Rend. Lincei Mat. Appl. 32 (2021), 317–334
DOI 10.4171/RLM/938

Partial Differential Equations — *Morrey estimates for a class of noncoercive elliptic systems with VMO-coefficients*, by Giuseppa Rita Cirmi, Salvatore D'Asero and Salvatore Leonardi, communicated on 12 February 2021.

Abstract. — We consider a non-coercive vectorial boundary value problem with non smooth coefficients and a drift term and we study the regularity of a solution u and its gradient in the framework of suitable Morrey spaces.

Key words: Elliptic systems, non-coercive, VMO coefficients, fractional differentiability

Mathematics Subject Classification: 35J47, 35B65

1. Introduction

In this paper we will study the regularity of a weak solution u of the following homogeneous Dirichlet vectorial problem (under the Einstein's convention over repeated indices)

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left[M_{ij}^{rs}(x)\dfrac{\partial u^r}{\partial x_i} - E_j^{rs}(x)u^r\right] = f^s(x) & \text{in } \Omega \\ u^s = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $s = 1, 2, \ldots, N$, with $N \geq 2$, Ω is a bounded open subset of $\mathbb{R}^n$, with $n > 2$, $M_{ij}^{rs} : \Omega \to \mathbb{R}$ are measurable and bounded entries of a symmetric, elliptic, non (necessarily) diagonal matrix M. Concerning the tensor $E = (E_j^{rs}(x))$ and the right-hand side $f = (f^s(x))$, we assume that they belong to some suitable Morrey spaces to be specified later on and we recover an estimate on the modulus of the gradient Du of a solution u of problem (1) in the corresponding Morrey space, as in the classical Morrey–Campanato's theory.

The operator we are dealing with presents at least three difficulties: it is a vectorial operator, it has non smooth coefficients and it is non coercive.

Namely, we will consider a weak solution u of the aforementioned linear system with the coefficients of the principal part belonging to the space VMO and, without assuming any further condition, we will prove the Morrey regularity of Du and its fractional differentiability by using the Campanato–Mingione approach.

Concerning the existence of weak solution we point out that, already in the case of one single equation, the main issue is due to the noncoercitivity of the operator $u \to -\text{div}[M(x)\nabla u - E(x)u]$ and it can be overcome by assuming a smallness condition on the $\| |E| \|_{L^n}$, as it was done by G. Stampacchia in his pioneering papers [53, 54].

In turn, the Morrey estimate obtained for $|Du|$ allows us to extend to the problem (1) the Calderon–Zygmund theory introduced in the paper [50] by G. Mingione (see also [1]). The results we will prove extend to the vectorial Dirichlet problem (1) those obtained in [11] in the scalar case (that is $N = 1$).

In the framework of regularity theory of weak solutions the reader can also refer to the following papers [5, 24, 29, 30, 31, 32, 33, 34, 35, 36, 51, 52, 16, 7, 8, 9, 10, 13, 18, 42, 44, 45, 27, 28].

2. Main notations, functions spaces and auxiliary lemmas

In this section, for reader's convenience, we recall some useful properties of functions spaces and we well use some lemmas that we are going to exploit.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n > 2$, with a sufficiently smooth boundary $\partial\Omega$ and diameter d_Ω and $N \in \mathbb{N}$, $N \geq 2$.

Given $x_0 \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x_0, r)$ the ball centered at x_0 with radius r.

Definition 2.1 (Morrey space). Let $p \geq 1$ and $0 \leq \lambda < n$. $L^{p,\lambda}(\Omega, \mathbb{R}^N)$ is the space of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that

$$\sup_{x_0 \in \Omega, 0<r\leq d_\Omega} r^{-\lambda} \int_{\Omega\cap B(x_0,r)} |u|^p \, dx < +\infty.$$

Definition 2.2 (Fractional Sobolev space). Let $t \in]0,1]$ and $p \geq 1$. $W^{t,p}(\Omega, \mathbb{R}^N)$ is the space of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that

$$\|u\|_{W^{t,p}(\Omega,\mathbb{R}^N)} = \|u\|_{L^p(\Omega,\mathbb{R}^N)} + [u]_{t,p,\Omega} < +\infty$$

where

$$[u]_{t,p,\Omega} = \begin{cases} \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+tp}} \, dx \, dy\right)^{\frac{1}{p}} & \text{if } t < 1 \\ \|Du\|_{L^p(\Omega,\mathbb{R}^N)} & \text{if } t = 1. \end{cases}$$

Remark 2.3. Some well-known features of Morrey spaces, tacitly used throughout the paper, are the following:

- $L^{p,\lambda}(\Omega) \not\subseteq L^{p+\varepsilon}(\Omega)$, $\forall\varepsilon > 0$;
- if $p \geq q$ and $\frac{N-\lambda}{p} \leq \frac{N-\mu}{q}$ then $L^{p,\lambda}(\Omega, \mathbb{R}^N) \hookrightarrow L^{q,\mu}(\Omega, \mathbb{R}^N)$.

Moreover, we introduce the notion of VMO class for matrix-valued function. Given a matrix-function $w \in L^1(\Omega, \mathbb{R}^{N^2})$ and $r > 0$, we define

$$V(x,r) \equiv \sup_{0<\rho\leq r} \frac{1}{|\Omega\cap B(x,\rho)|} \int_{\Omega\cap B(x,\rho)} |w(y) - (w)_{\Omega\cap B(x,\rho)}| \, dy$$

where for any measurable subset $B \subseteq \mathbb{R}^n$

$$(w)_B \equiv \frac{1}{|B|} \int_B w(x)\,\mathrm{d}x$$

denotes the average of a function w in B. We introduce the VMO-continuity modulus for w

$$V(r) \equiv \sup_{x \in \Omega} V(x, r).$$

DEFINITION 2.4 (Sarason VMO space). By VMO we denote the space of all matrix-functions $w \in L^1(\Omega, \mathbb{R}^{N^2})$ such that

$$V(r) < +\infty \quad \text{for all } 0 < r \leq \mathrm{d}_\Omega \quad \text{and} \quad \lim_{r \to 0} V(r) = 0.$$

We make the reader aware that in the sequel we will denote by c various positive constants depending only on the known data and whose values may vary from one line to another.

Next lemma concerns the product between a tensor-valued function and a vector-valued function belonging to Morrey spaces and it can be readily deduced from Lemma 5.1 of [23].

LEMMA 2.5. *Let $n-2 < \mu < n$, $E \in L^{2,\mu}(\Omega, \mathbb{R}^{nN^2})$ and $u \in L^{2,\nu+2}(\Omega, \mathbb{R}^N)$ such that $\mathrm{D}u \in L^{2,\nu}(\Omega, \mathbb{R}^{nN})$ for some $\nu \in [0, n-2[$. Then*

$$Eu \in L^{2,\mu+\nu-n+2}(\Omega, \mathbb{R}^{nN})$$

and moreover

$$\|Eu\|_{L^{2,\mu+\nu-n+2}(\Omega,\mathbb{R}^{nN})} \leq C\|E\|_{L^{2,\mu}(\Omega,\mathbb{R}^{nN^2})}(\|\mathrm{D}u\|_{L^{2,\nu}(\Omega,\mathbb{R}^{nN})} + \|u\|_{L^{2,2+\nu}(\Omega,\mathbb{R}^N)})$$

for some $C > 0$ independent of u and E.

Finally, the last result we state is a Sobolev–Morrey embedding Lemma for vector-valued functions whose proof follows applying component-wise Lemma 5.1 of [17].

LEMMA 2.6. *Assume that $\partial\Omega \in C^1$. Let $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that $\mathrm{D}u \in L^{2,\nu}(\Omega, \mathbb{R}^{nN})$ with $\nu \in]0, n-2[$.*

Then

$$u \in L^{2_\nu,\nu}(\Omega, \mathbb{R}^N) \quad \text{where } \frac{1}{2_\nu} = \frac{1}{2} - \frac{1}{n-\nu},$$

and moreover there exists a positive constant C depending on n, N, ν such that

$$\|u\|_{L^{2_\nu,\nu}(\Omega,\mathbb{R}^N)} \leq C\|\mathrm{D}u\|_{L^2(\Omega,\mathbb{R}^{nN})}.$$

3. Statement of the main results

Let $M : \Omega \to \mathbb{R}^{n^2N^2}$ be a matrix-function with measurable entries $M_{ij}^{rs}(x)$ such that for a.e. $x \in \Omega$

$$M_{ij}^{rs}(x) \in L^{\infty}(\Omega) \cap VMO(\Omega) \tag{2}$$

for $i, j = 1, 2, \ldots, n$ and $r, s = 1, 2, \ldots, N$,

$$\alpha|\xi|^2 \le M_{ij}^{rs}(x)\xi_i^r\xi_j^s \le \beta|\xi|^2, \quad M_{ij}^{sr}(x) = M_{ji}^{rs}(x), \quad \text{for any } \xi \in \mathbb{R}^{nN}. \tag{3}$$

Let $E : \Omega \to \mathbb{R}^{nN^2}$ be a matrix-valued function whose entries are the measurable functions $E_i^{rs}(x)$ such that

$$E_i^{rs}(x) \in L^{2,\mu}(\Omega) \quad \text{with } n-2 < \mu < n, \tag{4}$$

for $i = 1, \ldots, n$ $r, s = 1, \ldots, N$.

Let $f : \Omega \to \mathbb{R}^N$ be a vector-valued function such that

$$f \in L^{\frac{2n}{n+2}, \frac{2\lambda}{n+2}}(\Omega, \mathbb{R}^N) \quad \text{with } 0 < \lambda < n-2. \tag{5}$$

Finally, given a vector-valued function $u = (u^s)_{s=1,2,\ldots,N}$, Du denotes its gradient, that is

$$\mathrm{D}u \equiv \left(\frac{\partial u^s}{\partial x_i}\right)_{s=1,2,\ldots,N;\, i=1,2,\ldots,n} \equiv (\mathrm{D}_i u^s)_{s=1,2,\ldots,N;\, i=1,2,\ldots,n}.$$

Definition 3.1. By a weak solution of the problem (1) we mean a function u such that

$$\begin{cases} u \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ \displaystyle\int_\Omega M_{ij}^{rs}(x)\mathrm{D}_i u^r \mathrm{D}_j \varphi^s\,dx = \int_\Omega E_j^{rs}(x) u^r \mathrm{D}_j\varphi^s\,dx + \int_\Omega f^s\varphi^s\,dx \end{cases} \tag{6}$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Remark 3.2. We point out that the existence of a weak solution of the problem (1) can be ensured by assuming the additional hypotheses on the drift term $E \in L^n(\Omega, \mathbb{R}^{nN^2})$ and $\|E\|_{L^n(\Omega, \mathbb{R}^{nN^2})}$ sufficiently small, as it was done by G. Stampacchia in the papers [53, 54].

More recently, T. Del Vecchio, M. R. Posteraro [22] and L. Boccardo [2] retrieved the results proved by G. Stampacchia, weakening also the assumptions on the right-hand side, and without any smallness assumption on the norm of the drift term (see also [3, 4]).

In the vectorial case (that is $N \ge 2$), the smallness condition on $\|E\|_{L^n(\Omega, \mathbb{R}^{nN^2})}$ can be removed by assuming further "ad hoc" structural conditions for the principal part and the first order term, which recall back the so called "Landes condition" (see [12]).

We also stress that, for some constants $c > 0$, one has

$$|Eu| \le c(|E|^{\frac{n}{2}} + |u|^{\frac{2^*}{2}})$$

so that our lower order term falls in the case of "controlli limite" (see Campanato [6, pages 122 and 125]).

Here we state a regularity result similar to Campanato's one (see [6, page 91]) that can be proved for weak solutions of problem (1).

THEOREM 3.3. *Assume that conditions* (3), (4), (5) *hold and let* $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ *be a weak solution of the problem* (1). *Then,*

i) $\mathrm{D}u \in L^{2,\lambda}(\Omega, \mathbb{R}^{nN})$
ii) $u \in L^{2_\lambda,\lambda}(\Omega, \mathbb{R}^N)$ *where* $\dfrac{1}{2_\lambda} = \dfrac{1}{2} - \dfrac{1}{n-\lambda}$

with corresponding norms estimates.

REMARK 3.4. Observe that if $|E| = 0$ then we retrieve the result of Theorem 8.V, page 92 of [6].

Finally, we state a theorem on the fractional differentiability of Du.

THEOREM 3.5. *Assume that hypotheses* (3), (4), (5) *and*

$$M_{ij}^{rs}(x) \in C^{0,\eta}(\Omega), \quad 0 < \eta \le 1$$

hold. Let $u \in W_0^1(\Omega, \mathbb{R}^N)$ *be a weak solution of* (1). *Then*

$$\mathrm{D}u \in \mathrm{W}_{\mathrm{loc}}^{t,2}(\Omega, \mathbb{R}^{nN}) \tag{7}$$

for every $t \in [0, \eta\delta[$ *and for every* $\delta \in \left[0, \min\left\{1, \frac{\lambda}{2}\right\}\right[$.

Moreover, for every couple of open subset $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *there exists a constant* $c_6 = c_6(\alpha, \beta, n, N, \Omega, \|E\|_{L^{2,\mu}(\Omega,\mathbb{R}^{nN^2})}, \|f\|_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)})$, *independent of u, such that*

$$[\mathrm{D}u]^2_{W^{t,2}(\Omega',\mathbb{R}^{nN})} \le c_6 \left[\int_{\Omega''} |\mathrm{D}u|^2 \, dx + \|\mathrm{D}u\|^2_{L^{2,\lambda}(\Omega'',\mathbb{R}^{nN})} \right]. \tag{8}$$

Further details can also be found in [13, 14, 15, 19, 20, 21, 25, 26, 37, 38, 39, 40, 41, 43, 46, 47, 48, 49].

4. PROOFS OF THEOREMS 3.3 AND 3.5

PROOF OF THEOREM 3.3. Let $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of the problem (1). Let $x_0 \in \Omega$ and $R > 0$ be such that $B_R(x_0) \subset\subset \Omega$ and $v \in W_0^{1,2}(B_R(x_0),$

$\mathbb{R}^N$) be the solution of

$$\text{(9)}\qquad \begin{cases} \mathrm{D}_j(M_{ij}^{sr}(x)\mathrm{D}_i v^r) = 0 & \text{in } B_R \\ v^s = u^s & \text{on } \partial B_R,\ s = 1,\dots,N. \end{cases}$$

Then the function

$$w(x) = u(x) - v(x), \quad x \in B_R(x_0)$$

is the unique weak solution of the vectorial problem

$$\text{(10)}\qquad \begin{cases} -\mathrm{D}_j(M_{ij}^{sr}(x)\mathrm{D}_i w^r) = -\mathrm{D}_j(E_j^{sr}(x)u^r) + f^s & \text{in } B_R \\ w^s = 0 & \text{on } \partial B_R,\ s = 1,\dots,N. \end{cases}$$

Choosing w as test function in the weak formulation of the problem (10) and using hypothesis (3), Young's and Sobolev's inequality, for any $\sigma > 0$ we obtain

$$\text{(11)}\qquad \begin{aligned} \alpha \int_{B_R} |\mathrm{D}w|^2\,\mathrm{d}x &\le \frac{1}{2\alpha}\sum_{s=1}^{N}\sum_{j=1}^{n}\int_{B_R} |E_j^{sr}(x)u^r|^2\,\mathrm{d}x + \frac{\alpha}{2}\sum_{s=1}^{N}\sum_{j=1}^{n}\int_{B_R}|\mathrm{D}_j w^s|^2\,\mathrm{d}x \\ &\quad + C(\sigma)\Big(\int_{B_R}|f|^{\frac{2n}{n+2}}\,\mathrm{d}x\Big)^{\frac{n+2}{n}} + \sigma\mathscr{S}\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x \\ &\le \frac{1}{2\alpha}\int_{B_R}|E(x)u|^2\,\mathrm{d}x + \frac{\alpha}{2}\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x \\ &\quad + C(\sigma)\Big(\int_{B_R}|f|^{\frac{2n}{n+2}}\,\mathrm{d}x\Big)^{\frac{n+2}{n}} + \sigma\mathscr{S}\int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x \end{aligned}$$

where $|E(x)u|$ denotes the norm of tensor $E(x)u$ in $\mathbb{R}^{nN}$ and $\mathscr{S}$ is the Sobolev's constant.

Choosing a suitable $\sigma > 0$, we get

$$\text{(12)}\qquad \int_{B_R}|\mathrm{D}w|^2\,\mathrm{d}x \le c\Big[\int_{B_R}|E(x)u|^2\,\mathrm{d}x + \Big(\int_{B_R}|f|^{\frac{2n}{n+2}}\,\mathrm{d}x\Big)^{\frac{n+2}{n}}\Big].$$

On the other hand, in force of assumption (2), a solution v of the problem (9) satisfies the Saint Venaint's principle (see Theorem 5.1 of [46]), that is, there exist two positive constants $c = c(\alpha,\beta,n) > 0$ and $\gamma = \gamma\big(\frac{\alpha}{\beta}, n\big) \in\,]0,1[$ such that

$$\text{(13)}\qquad \int_{B_\rho}|\mathrm{D}v|^2\,\mathrm{d}x \le c\Big(\frac{\rho}{R}\Big)^{n-2+2\gamma}\int_{B_R}|\mathrm{D}v|^2\,\mathrm{d}x, \quad \text{for all } 0 < \rho \le R.$$

Therefore, by (12) and (13), we deduce for every $0 < \rho \le R$

$$\begin{aligned}(14)\qquad \int_{B_\rho} |\mathrm{D}u|^2\,\mathrm{d}x &= c\int_{B_\rho} |\mathrm{D}v|^2\,\mathrm{d}x + c\int_{B_\rho} |\mathrm{D}w|^2\,\mathrm{d}x \\ &\le c\Big(\frac{\rho}{R}\Big)^{n-2+2\gamma}\int_{B_R} |\mathrm{D}v|^2\,\mathrm{d}x + c\int_{B_R} |E(x)u|^2\,\mathrm{d}x \\ &\quad + c\Big(\int_{B_R} |f|^{\frac{2n}{n+2}}\,\mathrm{d}x\Big)^{\frac{n+2}{n}} \\ &\le c\Big(\frac{\rho}{R}\Big)^{n-2+2\gamma}\int_{B_R} |\mathrm{D}u|^2\,\mathrm{d}x + c\int_{B_R} |E(x)u|^2\,\mathrm{d}x \\ &\quad + c\Big(\int_{B_R} |f|^{\frac{2n}{n+2}}\,\mathrm{d}x\Big)^{\frac{n+2}{n}}.\end{aligned}$$

Since $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ then $u \in L^{2^*}(\Omega, \mathbb{R}^N) \subset L^{2,2}(\Omega, \mathbb{R}^N)$ and $Du \in L^2(\Omega, \mathbb{R}^{nN})$, consequently, by virtue of Lemma 2.5, one has

$$Eu \in L^{2,\mu_0}(\Omega, \mathbb{R}^{nN}) \quad \text{with } \mu_0 = \mu - n + 2$$

and from (14) we obtain

$$\begin{aligned}(15)\qquad \int_{B_\rho} |\mathrm{D}u|^2\,\mathrm{d}x &\le c\Big(\frac{\rho}{R}\Big)^{n-2+2\gamma}\int_{B_R} |Du|^2\,\mathrm{d}x \\ &\quad + c\|Eu\|^2_{L^{2,\mu_0}(\Omega,\mathbb{R}^{nN})}R^{\mu_0} + c\|f\|^2_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)}R^{\lambda} \\ &\le c\Big(\frac{\rho}{R}\Big)^{n-2+2\gamma}\int_{B_R} |Du|^2\,\mathrm{d}x \\ &\quad + c(\|Eu\|_{L^{2,\mu_0}(\Omega,\mathbb{R}^{nN})}, \|f\|_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)})R^{\mu_1}\end{aligned}$$

where

$$(16)\qquad \mu_1 = \min\{\mu_0, \lambda\} < n - 2.$$

Iterating the above inequality (see Campanato [6], Lemma 1.1, page 7), we establish that

$$\mathrm{D}u \in L^{2,\mu_1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN}),$$

with the corresponding norm estimate

$$(17)\quad \|\mathrm{D}u\|^2_{L^{2,\mu_1}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})} \le c[\|\mathrm{D}u\|^2_{L^2(\Omega,\mathbb{R}^{nN})} + \|Eu\|^2_{L^{2,\mu_0}(\Omega,\mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)}]$$

where $c > 0$ is independent of u, E and f.

Now, through an extension technique and successive standard "flattening and covering" arguments (see the Appendix), we get the regularity of Du up to the boundary of Ω, with the norm estimate

$$(18)\quad \|Du\|^2_{L^{2,\mu_1}(\Omega,\mathbb{R}^{nN})} \le c[\|Du\|_{L^2(\Omega,\mathbb{R}^{nN})} + \|Eu\|^2_{L^{2,\mu_0}(\Omega,\mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2n}{n+2},\frac{2\lambda}{n+2}}(\Omega,\mathbb{R}^N)}].$$

We now compare μ_0 with λ.

If $\mu_0 \ge \lambda$ then $\mu_1 = \lambda$ and the thesis follows. Otherwise $\mu_1 = \mu_0$ and we can apply Lemma 2.6 to the function $u \in W^{1,2}_0(\Omega,\mathbb{R}^N)$. Thus, since $Du \in L^{2,\mu_1}(\Omega, \mathbb{R}^{nN})$ we obtain

$$u \in L^{2_{\mu_1},\mu_1}(\Omega,\mathbb{R}^N) \quad \text{where } \frac{1}{2_{\mu_1}} = \frac{1}{2} - \frac{1}{n-\mu_1}.$$

On the other hand, $L^{2_{\mu_1},\mu_1}(\Omega,\mathbb{R}^N)$ is embedded into $L^{2,2+\mu_1}(\Omega,\mathbb{R}^N)$ therefore a new application of Lemma 2.5 gives us

$$Eu \in L^{2,\mu_0+\mu_1}(\Omega,\mathbb{R}^{nN})$$

with the norm estimate

$$(19)\quad \|Eu\|_{L^{2,\mu_0+\mu_1}(\Omega,\mathbb{R}^{nN})} \le c\|E\|_{L^{2,\mu}(\Omega,\mathbb{R}^{nN^2})}(\|Du\|_{L^{2,\mu_1}(\Omega,\mathbb{R}^{nN})} + \|u\|_{L^{2,2+\mu_1}(\Omega,\mathbb{R}^N)})$$

for some $c > 0$ independent of u and E.

Applying to (14) the improved norm estimate (19), we have

$$\begin{aligned}(20)\quad \int_{B_\rho} |Du|^2\,dx &\le c\left(\frac{\rho}{R}\right)^{n-2+2\gamma}\int_{B_R}|Du|^2\,dx + c\|Eu\|^2_{L^{2,\mu_0+\mu_1}(\Omega,\mathbb{R}^{nN})}R^{\mu_0+\mu_1}\\ &\quad + c\|f\|^2_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)}R^\lambda\\ &\le c\left(\frac{\rho}{R}\right)^{n-2+2\gamma}\int_{B_R}|Du|^2\,dx\\ &\quad + c(\|Eu\|^2_{L^{2,\mu_0+\mu_1}(\Omega,\mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)})R^{\mu_2}\end{aligned}$$

where

$$\mu_2 = \min\{\mu_0+\mu_1,\lambda\} < n-2.$$

As in the previous step, Lemma 1.1 from [6], provides us the local regularity

$$Du \in L^{2,\mu_2}_{loc}(\Omega,\mathbb{R}^{nN})$$

which can be extended up to the boundary of Ω, and we deduce that

$$Du \in L^{2,\mu_2}(\Omega,\mathbb{R}^{nN})$$

with the corresponding estimate

$$\|Du\|^2_{L^{2,\mu_2}(\Omega,\mathbb{R}^{nN})} \le c[\|Du\|^2_{L^2(\Omega,\mathbb{R}^{nN})} + \|Eu\|^2_{L^{2,\mu_0+\mu_1}(\Omega,\mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)}] \tag{21}$$

where $c > 0$ is a constant independent of u, E and f.

Iterating the previous procedure and setting for every $k = 1, 2, \ldots$

$$\mu_k = \min\{\mu_0 + \mu_{k-1}, \lambda\} < n - 2, \tag{22}$$

it follows

i) $Eu \in L^{2,\mu_0+\mu_k}(\Omega, \mathbb{R}^{nN})$, with the norm estimate

$$\|Eu\|_{L^{2,\mu_0+\mu_k}(\Omega,\mathbb{R}^{nN})} \le c_k \|E\|^k_{L^{2,\mu}(\Omega,\mathbb{R}^{nN^2})} (\|Du\|_{L^{2,\mu_k}(\Omega,\mathbb{R}^{nN})} + \|u\|_{L^{2,2+\mu_k}(\Omega,\mathbb{R}^N)}),$$

for some constant $c_k > 0$ independent of u and E, and

ii)

$$\int_{B_\rho} |Du|^2\,dx \le c\left(\frac{\rho}{R}\right)^{n-2+2\gamma} \int_{B_R} |Du|^2\,dx + c(\|Eu\|^2_{L^{2,\mu_0+\mu_k}(\Omega,\mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2n}{n+2},\frac{n\lambda}{n+2}}(\Omega,\mathbb{R}^N)}) R^{\mu_{k+1}}$$

iii) $Du \in L^{2,\mu_{k+1}}(\Omega, \mathbb{R}^{nN})$, with norm estimate

$$\|Du\|^2_{L^{2,\mu_{k+1}}(\Omega,\mathbb{R}^{nN})} \le c[\|Du\|^2_{L^2(\Omega,\mathbb{R}^{nN})} + \|Eu\|^2_{L^{2,\mu_0+\mu_k}(\Omega,\mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2n}{n+2},\frac{2\lambda}{n+2}}(\Omega,\mathbb{R}^N)}]$$

for some constant $c > 0$ independent of u, E and f.

After a finite number of steps we will have $\mu_0 + \mu_k > \lambda$, which implies $\mu_{k+1} = \lambda$, and in turn $Du \in L^{2,\lambda}(\Omega, \mathbb{R}^{nN})$.

Finally, a further application of Lemma 2.6 provides us

$$u \in L^{2_\lambda,\lambda}(\Omega, \mathbb{R}^N)$$

with $\frac{1}{2_\lambda} = \frac{1}{2} - \frac{1}{n-\lambda}$. □

Proof of Theorem 3.5. We will exploit the method introduced in [50] (see also [18]).

Let $B \subset\subset \Omega$ be a ball of radius R and let $\hat{B}$ be the enlarged ball of radius $32R$. We shall denote by $Q_{inn}(B)$ and $Q_{out}(B)$ the largest and the smallest cubes, concentric to B and with sides parallel to the coordinate axes, contained in B and containing B respectively. If we put

$$Q_{inn} = Q_{inn}(B), \quad Q_{out} = Q_{out}(B)$$

and

$$\hat{Q}_{inn} = Q_{inn}(\hat{B}), \quad \hat{Q}_{out} = Q_{out}(\hat{B}),$$

we have the following inclusions

$$Q_{inn} \subset B \subset\subset 4B \subset\subset \hat{Q}_{inn} \subset \hat{B} \subset \hat{Q}_{out} \tag{23}$$

(with kB we denote the ball with radius kR, $k \in \mathbb{N}$).

Let Ω' and Ω'' be a couple of open subset such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and $x_0 \in \Omega'$. For any $\tau \in]0,1[$ (that will be chosen later) we fix $h \in \mathbb{R}$ with $0 < |h| \ll \min\{1, d(\Omega', \partial\Omega'')\}$ such that, denoted with $B = B(x_0, |h|^\tau)$ the ball centered in x_0 and with radius $|h|^\tau$, the outer cube of B, $\hat{Q}_{out}$ is included in Ω''.

Moreover, given a vector-valued function $\omega : \Omega \to \mathbb{R}^N$ and a real number h, for any $i = 1, \dots, N$ we define the finite difference operator τ_{ih} as

$$\tau_{ih}(\omega)(x) = \omega(x + he_i) - \omega(x),$$

for $x \in \Omega$ such that $x + he_i \in \Omega$, where $\{e_i\}_{i=1,\dots,N}$ denotes the canonical basis of $\mathbb{R}^N$.

Let $v \in W^{1,2}(\hat{B}, \mathbb{R}^N)$ be the unique weak solution to the problem

$$\begin{cases} D_j(M_{ij}^{sr}(x) D_i v^r) = 0 & \text{in } \hat{B} \\ v^s = u^s & \text{on } \partial\hat{B} \end{cases} \quad s = 1, \dots, N, \tag{24}$$

and let $v_0 \in W^{1,2}(8B, \mathbb{R}^N)$ be the unique weak solution to the problem

$$\begin{cases} D_j(M_{ij}^{sr}(x_0) D_i v_0^r) = 0 & \text{in } 8B \\ v_0^s = v^s & \text{on } \partial 8B \end{cases} \quad s = 1, \dots, N. \tag{25}$$

Then we have

$$\int_B |\tau_{ih}(Du)|^2 \, dx \le c\left[\int_B |\tau_{ih}(Dv_0)|^2 \, dx + \int_{\hat{B}} |Du - Dv|^2 \, dx + \int_{8B} |Dv - Dv_0|^2 \, dx\right]. \tag{26}$$

The first term and the third term in the right-hand side of (26) can be estimated, respectively, as

$$\int_B |\tau_{ih}(Dv_0)|^2 \, dx \le c|h|^{2(1-\tau)} \int_{8B} |Dv_0 - z_0|^2 \, dx \quad \text{for all } z_0 \in \mathbb{R}^n, \tag{27}$$

and

$$\int_{8B} |Dv - Dv_0|^2 \, dx \le c|h|^{2\eta\tau} \int_{\hat{B}} |Du|^2 \, dx \tag{28}$$

(see [50]).

Finally, we have to estimate

$$\int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x.$$

Let us observe that the function $w = v - u \in \mathrm{W}_0^{1,2}(\hat{B}, \mathbb{R}^N)$ is the weak solution to the equation

$$\mathrm{D}_j[M_{ij}^{sr}(x)\mathrm{D}_i(w^r + u^r)] = 0, \quad s = 1, \ldots, N \quad \text{in } \hat{B}, \tag{29}$$

whence, by assumption (3), we deduce

$$\begin{aligned} \int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x &= \int_{\hat{B}} |\mathrm{D}w|^2 \,\mathrm{d}x \\ &\le \frac{1}{\alpha}\int_{\hat{B}} M_{ij}^{sr}(x)\mathrm{D}_i u^r \mathrm{D}_j w^s \,\mathrm{d}x \le \frac{\beta}{\alpha}\int_{\hat{B}} |\mathrm{D}u|\,|\mathrm{D}w| \,\mathrm{d}x \\ &\le c\left(\varepsilon \int_{\hat{B}} |\mathrm{D}(u-v)|^2 \,\mathrm{d}x + C(\varepsilon)\int_{\hat{B}} |\mathrm{D}u|^2 \,\mathrm{d}x\right) \end{aligned} \tag{30}$$

with ε, $C(\varepsilon)$ positive constants independent of the radius of $\hat{B}$.

In turn, for a sufficiently small ε, inequality (30) yields

$$\int_{\hat{B}} |\mathrm{D}u - \mathrm{D}v|^2 \,\mathrm{d}x \le c\int_{\hat{B}} |\mathrm{D}u|^2 \,\mathrm{d}x, \tag{31}$$

and the right hand side behaves as $|h|^{\tau\lambda}$ thanks to Theorem 3.3.

From this point on, we gather together inequalities (26), (27), (28) and (31) and we can argue as in the proof of Theorem 4 in [18], exploiting the method introduced in [50]. □

5. Appendix

Now we prove the Morrey regularity of $|\mathrm{D}u|$ in Ω. For this purpose, we follow the idea of G. M. Troianiello [55] adapted to the case of systems (cfr [47]).

We denote a vector of $\mathbb{R}^n$ by $x = (x_1, \ldots, x_{n-1}, x_n) \equiv (x', x_n)$.

If $y = (y', 0)$ we define

$$B_\rho^+(y) = \{x \in B(y,\rho) : x_n > 0\},$$
$$\Gamma_\rho(y) = \{x \in B(y,\rho) : x_n = 0\}.$$

Fixed $R_1 > 0$, let $\hat{M}(x) \equiv (\hat{M}_{ij}^{rs}(x))_{i,j=1,\ldots,n\; r,s=1,\ldots,N}$ be a matrix-valued function and let $\hat{E}(x) \equiv (\hat{E}_i^{rs}(x))_{i=1,\ldots,n\; r,s=1,\ldots,N}$ be a tensor and $\hat{f}(x) \equiv (\hat{f}^s(x))_{s=1,\ldots,N}$ be a vector-valued function defined in $\Omega = B_{R_1}^+(y)$.

We begin by investigating a solution of the problem

$$(32)\quad \begin{cases} \hat{u} \in W^{1,2}(B^+_{R_1}(y), \mathbb{R}^N) \\ \hat{u}_{|\Gamma_{R_1}(y)} = 0 \\ \displaystyle\int_{B^+_{R_1}(y)} \hat{M}^{rs}_{ij}(x) \mathrm{D}_j \hat{u}^r \mathrm{D}_i \varphi^s \, \mathrm{d}x = \int_{B^+_{R_1}} \hat{E}^{rs}_i(x) \hat{u}^r \mathrm{D}_i \varphi^s \, \mathrm{d}x + \int_{B^+_{R_1}} \hat{f}^s \varphi^s \, \mathrm{d}x \\ \quad \text{for all } \varphi \in W^{1,2}_0(B^+_{R_1}(y), \mathbb{R}^N). \end{cases}$$

We state the following

LEMMA 5.1. *Assume that tensor* $\hat{M}$, *satisfies* (2), (3), *tensor* $\hat{E}$ *and vector valued function* $\hat{f}$ *satisfy* (4) *and* (5) *respectively with* $\Omega = B^+_{R_1}$. *Let* $\mu_0 = \mu - n + 2$ *and let* $\hat{u}$ *be a solution of problem* (32).

Then, for every $R \in \,]0, R_1[$, *we have*

$$|\mathrm{D}\hat{u}| \in L^{2,\mu_1}(B^+_R, \mathbb{R}^{nN}) \quad \textit{with } \mu_1 = \min\{\lambda, \mu_0\},$$

and there exists a positive constant c *depending only on* α, β, n, N, R_1, *such that*

$$(33)\quad \|\mathrm{D}\hat{u}\|^2_{L^{2,\mu_1}(B^+_R, \mathbb{R}^{nN})} \le c[\|\mathrm{D}\hat{u}\|^2_{L^2(B^+_{R_1}, \mathbb{R}^{nN})} + \|\hat{E}\hat{u}\|^2_{L^{2,\mu_0}(B^+_{R_1}, \mathbb{R}^{nN})} + \|\hat{f}\|^2_{L^{\frac{2N}{N+2}, \frac{2\lambda}{N+2}}(B^+_{R_1}, \mathbb{R}^N)}].$$

PROOF. We extend the tensors

$$\hat{M}(x) \equiv (\hat{M}^{rs}_{ij}(x))_{i,j=1,\dots,n\ r,s=1,\dots,N}, \quad \hat{E}(x) \equiv (\hat{E}^{rs}_i(x))_{i=1,\dots,n\ r,s=1,\dots,N}$$

and the vector valued functions

$$\hat{f}(x) \equiv (\hat{f}^s(x))_{s=1,\dots,N}, \quad \hat{u}(x) \equiv (\hat{u}^s(x))_{s=1,\dots,N}$$

almost everywhere to $B_{R_1}(y)$ by setting

$$\overline{M^{rs}_{in}}(x', x_n) = \begin{cases} \hat{M}^{rs}_{in}(x', x_n) & \text{if } x_n > 0 \\ -\hat{M}^{rs}_{in}(x', -x_n) & \text{if } x_n < 0 \end{cases}$$

$$\overline{M^{rs}_{ni}}(x', x_n) = \begin{cases} \hat{M}^{rs}_{ni}(x', x_n) & \text{if } x_n > 0 \\ -\hat{M}^{rs}_{ni}(x', -x_n) & \text{if } x_n < 0 \end{cases}$$

for $i = 1, \dots, n-1$ and for $r, s = 1, \dots, N$,

$$\overline{M^{rs}_{ij}}(x', x_n) = \begin{cases} \hat{M}^{rs}_{ij}(x', x_n) & \text{if } x_n > 0 \\ \hat{M}^{rs}_{ij}(x', -x_n) & \text{if } x_n < 0 \end{cases}$$

for all the remaining values of i, j and for $r, s = 1, \dots, N$,

$$\overline{E_n^{rs}}(x', x_n) = \begin{cases} \hat{E}_n^{rs}(x', x_n) & \text{if } x_n > 0 \\ -\hat{E}_n^{rs}(x', -x_n) & \text{if } x_n < 0 \end{cases}$$

$$\overline{E_i^{rs}}(x', x_n) = \begin{cases} \hat{E}_i^{rs}(x', x_n) & \text{if } x_n > 0 \\ \hat{E}_i^{rs}(x', -x_n) & \text{if } x_n < 0 \end{cases}$$

$$\text{for } i = 1, \ldots, n-1,$$

for $s = 1, 2, \ldots, N$

$$\overline{f^s}(x', x_n) = \begin{cases} \hat{f}^s(x', x_n) & \text{if } x_n > 0 \\ -\hat{f}^s(x', -x_n) & \text{if } x_n < 0 \end{cases}$$

and finally for $s = 1, 2, \ldots, N$

$$\overline{u^s}(x', x_n) = \begin{cases} \hat{u}^s(x', x_n) & \text{if } x_n > 0 \\ -\hat{u}^s(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Note that the functions $\overline{M_{ij}}$ satisfy (2) and (3), the tensor $\bar{E}$ and the vector valued function $\bar{f}$ satisfy (4) and (5) respectively in $B_{R_1}(y)$, and $\bar{u} \in W_0^{1,2}(B_{R_1}(y), \mathbb{R}^N)$.

Given a function $v(x)$, with $x = (x', x_n) \in B_{R_1}(y)$, we set

$$\tilde{v}(x', x_n) \equiv v(x', -x_n).$$

Fixed a function $v \in C_0^1(B_{R_1}(y), \mathbb{R}^N)$, we note that $v - \tilde{v}$ belongs in $C_0^1(B_{R_1}^+(y), \mathbb{R}^N)$. Therefore, simple calculations show that

$$(34) \quad \int_{B_{R_1}(y)} \overline{M}(x) \mathrm{D}\bar{u} \mathrm{D}v \, \mathrm{d}x - \int_{B_{R_1}(y)} \bar{E}(x) \bar{u} \mathrm{D}v \, \mathrm{d}x$$

$$= \int_{B_{R_1}^+(y)} \hat{M}_{ij}^{rs}(x) \mathrm{D}\hat{u} \mathrm{D}(v - \tilde{v}) \, \mathrm{d}x - \int_{B_{R_1}^+(y)} \hat{E}_i^{rs}(x) \hat{u} \mathrm{D}(v - \tilde{v}) \, \mathrm{d}x$$

$$= \int_{B_{R_1}^+(y)} \hat{f}^s (v - \tilde{v}) \, \mathrm{d}x = \int_{B_{R_1}(y)} \bar{f} v \, \mathrm{d}x$$

and by a density argument we prove that the function $\bar{u}$ is solution of the problem (1) in $\Omega = B_{R_1}(y)$ with M_{ij}^{rs}, E_i^{rs} and f^s replaced by $\overline{M_{ij}^{rs}}$, $\overline{E_i^{rs}}$ and $\overline{f^s}$, respectively. Therefore $\bar{u}$ verifies (15) and hence (17) with $\Omega = B_{R_1}(y)$. Now, the Lemma follows by changing back the coordinates. □

Now we are able to prove the claimed global Morrey regularity of $\mathrm{D}u$.

Since $\partial\Omega \in C^1$, for each $\bar{y} \in \partial\Omega$ and $\delta_2 > 0$, there is a ball $B_{R_0}(\bar{y})$ and a $C^1(B_{R_0}(\bar{y}))$-diffeomorphism $\Lambda : \overline{B_{R_0}(\bar{y})} \to \overline{B_{\delta_2 R_0}(0)}$, which straighten $\partial\Omega \cap B_{R_0}(\bar{y})$ and such that

(1) $\Lambda(\bar{y}) = 0$
(2) $B^+_{\delta_1 R_0}(0) \subset \Lambda(B_{R_0}(\bar{y}) \cap \Omega) \subset B^+_{\delta_2 R_0}(0)$ for some $0 < \delta_1 \le \delta_2$.

Put $R_1 = \delta_1 R_0$. If $z \in B^+_{R_1}(0) \equiv B^+_{R_1}$, we set

$$\hat{M}^{rs}_{ij}(z) = M^{rs}_{hk}(\Lambda^{-1}(z)) \frac{\partial \Lambda_i}{\partial y_h}(\Lambda^{-1}(z)) \frac{\partial \Lambda_j}{\partial y_k}(\Lambda^{-1}(z)) J(z) \tag{35}$$

$$\begin{aligned} \hat{E}^{rs}_i(z) &= E^{rs}_h(\Lambda^{-1}(z)) \frac{\partial \Lambda_i}{\partial y_h}(\Lambda^{-1}(z)) J(z) \\ \hat{f}^s(z) &= f^s(\Lambda^{-1}(z)) J(z) \\ \hat{u}^s(z) &= u^s(\Lambda^{-1}(z)) J(z) \end{aligned} \tag{36}$$

where $z = \Lambda(y)$, $y = \Lambda^{-1}(z)$ and $J(z)$ denotes the absolute value of the Jacobian determinant of Λ^{-1} at z.

Let us observe that $\hat{M}^{rs}_{ij}$ belong to $L^\infty(B^+_{R_1}) \cap VMO$ (see Lemma 2.1 of [46]), $\hat{E}^{rs}_i$ belong to $L^n(B^+_{R_1}) \cap L^{2,\mu}(B^+_{R_1})$.

Moreover, from the definition (35), it follows that

$$\hat{M}^{rs}_{ij} \xi^r_i \xi^s_j = M^{rs}_{hk} \frac{\partial \Lambda_i}{\partial y_h} \frac{\partial \Lambda_j}{\partial y_k} \xi^r_i \xi^s_j J \ge \alpha \sum_{h=1}^n \Big(\sum_{r=1}^N \sum_{i=1}^n \frac{\partial \Lambda_i}{\partial y_h} \xi^r_i \Big)^2 \min_{\overline{B^+_{R_1}}} J \ge \alpha' |\xi|^2$$

for all $\xi \in \mathbb{R}^{nN}$.

Thus, a change of variables in (1) yields

$$\begin{cases} \hat{u} \in W^{1,2}(B^+_{R_1}, \mathbb{R}^N) \\ \hat{u}_{|\Gamma_{R_1}(0)} = 0 \\ \displaystyle\int_{B^+_{R_1}} \hat{M}^{rs}_{ij}(z) \mathrm{D}\hat{u} \mathrm{D}\varphi \, \mathrm{d}z = \int_{B^+_{R_1}} \hat{E}^{rs}_i(z) \hat{u} \mathrm{D}\varphi \, \mathrm{d}z + \int_{B^+_{R_1}} \hat{f}^s \varphi \, \mathrm{d}z \\ \quad \text{for all } \varphi \in W^{1,2}_0(B^+_{R_1}, \mathbb{R}^N). \end{cases} \tag{37}$$

To (37) we apply Lemma 5.1 and thus we get the membership of $\mathrm{D}\hat{u}$ in $L^{2,\mu_1}(B^+_R, \mathbb{R}^{nN})$, $0 < R < R_1$, with norm estimate (33).

We extend $\mathrm{D}\hat{u}$ a.e. to B_R setting $\mathrm{D}\hat{u}(x', x_n) = \mathrm{D}\hat{u}(x', -x_n)$ if $x_n < 0$. Thus $\mathrm{D}\hat{u} \in L^{2,\mu_1}(B_R, \mathbb{R}^{nN})$.

As a consequence, since for some $r > 0$ $B_r(\bar{y}) \cap \Omega \subset \Lambda^{-1}(B^+_R)$, the matrix-function $\mathrm{D}\hat{u}(\Lambda(y))$, $y \in B_r(\bar{y})$ belongs to $L^{2,\mu_1}(B_r(\bar{y}), \mathbb{R}^{nN})$ that is, by the chain rule, $\mathrm{D}u$ belongs to $L^{2,\mu_1}(\Omega \cap B_r(\bar{y}), \mathbb{R}^{nN})$; thus, thanks to (36), by changing back coordinates in (33), we deduce

$$\|\mathrm{D}u\|^2_{L^{2,\mu_1}(B_r(\bar{y}) \cap \Omega)} \le c[\|\mathrm{D}u\|^2_{L^2(\Omega, \mathbb{R}^{nN})} + \|Eu\|^2_{L^{2,\mu_0}(\Omega, \mathbb{R}^{nN})} + \|f\|^2_{L^{\frac{2N}{N+2}, \frac{2\lambda}{N+2}}(\Omega, \mathbb{R}^N)}]. \tag{38}$$

Because $\partial\Omega$ is compact, there is a finite number of balls such as $B_r(\bar{y})$, say $B^1, B^2, \ldots, B^m$, which cover $\partial\Omega$.

Moreover, there exists an open set $\Omega \setminus \bigcup_{i=1}^{m} B_i \subset H_0 \subset\subset \Omega$ such that $H_0, B^1, B^2, \dots, B^m$ cover $\overline{\Omega}$.

If $\{g_i\}_{i=0,1,\dots,m}$ is a partition of the unity relative to the above covering then it turns out

$$\|Du\|_{L^{2,\mu_1}(\Omega,\mathbb{R}^{nN})} \leq c\left[\|Du\|_{L^{2,\mu_1}(H_0,\mathbb{R}^{nN})} + \sum_{i=1}^{m}\|Du\|_{L^{2,\mu_1}(B_r^m(\bar{y})\cap\Omega,\mathbb{R}^{nN})}\right]. \tag{39}$$

Then from (39) by joining together (38) and the interior estimate (33) we derive the global Morrey estimate (18). □

ACKNOWLEDGMENTS. This work has been supported by Piano della Ricerca di Ateneo 2020-2022-PIACERI.

Project MO.S.A.I.C. "Monitoraggio satellitare, modellazioni matematiche e soluzioni architettoniche e urbane per lo studio, la previsione e la mitigazione delle isole di calore urbano", Project EEEP&DLaD.

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Received 12 October 2020,
and in revised form 24 January 2021

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