{\pm} \in W^{1,p}(\Omega), \;\; u = u^+ - u^-, \;\; |u| = u^+ + u^-.$$

Given $u, v \in W^{1,p}(\Omega)$ with $u(z) \le v(z)$ for a.a. $z \in \Omega$, we define

$$[u, v] = \{h \in W^{1,p}(\Omega) \,:\, u(z) \le h(z) \le v(z) \;\text{ for a.a. } z \in \Omega\}.$$

Also, by $int_{C^1(\bar{\Omega})}[u, v]$ we denote the interior in the $C^1(\bar{\Omega})$-norm topology of $[u, v] \cap C^1(\bar{\Omega})$.

We say that a set $S \subseteq W^{1,p}(\Omega)$ is a "downward directed" (resp. "upward directed") if for all $u_1, u_2 \in S$, we can find $u \in S$ such that $u \le u_1$, $u \le u_2$ (resp. for all $v_1, v_2 \in S$, we can find $v \in S$ such that $v_1 \le v$, $v_2 \le v$).

Finally let us recall some basic facts about critical groups which we will need in the sequel.

So, let X be a Banach space and $\varphi \in C^1(X; \mathbb{R})$. For $\eta \in \mathbb{R}$, we introduce the following sets

$$K_\varphi = \{u \in X \,:\, \varphi'(u) = 0\}, \quad K_\varphi^\eta = \{u \in K_\varphi : \varphi(u) = \eta\}, \quad \varphi^\eta = \{u \in X : \varphi(u) \le \eta\}.$$

Also for any topological pair (Y_1, Y_2), with $Y_2 \subseteq Y_1 \subseteq X$, and every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group with integer coefficients for the pair (Y_1, Y_2).

If $u \in K_\varphi$ is isolated and $\varphi(u) = c$ (that is, $u \in K_\varphi^c$), then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$.

The excision property of singular homology implies that the above definition is independent of the choice of the isolating neighborhood U.

Suppose that $\varphi(\cdot)$ satisfies the Cerami condition (C-condition, for short) and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

From the Second Deformation Theorem (see [18], Theorem 5.3.12, p. 386), we see that this definition is independent of the choice of $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define

$$M(t,u) = \sum_{k\geq 0} rank\, C_k(\varphi,u)\, t^k \quad \text{for all } t \in \mathbb{R} \text{ all } u \in K_\varphi,$$

$$P(t,u) = \sum_{k\geq 0} rank\, C_k(\varphi,\infty)\, t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t),$$

where $Q(t) = \sum_{k\geq 0} \beta_k\, t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

3. Constant sign solutions

In this section we prove the existence of constant sign smooth solutions for problem (1.1). We also show the existence of extremal constant sign solutions (that is, a smallest positive solution and a biggest negative solution). These extremal constant sign solutions will be helpful in producing a nodal solution, see Section 4.

We state the conditions on the source term $f(z,x)$. Recall that p^* is the critical Sobolev exponent for $p > 1$, that is $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$.

***H*₁**: Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(*i*) there exists a function $\alpha \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leq \alpha(z)[1 + |x|^{r-1}] \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R},\ p < r < p^*;$$

(*ii*)

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = -\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

(*iii*) there exists a function $\eta \in L^\infty$ such that

$$\eta(z) \geq \hat{\lambda}_1, \quad \text{for a.a. } z \in \Omega,\ \eta \neq \hat{\lambda}_1,$$

$$\liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \geq \eta(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Proposition 3.1. *If hypotheses $H(\xi), H(\beta), H_1$ hold, then problem (1.1) has at least two constant sign solutions*

$$u_0 \in int\, C_+ \quad \text{and} \quad v_0 \in -int\, C_+.$$

Proof. First we produce a positive solution.

Hypothesis H_1 implies that we can find $M_1 > 0$ such that

$$\frac{f(z,x)}{|x|^{p-2}x} \leq -\|\xi\|_{L^\infty(\Omega)} \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_1. \tag{3.1}$$

With $\theta > \|\xi\|_{L^\infty(\Omega)}$, we introduce the Carathéodory function $\hat{f}_+(z,x)$ defined by

$$\hat{f}_+(z,x) = \begin{cases} f(z,x^+) + \theta(x^+)^{p-1} & \text{if } x \le M_1 \\ f(z,M_1) + \theta M_1^{p-1} & \text{if } x > M_1. \end{cases} \tag{3.2}$$

We set $\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)\,ds$ and consider the C^1-functional $\hat{\varphi}_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{p}K(u) + \frac{\theta}{p}\|u\|^p_{L^p(\Omega)} - \int_\Omega \hat{F}_+(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega),$$

From (3.2) and since $\theta > \|\xi\|_{L^\infty(\Omega)}$, we see that $\hat{\varphi}_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem and the compactness of the trace map, we infer that $\hat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous. By the Weierstrass–Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf\left[\hat{\varphi}_+(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.3}$$

Let $F(z,x) = \int_0^x f(z,s)\,ds$. Hypothesis $H_1(iii)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in]0, M_1[$ such that

$$\frac{1}{p}[\eta(z) - \varepsilon]|x|^p \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| < \delta. \tag{3.4}$$

Recall that $\hat{u}_1 \in int\, C_+$. So, we can find $t \in]0,1[$ small such that

$$0 < t\,\hat{u}_1 \le \delta \quad \text{for all } z \in \bar{\Omega}. \tag{3.5}$$

Then from (3.2), (3.4) and (3.5), we see that

$$\begin{aligned} \hat{\varphi}_+(t\hat{u}_1) &\le \frac{t^p}{p}K(\hat{u}_1) - \frac{t^p}{p}\int_\Omega [\eta(z) - \varepsilon]\hat{u}_1^p\,dz \\ &= \frac{t^p}{p}\left[\int_\Omega [\hat{\lambda}_1 - \eta(z)]\hat{u}_1^p\,dz + \varepsilon\right] \quad (\text{recall } \|\hat{u}_1\|_{L^p(\Omega)} = 1). \end{aligned} \tag{3.6}$$

Hypothesis $H_1(iii)$ implies that

$$\tau_0 = \int_\Omega [\eta(z) - \hat{\lambda}_1]\hat{u}_1^p\,dz > 0.$$

From (3.6) we have

$$\begin{aligned} & \hat{\varphi}_+(t\hat{u}_1) \le \frac{t^p}{p}[-\tau_0 + \varepsilon] < 0 \quad (\text{choosing } \varepsilon \in]0, \tau_0[), \\ \Rightarrow\ & \hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \quad (\text{see } (3.3)), \\ \Rightarrow\ & u_0 \ne 0. \end{aligned}$$

From (3.3) we have

$$\begin{aligned} & \hat{\varphi}'_+(u_0) = 0, \\ \Rightarrow\ & \langle A(u_0), h\rangle + \int_\Omega [\xi(z) + \theta]|u_0|^{p-2}u_0 h\,dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h\,d\sigma = \int_\Omega \hat{f}_+(z,u_0)h\,dz \end{aligned} \tag{3.7}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.7) first we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & K(u_0^-) + \theta\|u_0^-\|^p_{L^p(\Omega)} = 0 \quad (\text{see } (3.2)), \\ \Rightarrow\ & c_1\,\|u_0^-\|^p_{W^{1,p}(\Omega)} \le 0 \quad \text{for some constant } c_1 > 0 \ (\text{since } \theta > \|\xi\|_{L^\infty(\Omega)}), \\ \Rightarrow\ & u_0 \ge 0,\ u_0 \ne 0. \end{aligned}$$

Next in (3.7) we choose $h = (u_0 - M_1)^+ \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned}
& \langle A(u_0), (u_0 - M_1)^+\rangle + \int_\Omega [\xi(z) + \theta] u_0^{p-1} (u_0 - M_1)^+ \, dz + \int_{\partial\Omega} \beta(z) u_0^{p-1} (u_0 - M_1)^+ \, d\sigma \\
= \; & \int_\Omega [f(z, M_1) + \theta M_1^{p-1}](u_0 - M_1)^+ \, dz \quad \text{(see (3.2))} \\
\le \; & \langle A(M_1), (u_0 - M_1)^+\rangle + \int_\Omega [\xi(z) + \theta] M_1^{p-1} (u_0 - M_1)^+ \, dz + \int_{\partial\Omega} \beta(z) M_1^{p-1} (u_0 - M_1)^+ \, d\sigma \\
& \text{(see (3.1) and hypothesis } H(\beta)), \\
\Rightarrow \; & u_0 \le M_1 \quad (\text{recall that } \theta > \|\xi\|_{L^\infty(\Omega)}).
\end{aligned}$$

So we have proved that

$$u_0 \in [0, M_1], \; u_0 \neq 0. \tag{3.8}$$

From (3.2), (3.7) and (3.8) we infer that

$$\begin{cases} -\Delta_p u_0(z) + \xi(z) u_0(z)^{p-1} = f(z, u_0(z)) & \text{in } \Omega \\ \dfrac{\partial u_0}{\partial n_p} + \beta(z) u_0^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

From Eq. (3.9) and Proposition 2.10 of Papageorgiou–Radulescu [14], we have that

$$u_0 \in L^\infty(\Omega).$$

Then Theorem 2 of Lieberman [10] implies that

$$u_0 \in C_+ \setminus \{0\}.$$

Let $\rho = \|u_0\|_{L^\infty(\Omega)}$. Hypotheses $H_1(i), (iii)$ imply that we can find $\hat{\xi}_\rho > 0$ such that

$$f(z, x) + \hat{\xi}_\rho x^{p-1} \ge 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \rho].$$

Then from (3.9) we have

$$\begin{aligned}
& \Delta_p u_0(z) \le \left[\|\xi\|_{L^\infty(\Omega)} + \hat{\xi}_\rho\right] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega \\
\Rightarrow \; & u_0 \in int\, C_+ \quad \text{(by the nonlinear maximum principle).}
\end{aligned}$$

Similarly, we obtain a negative solution $v_0 \in -int\, C_+$, using this time the Carathéodory function

$$\hat{f}_-(z, x) = \begin{cases} f(z, -M_1) - \theta M_1^{p-1} & \text{if } x < -M_1 \\ f(z, -x^-) - \theta (x^-)^{p-1} & \text{if } x \ge -M_1. \end{cases} \quad \square$$

In fact we can produce extremal constant sign solutions for problem (1.1), that is, a smallest positive solution and a biggest negative solution.

To this end, we proceed as follows. Hypotheses $H_1(i), (iii)$ imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$f(z, x)x \ge [\eta(z) - \varepsilon]|x|^p - c_\varepsilon |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.10}$$

The unilateral growth restriction on $f(z, \cdot)$ leads to the following auxiliary nonlinear Robin problem

$$\begin{cases} -\Delta_p u_{(z)} + \xi(z)|u(z)|^{p-2} u(z) = [\eta(z) - \varepsilon]|u(z)|^{p-2} u(z) - c_\varepsilon |u(z)|^{r-2} u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Proposition 3.2. *If hypotheses $H(\xi), H(\beta)$ hold and $\varepsilon > 0$ is small, then problem (3.11) admits a unique positive solution $\tilde{u} \in intC_+$ and, since the equation is odd, $\tilde{v} = -\tilde{u} \in -intC_+$ is the unique negative solution of (3.11).*

Proof. First we show the existence of a positive solution.

We consider the C^1-functional $\psi_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}K(u) + \frac{\theta}{p}\|u^-\|^r_{L^r(\Omega)} - \frac{1}{p}\int_\Omega [\eta(z) - \varepsilon](u^+)^p\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

As before (see the proof of Proposition 3.1), $\theta > \|\xi\|_{L^\infty(\Omega)}$. Since $r > p$ and $\theta > \|\xi\|_{L^\infty(\Omega)}$ we see that $\psi_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\psi_+(\tilde{u}) = \inf\left\{\psi_+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.12}$$

Let $\varepsilon \in]0, \tau_0[$. Then as in the proof of Proposition 3.1 and since $r > p$, we have that

$$\begin{aligned} &\psi_+(\tilde{u}) < 0 = \psi_+(0),\\ \Rightarrow\quad &\tilde{u} \neq 0. \end{aligned}$$

Also, from (3.2) it follows that

$$\begin{aligned} &\psi'_+(\tilde{u}) = 0,\\ \Rightarrow\quad &\langle A(\tilde{u}), h\rangle + \int_\Omega \xi(z)|\tilde{u}|^{p-2}\tilde{u}\,h\,dz + \int_{\partial\Omega}\beta(z)|\tilde{u}|^{p-2}\tilde{u}\,h\,d\sigma - \theta\int_\Omega(\tilde{u}^-)^{p-1}h\,d\sigma\\ =\quad &\int_\Omega[\eta(z) - \varepsilon](\tilde{u}^+)^{p-1}h\,dz - \int_\Omega c_\varepsilon(\tilde{u}^+)^{r-1}h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.13}$$

In (3.13) we choose $h = -\tilde{u}^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &K(\tilde{u}^-) + \theta\|\tilde{u}^-\|^p_{L^p(\Omega)} = 0,\\ \Rightarrow\quad &c_2\|\tilde{u}^-\|^p_{W^{1,p}(\Omega)} \le 0 \quad \text{for some constant } c_2 > 0 \text{ (since } \theta > \|\xi\|_{L^\infty(\Omega)}),\\ \Rightarrow\quad &\tilde{u} \ge 0,\ \tilde{u} \neq 0. \end{aligned}$$

From (3.13) we infer that

$$\left\{\begin{array}{ll} -\Delta_p\tilde{u}_{(z)} + \xi(z)\tilde{u}(z)^{p-1} = [\eta(z) - \varepsilon]\tilde{u}(z)^{p-1} - c_\varepsilon\tilde{u}(z)^{r-1} & \text{in } \Omega\\ \dfrac{\partial\tilde{u}}{\partial n_p} + \beta(z)\tilde{u}^{p-1} = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.14}$$

Then the nonlinear regularity theory (see the proof of Proposition 3.1) implies that $\tilde{u} \in C_+\backslash\{0\}$. Moreover, from (3.14) we have

$$\begin{aligned} &\Delta_p\tilde{u}(z) \le [c_\varepsilon\|\tilde{u}\|^{r-p}_{L^\infty(\Omega)} + c_\varepsilon]\tilde{u}(z)^{p-1}\\ \Rightarrow\quad &\tilde{u} \in C_+ \quad \text{(by the nonlinear maximum principle).} \end{aligned}$$

Next we show that this positive solution is unique. Suppose that $\tilde{y}$ is another positive solution of problem (3.11). Again we have that $\tilde{y} \in intC_+$. Let $t > 0$ be the biggest positive real such that $t\tilde{y} \le \tilde{u}$. Suppose $0 < t < 1$. Let $\rho = \|\tilde{u}\|_{L^\infty(\Omega)}$ and choose $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \to [\eta(z) - \varepsilon]x^{p-1} - c_\varepsilon x^{r-1} + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$. We have

$$\begin{aligned} &-\Delta_p(t\tilde{y}) + [\xi(z) + \hat{\xi}_\rho](t\tilde{y})^{p-1}\\ =\quad &t^{p-1}[-\Delta_p\tilde{y} + (\xi(z) + \hat{\xi}_\rho)\tilde{y}^{p-1}]\\ =\quad &t^{p-1}[(\eta(z) - \varepsilon)\tilde{y}^{p-1} - c_\varepsilon\tilde{y}^{r-1} + \hat{\xi}_\rho\tilde{y}^{p-1}]\\ \le\quad &(\eta(z) - \varepsilon)(t\tilde{y})^{p-1} - c_\varepsilon(t\tilde{y})^{r-1} + \hat{\xi}_\rho(t\tilde{y})^{p-1} \quad \text{(since } 0 < t < 1,\ p < r)\\ \le\quad &(\eta(z) - \varepsilon)\tilde{u}^{p-1} - c_\varepsilon\tilde{u}^{r-1} + \hat{\xi}_\rho\tilde{u}^{p-1} \quad \text{(since } t\tilde{y} \le \tilde{u})\\ =\quad &-\Delta_p\tilde{u} + [\xi(z) + \hat{\xi}_\rho]\tilde{u}^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{3.15}$$

Note that since $\tilde{y} \in intC_+$, $p < r$, $0 < t < 1$, we can find a constant $c_4 > 0$ such that

$$c_\varepsilon(t^{p-1} - t^{r-1})\tilde{y}(z)^{r-1} \geq c_4 > 0 \quad \text{for a.a. } z \in \Omega.$$

Therefore from (3.15) and Proposition 2.10 of Papageorgiou–Radulescu–Repovs [17], we have

$$\tilde{u} - t\tilde{y} \in D_+,$$

which contradicts the maximality of $t > 0$. So, we must have $t \geq 1$ which implies that

$$\tilde{y} \leq \tilde{u}.$$

Reversing the roles of $\tilde{u}$ and $\tilde{y}$ in the above argument, we also have

$$\begin{aligned} & \tilde{u} \leq \tilde{y}, \\ \Rightarrow \quad & \tilde{u} = \tilde{y} \end{aligned}$$

proving the uniqueness of the positive solution $\tilde{u} \in C_+$ of problem (3.11). Since the problem is odd, $\tilde{v} = -\tilde{u} \in -intC_+$ is the unique negative solution of (3.11). □

Remark 3.3. We present an alternative proof of the uniqueness part based on the nonlinear Picone's identity due to Allegretto–Huang [1]. So for $v, w \in intC_+$, let

$$R(v,w) \equiv R(v(z), w(z)) := |Dv(z)|^p - \left(|Dw(z)|^{p-2}Dw(z), D\left(\frac{v(z)^p}{w(z)^{p-1}}\right)\right)_{\mathbb{R}^N}.$$

We know that $0 \leq R(v,w)$ for a.a. $z \in \Omega$ (see [1]). Suppose that $\tilde{u}, \tilde{y}$ are two positive solutions of (3.11). We have $\tilde{u}, \tilde{y} \in intC_+$. Then

$$\begin{aligned} & \int_\Omega ([\eta(z) - \varepsilon] - c_\varepsilon \tilde{u}^{r-p})(\tilde{u}^p - \tilde{y}^p)\,dz \\ = \; & \int_\Omega ([\eta(z) - \varepsilon]\tilde{u}^{p-1} - c_\varepsilon \tilde{u}^{r-1})\left(\tilde{u}^p - \frac{\tilde{y}^p}{\tilde{u}^{p-1}}\right) dz \\ = \; & \int_\Omega (-\Delta_p \tilde{u} + \xi(z)\tilde{u}^{p-1})\left(\tilde{u}^p - \frac{\tilde{y}^p}{\tilde{u}^{p-1}}\right) dz \\ = \; & \int_\Omega |D\tilde{u}|^{p-2}\left(D\tilde{u}, D\left(\tilde{u}^p - \frac{\tilde{y}^p}{\tilde{u}^{p-1}}\right)\right)_{\mathbb{R}^N} dz + \int_\Omega \xi(z)\tilde{u}^{p-1}\left(\tilde{u}^p - \frac{\tilde{y}^p}{\tilde{u}^{p-1}}\right) dz \\ & + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1}\left(\tilde{u}^p - \frac{\tilde{y}^p}{\tilde{u}^{p-1}}\right) d\sigma \quad \text{(using the nonlinear Green's identity, see [18] p.35)} \\ = \; & \|D\tilde{u}\|^p_{L^p(\Omega)} - \|D\tilde{y}\|^p_{L^p(\Omega)} + \int_\Omega R(\tilde{y}, \tilde{u})\,dz + \int_\Omega \xi(z)(\tilde{y}^p - \tilde{u}^p)\,dz + \int_{\partial\Omega} \beta(z)(\tilde{y}^p - \tilde{u}^p)\,d\sigma. \end{aligned} \tag{3.16}$$

Interchanging the roles of $\tilde{u}$ and $\tilde{y}$ in the move argument, we also have

$$\begin{aligned} & \int_\Omega ([\eta(z) - \varepsilon] - c_\varepsilon \tilde{y}^{r-p})(\tilde{y}^p - \tilde{u}^p)\,dz \\ = \; & \|D\tilde{y}\|^p_{L^p(\Omega)} - \|D\tilde{u}\|^p_{L^p(\Omega)} + \int_\Omega R(\tilde{u}, \tilde{y})\,dz + \int_\Omega \xi(z)(\tilde{u}^p - \tilde{y}^p)\,dz + \int_{\partial\Omega} \beta(z)(\tilde{u}^p - \tilde{y}^p)\,d\sigma. \end{aligned} \tag{3.17}$$

Adding together (3.16) and (3.17), we obtain

$$\begin{aligned} 0 \; \leq \; & \int_\Omega [R(\tilde{y}, \tilde{u}) + R(\tilde{u}, \tilde{y})]\,dz \\ = \; & \int_\Omega c_\varepsilon(\tilde{y}^{r-p} - \tilde{u}^{r-p})(\tilde{u}^{r-p} - \tilde{y}^{r-p})\,dz, \\ & \Rightarrow \tilde{u} = \tilde{y}. \quad \square \end{aligned}$$

Now let S_+ (resp. S_-) be the set of positive (resp. negative) solutions of problem (1.1). From Proposition 3.1 we know that

$$\emptyset \neq S_+ \subseteq intC_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -intC_+.$$

Proposition 3.4. *If Hypotheses $H(\xi), H(\beta), H_1$ hold, then $\tilde{u} \leq u$ for all $u \in S_+$ and $v \leq \tilde{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+ \subseteq C_+$ and consider the following Carathéodory function

$$g_+(z,x) = \begin{cases} [\eta(z) - \varepsilon](x^+)^{p-1} - c_\varepsilon (x^+)^{r-1} + \theta(x^+)^{p-1} & \text{if } x \leq u(z) \\ [\eta(z) - \varepsilon]\, u(z)^{p-1} - c_\varepsilon u(z)^{r-1} + \theta u(z)^{p-1} & \text{if } x > u(z), \end{cases} \tag{3.18}$$

with $0 < \varepsilon < \min\{\tau_0, \min_{\overline{\Omega}} u\}$ (since $u \in intC_+$) and as before $\theta > \|\xi\|_{L^\infty(\Omega)}$.

We set $G_+(z,x) = \int_0^x g_+(z,s)\, ds$ and consider the C^1-functional $\hat{\psi}_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{1}{p} K(u) + \frac{\theta}{p} \|u\|^p_{L^p(\Omega)} - \int_\Omega G_+(z,u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $\hat{\psi}_+(\cdot)$ is coercive (see (3.18) and recall that $\theta > \|\xi\|_{L^\infty(\Omega)}$). Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_+(\tilde{u}_0) = inf \left[\hat{\psi}_+(u) : u \in W^{1,p}(\Omega) \right]. \tag{3.19}$$

As before, since $0 < \varepsilon < \min\{\tau_0, \min_{\overline{\Omega}} u\}$, we have

$$\begin{aligned} & \hat{\psi}_+(\tilde{u}_0) < 0 = \hat{\psi}_+(0), \\ \Rightarrow\ & \tilde{u}_0 \neq 0. \end{aligned}$$

From (3.19) we have

$$\begin{aligned} & \hat{\psi}'_+(\tilde{u}_0) = 0, \\ \Rightarrow\ & \langle A(\tilde{u}_0), h \rangle + \int_\Omega [\xi(z) + \theta] |\tilde{u}_0|^{p-2} u_0 h\, dz + \int_{\partial\Omega} \beta(z) |\tilde{u}_0|^{p-2} u_0 h\, d\sigma = \int_\Omega g(z, \tilde{u}_0) h\, dz \end{aligned} \tag{3.20}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.20) first we choose $h = -\tilde{u}_0^- \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned} & K(\tilde{u}_0^-) + \theta \|\tilde{u}_0^-\|^p_{L^p(\Omega)} = 0 \\ \Rightarrow\ & c_5 \|\tilde{u}_0^-\|^p_{W^{1,p}(\Omega)} \leq 0 \quad \text{for some constant } c_5 > 0 \text{ (since } \theta > \|\xi\|_{L^\infty(\Omega)}), \\ \Rightarrow\ & \tilde{u}_0 \geq 0,\ \ \tilde{u}_0 \neq 0. \end{aligned}$$

Next in (3.20) we choose $h = (\tilde{u}_0 - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(\tilde{u}_0), (\tilde{u}_0 - u)^+ \rangle + \int_\Omega [\xi(z) + \theta] \tilde{u}_0^{p-1} (\tilde{u}_0 - u)^+\, dz + \int_{\partial\Omega} \beta(z) \tilde{u}_0^{p-1} (\tilde{u}_0 - u)^+\, d\sigma \\ = & \int_\Omega [(\eta(z) - \varepsilon) u^{p-1} - c_\varepsilon u^{r-1} + \theta u^{p-1}] (\tilde{u}_0 - u)^+\, dz \quad \text{(see (3.18))} \\ \leq & \int_\Omega [f(z,u) + \theta u^{p-1}] (\tilde{u}_0 - u)^+\, dz \quad \text{(see (3.10))} \\ = & \langle A(u), (\tilde{u}_0 - u)^+ \rangle + \int_\Omega [\xi(z) + \theta] u^{p-1} (\tilde{u}_0 - u)^+\, dz + \int_{\partial\Omega} \beta(z) u^{p-1} (\tilde{u}_0 - u)^+\, d\sigma \\ & \text{(since } u \in S_+) \\ \Rightarrow\ & \tilde{u}_0 \leq u. \end{aligned}$$

Hence we have proved that

$$\tilde{u}_o \in [0, u], \quad \tilde{u}_0 \neq 0. \tag{3.21}$$

From (3.18), (3.21) and (3.20), it follows that $\tilde{u}$ is a positive solution of problem (3.11), therefore

$$\begin{aligned} & \tilde{u}_0 = \tilde{u} \in intC_+ \quad \text{(see Proposition 3.2)}, \\ \Rightarrow \quad & \tilde{u} \leq u \quad \text{for all } u \in S_+. \end{aligned}$$

Similarly using te Carathéodory function

$$g_-(z,x) = \begin{cases} -[\eta(z) - \varepsilon](x^-)^{p-1} - c_\varepsilon(x^-)^{r-1} - \theta(x^-)^{p-1} & \text{if } x \leq v(z) \\ -[\eta(z) - \varepsilon](-v(z))^{p-1} - c_\varepsilon(-v(z))^{r-1} - \theta(-v(z))^{p-1} & \text{if } x > v(z) \end{cases}$$

we show that $v \leq \tilde{v}$ for all $v \in S_-$. □

Now we can prove the existence of extremal constant sign solutions for the problem (1.1). These solutions are important in producing a nodal solution (see Section 4).

Proposition 3.5. *If hypotheses $H(\xi), H(\beta), H_1$ hold, the problem (1.1) has a smallest positive solution $u_* \in S_+ \subseteq intC_+$ and a biggest negative solution $v_* \in S_- \subseteq -intC_+$ (that is, $u_* \leq u$ for all $u \in S_+$ and $v \leq v_*$ for all $v \in S_-$).*

Proof. From Papageorgiou–Radulescu–Repovs [16] (see the proof of Proposition 7), we know that S_+ is downward directed. So, invoking Lemma 3.10, p. 178, of Hu–Papageorgiou [6], we can find $\{u_n\}_{n\geq 1} \subseteq S_+$ decreasing such that

$$\inf_{n\geq 1} u_n = \inf S_+.$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, d\sigma = \int_\Omega f(z, u_n) h\, dz \tag{3.22}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in N$,

$$\tilde{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N} \text{ (see Proposition 3.4)}. \tag{3.23}$$

If in (3.22) we choose $h = u_n \in W^{1,p}(\Omega)$ and use (3.23) and hypothesis $H_1(i)$, we infer that $\{u_n\}_{n\geq 1} \subseteq S_+$ is bounded. This fact and the monotonicity of the sequence $\{u_n\}_{n\geq 1}$, imply that

$$u_n \xrightarrow{w} u_* \quad \text{in } W^{1,p}(\Omega). \tag{3.24}$$

In (3.22) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.20). Then

$$\begin{aligned} & \lim_{n\to+\infty} \langle A(u_n), u_n - u_*\rangle = 0, \\ \Rightarrow \quad & u_n \to u_* \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.4)}. \end{aligned} \tag{3.25}$$

So, if in (3.22) we pass to the limit and use (3.25) and (3.23), we obtain

$$\begin{aligned} & \langle A(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) u_*^{p-1} h\, d\sigma = \int_\Omega f(z, u_*) h\, dz \quad \text{for all } h \in W^{1,p}(\Omega), \\ & \tilde{u} \leq u_*, \\ \Rightarrow \quad & u_* \in S_+ \subseteq intC_+ \quad \text{and} \quad u^* = \inf S_+. \end{aligned}$$

Similarly we produce the biggest negative solution $v_* \in S_- \subseteq -intC_+$.
Note that S_- is upward directed (see (3.10)). □

4. Nodal solutions

In this section, using the extremal constant sign solutions from Proposition 3.5, we will produce a nodal solution. We have two such results which differ in the growth of $f(z,\cdot)$ near zero.

In the first we require the presence of a concave term near zero. So, we strengthen hypothesis $H_1(iii)$. The new conditions on the source term $f(z,x)$ are the following:

$\boldsymbol{H_2}$: Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) there exists a function $\alpha\in L^\infty(\Omega)$ such that

$$|f(z,x)|\le\alpha(z)[1+|x|^{r-1}]\quad\text{for a. a. } z\in\Omega,\text{ all } x\in\mathbb{R},\ p<r<p^*;$$

(ii)

$$\lim_{x\to\pm\infty}\frac{f(z,x)}{|x|^{p-2}x}=-\infty\quad\text{uniformly for a.a. } z\in\Omega,$$

(iii) there exist $q\in\,]0,1[$, $\delta>0$ and $\hat{c}>0$ such that

$$\hat{c}|x|^q\le f(z,x)x\le qF(z,x)\qquad\text{for a.a. } z\in\Omega,\text{ all } |x|\le\delta$$

(see (3.4) for definition of $F(z,x)$).

Remark 4.1. Hypothesis $H_2(iii)$ is more restrictive than hypothesis $H_1(iii)$ and implies the presence of a concave nonlinearity near zero.

In what follows $u_*\in intC_+$ and $v_*\in -intC_+$ are the two extremal constant sign solutions of problem (1.1) produced in Proposition 3.5.

Proposition 4.2. *If hypotheses $H(\xi),H(\beta),H_2$ hold, then problem* (1.1) *admits a nodal solution* $y_0\in[v_*,u_*]\cap C^1(\Omega)$.

Proof. Using the two extremal constant sign solutions $u_*\in intC_+$ and $v_*\in -intC_+$ we define the following Carathéodory function

$$j(z,x)=\begin{cases} f(z,v_*(z))+\theta|v_*(z)|^{p-2}v_*(z) & \text{if } x<v_*(z)\\ f(z,x)+\theta|x|^{p-2}x & \text{if } v_*\le x\le u_*(z)\\ f(z,u_*(z))+\theta u_*(z)^{p-1} & \text{if } x>u_*(z).\end{cases}\tag{4.1}$$

(as before $\theta>\|\xi\|_{L^\infty(\Omega)}$). Also we consider the positive and negative truncations of $j(z,x)$, namely the Carathéodory functions

$$j_\pm(z,x)=j(z,x^\pm).\tag{4.2}$$

We set $J(z,x)=\int_0^x j(z,s)\,ds$ and $J_\pm(z,x)=\int_0^x j_\pm(z,s)\,ds$ and consider the C^1-functionals $\hat{\varphi},\hat{\varphi}_\pm: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\varphi}(u)=\frac{1}{p}K(u)+\frac{\theta}{p}\|u\|^p_{L^p(\Omega)}-\int_\Omega J(z,u)\,dz,$$

$$\hat{\varphi}_\pm(u)=\frac{1}{p}K(u)+\frac{\theta}{p}\|u\|^p_{L^p(\Omega)}-\int_\Omega J_\pm(z,u)\,dz$$

for all $u\in W^{1,p}(\Omega)$.

Using (4.1), (4.2) and the nonlinear regularity theory, we show that

$$K_{\hat{\varphi}}\subseteq[v_*,u_*]\cap C^1(\bar{\Omega}),\ \ K_{\hat{\varphi}_+}\subseteq[0,u_*]\cap C_+,\ \ K_{\hat{\varphi}_-}\subseteq[v_*,0]\cap(-C_+).$$

The extremality of u_* and v_* implies that

$$K_{\hat{\varphi}} \subseteq [v_*, u_*] \cap C^1(\bar{\Omega}), \;\; K_{\hat{\varphi}_+} = \{0, u_*\}, \;\; K_{\hat{\varphi}_-}\{0, v_*\}. \tag{4.3}$$

The functional $\hat{\varphi}_+$ is coercive and sequentially weakly lower semicontinuous. So we can find $\hat{u}_* \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(\hat{u}_*) = \inf\{\hat{\varphi}_+(u) : u \in W^{1,p}(\Omega)\}. \tag{4.4}$$

From hypothesis $H_2(iii)$ and since $q < p$, we see that for $t \in]0, 1[$ small, we have

$$\begin{aligned} & \hat{\varphi}_+(t\hat{u}_1) < 0, \\ \Rightarrow \quad & \hat{\varphi}_+(\hat{u}_*) < 0 = \hat{\varphi}_+(0) \quad \text{(see (4.4))}, \\ \Rightarrow \quad & \hat{u}_* \neq 0. \end{aligned} \tag{4.5}$$

From (4.4) we have that $\hat{u}_* \in K_{\hat{\varphi}_+} = \{0, u_*\}$ (see (4.3)). Hence $\hat{u}_* = u_*$ (see (4.5)). Note that

$$\begin{aligned} & \hat{\varphi}\big|_{C_+} = \hat{\varphi}_+\big|_{C_+} \quad \text{(see (4.1), (4.2))}, \\ \Rightarrow \quad & u_* \in intC_+ \quad \text{is a local } C^1\text{-minimizer of } \hat{\varphi}, \\ \Rightarrow \quad & u_* \in intC_+ \quad \text{is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\varphi} \\ & \text{(see Papageorgiou–Radulescu [14], Proposition 2.12).} \end{aligned} \tag{4.6}$$

Similarly, using this time the functional $\hat{\varphi}_-$, we show that

$$v_* \in -intC_+ \quad \text{is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\varphi}. \tag{4.7}$$

We may assume that $\hat{\varphi}(v_*) \leq \hat{\varphi}(u_*)$ (the reasoning is the same if the opposite inequality holds, using (4.7) instead of (4.6)). We assume that $K_{\hat{\varphi}}$ is finite. Otherwise, on account of (4.3) and (4.1), we already have an infinity of distinct smooth nodal solutions of (1.1) and so we are done. Then, on account of (4.6) and using Theorem 5.7.6, p. 449, of Papageorgiou–Radulescu–Repovs [18], we can find $\rho \in]0, 1[$ small such that

$$\hat{\varphi}(v_*) \leq \hat{\varphi}(u_*) < \inf\{\hat{\varphi}(u) : \|u - u_*\|_{W^{1,p}(\Omega)} = \rho\} = \hat{m}, \quad \|v_* - u_*\|_{W^{1,p}(\Omega)} > \rho. \tag{4.8}$$

It is clear from (4.1) that $\hat{\varphi}$ is coercive. Therefore

$$\hat{\varphi}(\cdot) \quad \text{satisfies the } C\text{-condition} \tag{4.9}$$

(see Papageorgiou–Radulescu–Repovs [18], Proposition 5.1.15, p. 369). Because of (4.8) and (4.9), we see that we can apply the mountain pass theorem. So, we can find $y_* \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\hat{\varphi}} \subseteq [v_*, u_*] \cap C^1(\bar{\Omega}) \quad \text{(see (4.3))}, \quad \hat{m} \leq \hat{\varphi}(y_0). \tag{4.10}$$

From (4.10) and (4.8), we see that

$$y_0 \neq u_* \quad \text{and} \quad y_0 \neq v_*. \tag{4.11}$$

Also since y_0 is a critical point of $\hat{\varphi}$ of mountain pass type, from Theorem 6.5.8, p. 431, of Papageorgiou–Radulescu–Repovs [18], we have

$$C_1(\hat{\varphi}, y_0) \neq 0. \tag{4.12}$$

On the other hand hypothesis $H_0(iii)$ and Proposition 3.7 of Papageorgiou–Radulescu–Repovs [13] (see also Leonardi–Papageorgiou [8], Proposition 6, for a more general result) imply that

$$C_k(\hat{\varphi}, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.13}$$

From (4.12) and (4.13) it follows that $y_0 \neq 0$. Hence

$$y_0 \notin \{v_*, u_*, 0\} \quad \text{see (4.11)}.$$

Since $y_0 \in [v_*, u_*] \cap C^1(\bar{\Omega})$, the extremality of u_* and v_* implies that y_0 is nodal. $\square$

We can also produce a nodal solution, if we assume that $f(z,\cdot)$ is $(p-1)$-linear near zero. More precisely, the new conditions on the source term are the following:

$\boldsymbol{H_3}$: Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(z,0)=0$ for a.a. $z \in \Omega$ and

(*i*) there exists a function $\alpha \in L^\infty(\Omega)$ such that

$$|f(z,x)| \le \alpha(z)[1+|x|^{r-1}] \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R},\ p<r<p^*;$$

(*ii*)

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = -\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

(*iii*)

$$\hat{\lambda}_2 < \eta_0 \le \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \limsup_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \hat{\eta}_0 \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 4.3. Hypothesis $H_3(iii)$ is more restrictive than $H_1(iii)$. Not only it requires that asymptotically as $x \to 0$ the quotient $\dfrac{f(z,x)}{|x|^{p-2}x}$ stays strictly above $\hat{\lambda}_2 > \hat{\lambda}_1$, but in addition $f(z,\cdot)$ is necessarily $(p-1)$-linear near zero.

Proposition 4.4. *If hypotheses $H(\xi), H(\beta), H_3$ hold then problem (1.1) admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C^1(\bar{\Omega}).$$

Proof. Using the functional $\hat{\varphi}, \hat{\varphi}_\pm$ from the proof of Proposition 4.2 and reasoning as in that proof, we can produce $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\hat{\varphi}} \subseteq [v_*, u_*] \cap C^1(\bar{\Omega}) \quad (\text{see } (4.3)) \quad \text{and } y_0 \notin \{u_*, v_*\}. \tag{4.14}$$

The critical point y_0 was produced using the mountain pass theorem. So, we have

$$\hat{\varphi}(y_0) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \hat{\varphi}(\gamma(t)). \tag{4.15}$$

According to (4.15), if we can find $\tilde{\gamma}_* \in \Gamma$ such that $\hat{\varphi}\big|_{\tilde{\gamma}_*} < 0$ then $\hat{\varphi}(y_0) < 0 = \hat{\varphi}(0)$ and so $y_0 \ne 0$. This fact, combined with (4.14), implies that $y_0 \in C^1(\bar{\Omega})$ is a nodal solution of (1.1).

Recall (see Section 2) that $\partial B_1^{L^p(\Omega)} = \{u \in L^p(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ and $M = W^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)}$. Also we set $M_C = M \cap C^1(\bar{\Omega})$ and consider the following two sets of continuous paths

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M) : \hat{\gamma}(-1) = -\hat{u}_1, \hat{\gamma}(1) = \hat{u}_1\},$$
$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M_C) : \hat{\gamma}(-1) = -\hat{u}_1, \hat{\gamma}(1) = \hat{u}_1\}$$

Claim. *$\hat{\Gamma}_C$ is dense in $\hat{\Gamma}$.*

Let $\hat{\gamma} \in \hat{\Gamma}$ and $\varepsilon > 0$. We consider the multifunction $K_\varepsilon : [-1,1] \to 2^{C^1(\bar{\Omega})}$ defined by

$$K_\varepsilon(t) = \begin{cases} \{u \in C^1(\bar{\Omega}) : \|u - \hat{\gamma}(t)\|_{W^{1,p}(\Omega)} < \varepsilon\} & \text{if } -1 < t < 1 \\ \{\pm\hat{u}_1\} & t = \pm 1. \end{cases}$$

Evidently, this multifunction has nonempty and convex values. For $t \in \,]0,1[$ the set $K_\varepsilon(t)$ is open, while the sets $K_\varepsilon(\pm 1)$ are singletons. In addition, the continuity of $\hat{\gamma}(\cdot)$ implies that the multifunction $K_\varepsilon(\cdot)$ is lower semicontinuous (see Papageorgiou–Radulescu–Repovs [18], Theorem 2.5.4, p. 101). So, we can apply the *selection Theorem of Michael [11] (Theorem 8.1"'), see also Papageorgiou–Radulescu–Repovs [18], Theorem 2.5.17, p. 106, and find a continuous path* $\hat{\gamma}_\varepsilon : [-1,1] \to C^1(\bar{\Omega})$ *such that* $\hat{\gamma}(t) \in K_\varepsilon(t)$ *for all* $t \in [-1,1]$.

Now, let $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, *and let* $\{\hat{\gamma}_n = \hat{\gamma}_{\varepsilon_n}\}_{n\geq 1} \subseteq C([-1,1], C^1(\bar{\Omega}))$ *be the sequence of paths produced by the previous argument. We have*

$$\|\hat{\gamma}_n(t) - \hat{\gamma}(t)\|_{W^{1,p}(\Omega)} < \frac{1}{n} \quad \textit{for all } t \in]0,1[\quad \hat{\gamma}_n(\pm 1) = \pm \hat{u}_1, \ \textit{for all } n \in \mathbb{N}. \tag{4.16}$$

Recall that $\hat{\gamma}(t) \in \partial B_1^{L^p(\Omega)}$ *for all* $t \in [-1,1]$. *So, from (4.16) we see that we may assume that* $\|\hat{\gamma}(t)\|_{L^p(\Omega)} \neq 0$ *for all* $t \in [-1,1]$, *all* $n \in \mathbb{N}$. *We define*

$$\hat{\gamma}_n^*(t) = \frac{\hat{\gamma}_n(t)}{\|\hat{\gamma}(t)\|_{L^p(\Omega)}} \quad \textit{for all } t \in [-1,1], \textit{ all } n \in \mathbb{N}. \tag{4.17}$$

We have

$$\hat{\gamma}_n^* \in C([-1,1], M_C) \quad \textit{and} \quad \hat{\gamma}_n^*(\pm 1) = \pm \hat{u}_1 \quad \textit{for all } n \in \mathbb{N}.$$

Using (4.16) and (4.17) *we have*

$$\begin{aligned} \|\hat{\gamma}_n^*(t) - \hat{\gamma}(t)\|_{W^{1,p}(\Omega)} &\leq \|\hat{\gamma}_n^*(t) - \hat{\gamma}_n(t)\|_{W^{1,p}(\Omega)} + \|\hat{\gamma}_n(t) - \hat{\gamma}(t)\|_{W^{1,p}(\Omega)} \\ &\leq \frac{|1 - \|\hat{\gamma}_n(t)\|_{L^p(\Omega)}|}{\|\hat{\gamma}_n(t)\|_{L^p(\Omega)}} \|\hat{\gamma}_n(t)\|_{W^{1,p}(\Omega)} + \frac{1}{n} \quad \textit{for all } t \in [-1,1], \textit{ all } n \in \mathbb{N}. \end{aligned} \tag{4.18}$$

Recall that $\hat{\gamma}(t) \in M$ *for all* $t \in [-1,1]$. *Therefore we have*

$$\begin{aligned} & \max_{t\in[-1,1]} |1 - \|\hat{\gamma}_n(t)\|_{L^p(\Omega)}| \\ = & \max_{t\in[-1,1]} |\|\hat{\gamma}(t)\|_{L(\Omega)} - \|\hat{\gamma}_n(t)\|_{L^p(\Omega)}| \\ \leq & \max_{t\in[-1,1]} \|\hat{\gamma}(t) - \hat{\gamma}_n(t)\|_{L^p(\Omega)} \\ \leq & c_6 \max_{t\in[-1,1]} \|\hat{\gamma}(t) - \hat{\gamma}_n(t)\|_{W^{1,p}(\Omega)} \quad \textit{for some } c_6 > 0 \\ \leq & \frac{c_6}{n} \quad \textit{for all } n \in \mathbb{N} \textit{ (see (4.16))}, \\ \Rightarrow & \ \hat{\gamma}_n^* \to \hat{\gamma} \quad \textit{in } C([-1,1], M) \textit{ (see (4.18)) and } \hat{\gamma}_n^* \in C([-1,1], M_C) \textit{ for all } n \in \mathbb{N}. \end{aligned}$$

This proves the claim.

Proposition 2.3 and the Claim imply that given $\delta > 0$ small, we can find $\hat{\gamma} \in \hat{\Gamma}_C$ such that

$$\max_{t\in[-1,1]} K(\hat{\gamma}_0(t)) \leq \hat{\lambda}_2 + \delta. \tag{4.19}$$

On account of hypothesis $H_3(iii)$, we see that if $\delta > 0$ is small, then we can find $\tilde{\eta}_0 \in]\hat{\lambda}_2 + \delta, \eta_0[$ and $0 < \delta_0 < \delta \leq \min\{\min_{\bar{\Omega}} u_*, \min_{\bar{\Omega}}(-v_*)\}$ (recall that $u_*, -v_* \in intC_+$) such that

$$\frac{1}{p}\tilde{\eta}_0|x|^p \leq f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0. \tag{4.20}$$

Recall that $\hat{\gamma}_0 \in \hat{\Gamma}_C$ and that $u_* \in intC$, $v_* \in -intC_+$. So, we can find $\mu > 0$ small such that

$$\mu_0\hat{\gamma}_0(t) \in [v_*, u_*] \quad \text{and} \quad \mu_0|\hat{\gamma}_0(t)(z)| \leq \delta_0 \quad \text{for all } t \in [-1,1], \text{ all } z \in \bar{\Omega}. \tag{4.21}$$

Then we have

$$\begin{aligned} \varphi(\mu_0\hat{\gamma}_0(t)) &= \frac{\mu_0^p}{p} K(\hat{\gamma}_0(t)) - \int_\Omega F(z, \mu_0\hat{\gamma}_0(t))\, dz \quad \text{(see (4.1) and (4.18))} \\ &\leq \frac{\mu_0^p}{p}[\hat{\lambda}_2 + \delta - \tilde{\eta}_0] \quad \text{(see (4.19), (4.20), (4.21))} \\ &< 0 \quad \text{for all } t \in [-1,1]. \end{aligned}$$

So, if we set $\tilde{\gamma}_0 = \mu_0\hat{\gamma}_0$, then $\tilde{\gamma}_0$ is a continuous path in $C^1(\bar{\Omega}) \subseteq W^{1,p}(\Omega)$ which connects $-\mu_0\hat{u}_1$ and $\mu_0\hat{u}_1$ such that

$$\hat{\varphi}\big|_{\tilde{\gamma}_0} < 0. \tag{4.22}$$

Next we will produce another continuous path in $W^{1,p}(\Omega)$ connecting $\mu_0\hat{u}_1$ and u_* and $\hat{\varphi}$ is negative along this new path.

Let $a = \hat{\varphi}_+(u_*) = \inf \hat{\varphi}_+ < 0 = \hat{\varphi}_+(0)$. From (4.3) we have

$$K^0_{\hat{\varphi}_+} = \{0\} \quad \hat{\varphi}^a_+ = \{u_*\}. \tag{4.23}$$

Using the Second Deformation Theorem (see Papageorgiou–Radulescu–Repovs [18], Theorem 5.3.12, p. 317), we can find a deformation $\hat{h} : [0,1] \times (\hat{\varphi}^0_+ \setminus K_{\hat{\varphi}^0_+}) \to \hat{\varphi}^0_+$ such that

$$\hat{h}(0,u) = u \quad \text{for all } u \in \hat{\varphi}^0_+ \setminus K^0_{\hat{\varphi}_+} = \hat{\varphi}^0_+ \setminus \{0\} \text{ (see (4.23))}, \tag{4.24}$$

$$\hat{h}(1, \hat{\varphi}^0_+ \setminus K^0_{\hat{\varphi}_+}) \subseteq \hat{\varphi}^a_+ = \{u_*\} \quad \text{(see (4.23))}, \tag{4.25}$$

$$\hat{\varphi}_+(\hat{h}(t,u)) \le \hat{\varphi}_+(\hat{h}(s,u)) \quad \text{for all } 0 \le s \le t \le 1, \text{ all } u \in \hat{\varphi}^0_+ \setminus \{0\}. \tag{4.26}$$

Note that

$$\begin{aligned} &\hat{\varphi}_+(\mu_0\hat{u}_1) = \hat{\varphi}(\mu_0\hat{u}_1) = \hat{\varphi}(\tilde{\gamma}_0(1)) < 0 \quad \text{(see (4.22))}, \\ \Rightarrow\ &\mu_0\hat{u}_1 \in \hat{\varphi}^0_+ \setminus \{0\}. \end{aligned} \tag{4.27}$$

Therefore we can define

$$\tilde{\gamma}_+(t) = \hat{h}(t, \mu_0\hat{u}_1)^+ \quad \text{for all } t \in [0,1].$$

Evidently this is a continuous path in $W^{1,p}(\Omega)$. Moreover, we have

$$\begin{aligned} &\tilde{\gamma}_+(0) = \mu_0\hat{u}_1 \quad \text{(see (4.24))}, \\ &\tilde{\gamma}_+(1) = \hat{h}(1, \mu_0\hat{u}_1) = u_* \quad \text{(see (4.25), (4.27))} \\ &\hat{\varphi}(\tilde{\gamma}_0(t)) = \hat{\varphi}_+(\tilde{\gamma}_0(t)) \le \hat{\gamma}_+(\mu_0\hat{u}_1) = \hat{\varphi}(\mu_0\hat{u}_1) < 0 \end{aligned}$$

for all $t \in [0,1]$ (see (4.26), (4.22)).

Therefore $\tilde{\gamma}_+$ is a continuous path in $W^{1,p}(\Omega)$ connecting $\mu_0\hat{u}_1$ and u_* and such that

$$\hat{\varphi}\big|_{\tilde{\gamma}_+} < 0. \tag{4.28}$$

Similarly we produce anther continuous path $\tilde{\gamma}_-$ in $W^{1,p}(\Omega)$ connecting $-\mu_0\hat{u}_1$ and v^* and such that

$$\hat{\varphi}\big|_{\tilde{\gamma}_-} < 0. \tag{4.29}$$

We concatenate $\tilde{\gamma}_-, \tilde{\gamma}_0, \tilde{\gamma}_+$ and generate a path $\tilde{\gamma}_* \in \Gamma$ such that

$$\begin{aligned} &\hat{\varphi}\big|_{\tilde{\gamma}_*} < 0 \quad \text{((see (4.22), (4.28), (4.29))} \\ \Rightarrow\ &y_0 \ne 0 \quad \text{and so } y_0 \in [v_*, u_*] \cap C^1(\bar{\Omega}) \text{ is nodal.} \quad \square \end{aligned}$$

We can state the following multiplicity theorem for problem (1.1).

Theorem 4.5. *If hypotheses $H(\xi), H(\beta), H_2$ or H_3 hold then problem (1.1) has at least three nontrivial smooth solutions*

$$u_0 \in intC_+, \quad v_0 \in -intC_+, \quad \text{and } y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega}) \text{ nodal.}$$

5. A particular case

In this section we consider the following parametric Robin problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \lambda\,|u(z)|^{p-2}u(z) - g(z,u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

In this problem $\lambda \in \mathbb{R}$ is a parameter and $g(z,x)$ is a perturbation of the parametric term, which satisfies the following conditions

$\hat{\boldsymbol{H}}$: Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $g(z,0) = 0$ for a.a. $z \in \Omega$ and

(*i*) there exists a function $\alpha \in L^\infty(\Omega)$ such that

$$|g(z,x)| \le \alpha(z)[1 + |x|^{r-1}] \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R},\ p < r < p^*;$$

(*ii*)

$$\lim_{x\to\pm\infty} \frac{g(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

(*iii*)

$$\lim_{x\to 0} \frac{g(z,x)}{|x|^{p-2}x} = 0 \quad \text{uniformly for a.a. } z \in \Omega.$$

Those conditions make the right-hand side of (5.1) a logistic reaction of equidiffusive type.

Using Theorem 4.5 (the case with hypotheses H_3), we have the following multiplicity theorem for problem (5.1).

Theorem 5.1. *If hypotheses $H(\xi), H(\beta), \hat{H}$ hold and $\lambda > \hat{\lambda}_2$, then problem (5.1) has at least three nontrivial smooth solutions $u_0 \in intC_+$, $v_0 \in -intC_+$, $y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega})$ nodal.*

6. Semilinear problem

In this section we deal with the semilinear problem (that is $p = 2$). So, the problem under consideration is the following

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z,u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

The conditions on the data of this problem are:
H(ξ)': $\xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^\infty(\Omega)$.

Remark 6.1. Now in addition of $\xi(\cdot)$ being indefinite, we can have that it is also unbounded from below.

H(β)': $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \ge 0$, $\forall z \in \partial\Omega$.

Remark 6.2. The case $\beta \equiv 0$ is also included and corresponds to the Neumann problem.

$\boldsymbol{H_4}$: Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(*i*) there exists a function $\alpha \in L^\infty(\Omega)$ such that

$$|f(z,x)| \le \alpha(z)[1 + |x|^{r-1}] \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R},\ 2 < r < 2^*;$$

(ii)

$$\lim_{x\to\pm\infty}\frac{f(z,x)}{x}=-\infty \quad \text{uniformly for a.a. } z\in\Omega,$$

(iii)

$$\hat{\lambda}_2<\eta_0\le\liminf_{x\to 0}\frac{f(z,x)}{x}\le\limsup_{x\to 0}\frac{f(z,x)}{x}\le\hat{\eta}_0 \quad \text{uniformly for a.a. } z\in\Omega.$$

(iv) for every $\rho>0$ there exists $\hat{\xi}_\rho>0$ such that, for a.a. $z\in\Omega$, the function

$$x\to f(z,x)+\hat{\xi}_\rho x$$

is nondecreasing on $[-\rho,\rho]$.

Remark 6.3. We have added on more condition (hypothesis $H_4(iv)$, a one-sided Lipschitz condition on $f(z,\cdot)$) compared to hypotheses H_3 in order to have a stronger conclusion concerning the nodal solution. The stronger conclusion will be used in the sequel, together with a stronger regularity on $f(z,\cdot)$, in order to produce a second nodal solution.

In this case $K: H^1(\Omega)\to\mathbb{R}$ is defined by

$$K(u)=\|Du\|^2_{L^p(\Omega)}+\int_\Omega \xi(z)u^2\,dz+\int_{\partial\Omega}\beta(z)u^2\,d\sigma \quad \text{for all } u\in H^1(\Omega).$$

From D'Agui–Marano–Papageorgiou [4], we know that there exists $\mu>0$ such that

$$K(u)+\mu\|u\|^2_{L^2(\Omega)}\ge\hat{c}_0\|u\|^2_{L^2(\Omega)} \quad \text{for all } u\in H^1(\Omega), \text{ some constant } \hat{c}_0>0. \tag{6.2}$$

In what follows, by $E(\hat{\lambda}_i)$ we denote the eigenspace corresponding to $\hat{\lambda}_i$, $I\in\mathbb{N}$. We know that $E(\hat{\lambda}_i)\subset C^1(\bar{\Omega})$ and it is finite dimensional. We have

$$H^1(\Omega)=\overline{\bigoplus_{i\ge1}E(\hat{\lambda}_i)}.$$

For $m\in\mathbb{N}$ we set

$$\bar{H}_m=\bigoplus_{i=1}^{m}E(\hat{\lambda}_i) \quad \text{and} \quad \hat{H}_m=\bar{H}_m^{\perp}=\overline{\bigoplus_{i\ge m+1}^{m}E(\hat{\lambda}_i)}.$$

We have the orthogonal direct sum decomposition

$$H^1(\Omega)=\bar{H}_m\bigoplus\hat{H}_m.$$

So, every $u\in H^1(\Omega)$ admits a unique decomposition

$$u=\bar{u}+\hat{u} \quad \text{with } \bar{u}\in\bar{H}_m,\ \hat{u}\in\hat{H}_m.$$

We recall that each eigenspace $E(\hat{\lambda}_i)$ has the unique continuation property (u.c.p for short) that is, if $u\in E(\hat{\lambda}_i)$ and vanishes on a set of positive Lebesgue measure, then $u\equiv 0$.

Theorem 6.4. *If hypotheses $H(\xi)', H(\beta)', H_4$ hold then problem (6.1) has at least three nontrivial smooth solutions*

$$u_0\in intC_+,\ \ v_0\in -intC_+,\ \ y_0\in int_{C^1(\bar{\Omega})}[v_0,u_0].$$

Proof. The existence of three solutions

$$u_0 \in intC_+, \;\; v_0 \in -intC_+, \;\; y_0 \in [v_0, u_0] \cap C^1(\bar{\Omega})$$

is guaranteed by Theorem 4.5. Note that in the arguments leading to that multiplicity Theorem, we replace $\theta > \|\xi\|_{L^\infty(\Omega)}$ by $\mu > 0$ as in (6.2). Moreover, instead of the regularity theory of Lieberman [23] (nonlinear problems), we use the one by Wang [25] (semilinear problem, for this reason the conditions on $\xi(\cdot)$ and $\beta(\cdot)$ are less restrictive). With these two changes, all the previous proofs remain valid and eventually give us the three nontrivial smooth solutions mentioned above.

Let $\rho = \max\{\|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_4(iv)$. We have

$$\begin{aligned}
& -\Delta y_0(z) + [\xi(z) + \hat{\xi}_\rho] y_0(z) \\
= \;& f(z, y_0(z)) + \hat{\xi}_\rho y_0(z) \\
\leq \;& f(z, u_0(z)) + \hat{\xi}_\rho u_0(z) \quad \text{(see (3.3) and hypothesis } H_4(iv)) \\
= \;& -\Delta u_0(z) + [\xi(z) + \hat{\xi}_\rho] u_0(z) \quad \text{for a.a. } z \in \Omega, \\
\Rightarrow \;& \Delta(u_0 - y_0)(z) \leq [\xi(z) + \hat{\xi}_\rho](u_0 - y_0)(z) \quad \text{for a.a. } z \in \Omega, \\
\Rightarrow \;& u_0 - y_0 \in intC_+ \quad \text{(by Hopf's maximum principle).}
\end{aligned}$$

Similarly we show that

$$y_0 - v_0 \in intC_+.$$

So, we conclude that

$$y_0 \in int_{C^1(\bar{\Omega})}[v_0, u_0]. \quad \square$$

We can improve the conclusion of this theorem by strengthening the regularity of $f(z, \cdot)$. More precisely, the new conditions on $f(z, \cdot)$ are the following:

***H*$_5$**: Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) there exists a function $\alpha \in L^\infty(\Omega)$ such that

$$|f'_x(z, x)| \leq \alpha(z)[1 + |x|^{r-1}] \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \; 2 < r < 2^*;$$

(ii)

$$\lim_{x \to \pm\infty} \frac{f(z, x)}{x} = -\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

(iii) there exist $m \in \mathbb{N}$, $m \geq 2$, $\eta \in L^\infty(\Omega)$ and $\delta > 0$ such that

$$\eta(z)x^2 \leq f(z, x)x \leq \hat{\lambda}_{m+1}x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta, \tag{6.3}$$

if $m \geq 3$, then $\hat{\lambda}_m \leq \eta(z)$ for a.a. $z \in \Omega$, $\hat{\lambda}_m \neq \eta$,
if $m = 2$, then $\hat{\lambda}_2 < \operatorname{ess\,inf}_\Omega \eta$;
moreover, for every $x \neq 0$, the second inequality is strict on a set of positive Lebesgue measure.

Remark 6.5. In this case, due to the differentiability of $f(z, \cdot)$, hypothesis $H_4(iv)$ is automatically satisfied.

Theorem 6.6. *If hypotheses $H(\xi)'$, $H(\beta)'$, H_5 hold, then problem (6.1) admits at least four nontrivial smooth solutions*

$$u_0 \in intC_+, \;\; v_0 \in -intC_+, \;\; y_0, y \in int_{C^1(\bar{\Omega})}[v_0, u_0] \;\; \textit{nodal.}$$

Proof. From Theorem 6.4, we already have three nontrivial smooth solutions

$$u_0 \in intC_+, \;\; v_0 \in -intC_+, \;\; y_0 \in int_{C^1(\bar{\Omega})}[v_0, u_0] \text{ nodal.}$$

Let $\lambda \in]\hat{\lambda}_m, \hat{\lambda}_{m+1}[$ and consider the C^2-functional $\hat{\psi} : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}K(u) - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1(\Omega).$$

From Proposition 6.2.6, p. 479, of Papageorgiou–Radulescu–Repovs [18], we have

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m = dim\bar{H}_m. \tag{6.4}$$

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (6.1) defined by

$$\varphi(u) = \frac{1}{2}K(u) - \int_\Omega F(z, u)\, dz \quad \text{for all } u \in H^1(\Omega).$$

We know that $\varphi \in C^2(H^1(\Omega))$ and we consider the homotopy $\hat{h}_t(u)$ defined by

$$\hat{h}_t(u) = (1 - t)\varphi(u) + t\psi(u) \quad \text{for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

For $0 < t \le 1$ we have

$$\langle(\hat{h}_t)'(u), h\rangle = (1 - t)\langle\varphi'(u), h\rangle + t\langle\psi'(u), h\rangle \quad \text{for all } u, h \in H^1(\Omega). \tag{6.5}$$

If $h = \hat{u} - \bar{u}$, then the orthogonality of the component spaces implies

$$\langle\varphi'(u), \hat{u} - u\rangle = K(\hat{u}) - K(\bar{u}) - \int_\Omega f(z, u)(\hat{u} - \bar{u})\, dz. \tag{6.6}$$

By hypothesis $H_5(iii)$, we have

$$\eta(z) \le \frac{f(z, x)}{x} \le \hat{\lambda}_{m+1} \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta.$$

So, if $u \in C^1(\bar{\Omega})$ and $\|u\|_{C^1(\bar{\Omega})} \le \delta$, then

$$\begin{aligned} f(z, u(z))(\hat{u} - \bar{u})(z) &= \begin{cases} \hat{\lambda}_{m+1}(\hat{u}^2 - \bar{u}^2)(z) & \text{if } u(z)(\hat{u} - \bar{u})(z) \ge 0 \\ \eta(z)(\hat{u}^2 - \bar{u}^2)(z) & \text{if } u(z)(\hat{u} - \bar{u})(z) < 0, \end{cases} \\ \Rightarrow \quad f(z, u)(\hat{u}^2 - \bar{u}^2)(z) &\le \hat{\lambda}_{m+1}\hat{u}^2 - \eta(z)\bar{u}^2 \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{6.7}$$

We use (6.7) in (6.6) and we have

$$\langle\varphi'(u), \hat{u} - \bar{u}\rangle \ge K(\hat{u}) - \hat{\lambda}_{m+1}\|\hat{u}\|^2_{L^2(\Omega)} - K(\bar{u}) + \int_\Omega \eta(z)\bar{u}^2\, dz \ge 0.$$

Since $\lambda \in]\hat{\lambda}_m, \hat{\lambda}_{m+1}[$ we have

$$\langle\psi'(u), \hat{u} - \bar{u}\rangle \ge c_6\|u\|^2_{H^1(\Omega)} \quad \text{for some constant } c_6 > 0, \text{ all } u \in H^1(\Omega).$$

Therefore for $0 < t \le 1$ and $u \in C^1(\bar{\Omega})$, with $0 < \|u\|_{C^1(\bar{\Omega})} \le \delta$, we have

$$(1 - t)\langle\varphi'(u), \hat{u} - \bar{u}\rangle + t\langle\psi'(u), \hat{u} - \bar{u}\rangle \ge tc_6\|u\|^2_{H^1(\Omega)}.$$

Since $K_{\hat{h}_t} \subseteq C^1(\bar{\Omega})$ for all $t \in [0, 1]$ (regularity theory, see Wang [25]), we infer that

$$u = 0 \text{ is an isolated critical point of } \hat{h}_t(\cdot) \text{ for all } 0 < t \le 1.$$

For $t = 0$, we have $\hat{h}_0 = \varphi$. Suppose we could find $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$u_n \to 0 \quad \text{in } H^1(\Omega) \quad \text{and} \quad \varphi'(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{6.8}$$

We have

$$\Delta u_n + \xi(z)u_n = f(z, u_n) \quad \text{in } \Omega, \quad \frac{\partial u_n}{\partial n} + \beta(z)u_n = 0 \quad \text{on } \partial\Omega. \tag{6.9}$$

From (6.9) and the regularity theory of Wang [25] we know that we can find $\alpha \in]0, 1[$ and constant $c_7 > 0$ such that

$$u_n \in C^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\bar{\Omega})} \leq c_7 \quad \text{for all } n \in \mathbb{N}. \tag{6.10}$$

Recall that $C^{1,\alpha}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$ compactly. So, from (6.8) and (6.10) it follows that

$$\begin{aligned}
& u_n \to 0 \quad \text{in } C^1(\bar{\Omega}), \\
\Rightarrow\ & \|u_n\|_{C^1(\bar{\Omega})} \leq \delta \quad \text{for all } n \geq n_0, \\
\Rightarrow\ & f(z, u_n)(\hat{u}_n - \bar{u}_n) \leq \hat{\lambda}_{m+1}\hat{u}_n^2 - \eta(z)\bar{u}_n^2 \quad \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0 \text{ (see (6.7))}.
\end{aligned}$$

So, we have

$$\begin{aligned}
& 0 = \langle \varphi'(u_n), \hat{u}_n - \bar{u}_n \rangle \geq K(\hat{u}_n) - \hat{\lambda}_{m+1}\|\hat{u}_n\|^2_{L^2(\Omega)} - K(\bar{u}_n) + \int_\Omega \eta(z)\bar{u}_n^2\, dz \\
\geq\ & c_8\|\bar{u}_n\|^2_{H^1(\Omega)} \quad \text{for some constant } c_8 > 0, \text{ all } n \geq n_0 \\
& \text{(see D'Agui–Marano–Papageorgiou [4], Lemma 2.2)}, \\
\Rightarrow\ & \bar{u}_n = 0 \quad \text{for all } n \geq n_0, \\
\Rightarrow\ & K(\hat{u}_n) = \hat{\lambda}_{m+1}\|\hat{u}_n\|^2_{L^2(\Omega)}, \\
\Rightarrow\ & \hat{u}_n \in E(\hat{\lambda}_{m+1}) \setminus \{0\} \quad \text{for all } n \geq n_0.
\end{aligned}$$

By the unique continuation property, we have $\hat{u}_n(z) \neq 0$ for a.a. $z \in \Omega$. So, by hypothesis $H_5(iii)$ we have

$$K(\hat{u}_n) = \hat{\lambda}_{m+1}\|\hat{u}_n\|^2_{L^2(\Omega)} > \int_\Omega f(z, u_n)\hat{u}_n\, dz \quad \text{for all } n \geq n_0,$$

a contradiction. This proves that $u = 0$ is an isolated critical point of $\varphi = \hat{h}_0$. Now we apply the homotopy invariance property of critical groups (see Papageorgiou–Radulescu–Repovs [18], Theorem 6.3.8, p. 505) and have

$$\begin{aligned}
& C_k(\varphi|_{C^1(\bar{\Omega})}, 0) = C_k(\psi|_{C^1(\bar{\Omega})}, 0) \quad \text{for all } k \in \mathbb{N}_0, \\
\Rightarrow\ & C_k(\varphi, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0 \\
& \text{(since } C^1(\bar{\Omega}) \hookrightarrow H^1(\Omega) \text{ densely, see Papageorgiou–Radulescu–Repovs [18], Th. 6.2.26)}, \\
\Rightarrow\ & C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (6.4))}.
\end{aligned}$$

If $\hat{\varphi}(\cdot)$ is as in the proof of Proposition 4.2 (with u_*, v_* replaced by u_0, v_0 respectively), then again by Theorem 6.6.26, p. 545, of [18], we have

$$C_k(\hat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0\ . \tag{6.11}$$

Since $y_0 \in int_{C^1(\bar{\Omega})}[v_0, u_0]$ and $y_0 \in K_{\hat{\varphi}}$ is of mountain pass type, we have

$$C_k(\varphi, y_0) = C_k(\hat{\varphi}, y_0) \quad \text{for all } k \in \mathbb{N}_0, \quad C_1(\hat{\varphi}, y_0) \neq 0. \tag{6.12}$$

But $\varphi \in C^2(H^1(\Omega))$, so from (6.12) and Theorem 6.5.11, p. 530, of Papageorgiou–Radulescu–Repovs [18], we have

$$C_k(\hat{\varphi}, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0\ . \tag{6.13}$$

From the proof of Proposition 4.2 we know that u_0, v_0 are local minimizers of $\hat{\varphi}$. Hence we have

$$C_k(\hat{\varphi}, u_0) = C_k(\hat{\varphi}, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \ . \tag{6.14}$$

Finally recall that $\hat{\varphi}(\cdot)$ is coercive (see (4.1)). Therefore

$$C_k(\hat{\varphi}, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \ . \tag{6.15}$$

Suppose that $K_{\hat{\varphi}} = \{0, y_0, u_0, v_0\}$. Then from (6.11), (6.13), (6.14), (6.15) and the Morse relation with $t = -1$, we have

$$\begin{aligned} &(-1)^{d_m} + (-1)^1 + 2(-1)^0 = (-1)^0, \\ \Rightarrow \quad &(-1)^{d_m} = 0, \quad \text{a contradiction.} \end{aligned}$$

So, there exists $\hat{y} \in K_{\hat{\varphi}} \subseteq [v_0, u_0] \cap C^1(\bar{\Omega})$ (see (4.3)) such that $\hat{y} \notin \{0, y_0, v_0, u_0\}$. Evidently $\hat{y}$ is the second nodal Smoot solution of (6.1) and as in the proof of Theorem 6.4, we have $\hat{y} \in int_{C^1(\bar{\Omega})}[v_0, u_0]$. □

Now we consider the following semilinear equidiffusive logistic Robin problem

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \lambda u(z) - g(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.16}$$

The conditions on the perturbation $g(z, x)$ are the following:

$\tilde{\boldsymbol{H}}$: Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $g(z, 0) = 0$ for a.a. $z \in \Omega$, $g(z, \cdot) \in C^1(\mathbb{R})$ and

(*i*) there exists a function $\alpha \in L^\infty(\Omega)$ such that

$$|g'_x(z, x)| \leq \alpha(z)[1 + |x|^{r-1}] \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R},\ 2 < r < 2^*;$$

(*ii*)

$$\lim_{x \to \pm\infty} \frac{g(z, x)}{x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

(*iii*)

$$g'_x(z, x) = \lim_{x \to 0} \frac{g(z, x)}{x} = 0 \quad \text{uniformly for a.a. } z \in \Omega.$$

Then Theorem 6.6 leads to the following multiplicity theorem for problem (6.16)

Theorem 6.7. *If hypotheses $H(\xi)', H(\beta)', \tilde{H}$ hold and $\lambda > \hat{\lambda}_2$ then problem (6.16) has at least four nontrivial smooth solutions*

$$u_0 \in intC_+, \quad v_0 \in -intC_+, \quad y_0, \hat{y} \in int_{C^1(\bar{\Omega})}[v_0, u_0] \ \text{ nodal.}$$

Remark 6.8. Theorem 6.7 extends the works of Ambrosetti–Lupo [2], Ambrosetti–Mancini [3] and Struwe [23,24] (Theorem 10.5, p.147) which deal with the Dirichlet problem, $\xi \equiv 0$ (no potential term) and produce three solutions but not nodal solutions

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