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Journal of Differential Equations 271 (2021) 849–863

Journal of Differential Equations

www.elsevier.com/locate/jde

Infinitely many solutions to singular convective Neumann systems with arbitrarily growing reactions

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Received 13 July 2020; accepted 16 September 2020

Abstract

An existence result for Neumann elliptic systems with singular, convective, sign-changing, arbitrarily growing reactions is established. Proofs are chiefly based on sub-super-solution and truncation techniques, nonlinear regularity theory, and fixed point arguments. As a consequence, infinitely many solutions are obtained through appropriate sequences of sub-super-solution pairs.

MSC: 35J57; 35J62; 35J75

Keywords: Neumann problem; Quasilinear elliptic system; Gradient dependence; Singular term; Arbitrary growth

1. Introduction

In this paper, we investigate the following homogeneous Neumann problem:

$$\begin{cases} -\Delta_p u = f(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ -\Delta_q v = g(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

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https://doi.org/10.1016/j.jde.2020.09.024

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, is a bounded domain with C^2-boundary $\partial\Omega$ having outer normal ν, $1 < p, q < +\infty$, Δ_r denotes the r-Laplacian, namely

$$\Delta_r u := \operatorname{div}\left(|\nabla u|^{r-2}\nabla u\right) \ \ \forall u \in W^{1,r}(\Omega),$$

and $f, g : \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are Carathéodory functions; see Section 3 for details.

The parabolic Neumann problem associated with (P), whose simplest form is the well-known Gierer-Meinhardt system, arises in biological pattern formation by auto- and cross-catalysis, morphogenesis, and cellular differentiation (cf. [12,13]). In particular, it is used to model the head formation of hydra, a freshwater animal long about 15 millimeters. Much attention received also its elliptic counterpart; see, e.g., [8].

From a mathematical point of view, (P) exhibits several difficulties. In particular,

- $-\Delta_p$ and $-\Delta_q$ are not maximal monotone under Neumann boundary conditions.
- f, g can be singular at zero, sign-changing, and with arbitrary behavior.
- f, g depend on the gradient of solution, which prevents to apply variational methods.

We took inspiration from the works [20,14] where Dirichlet and Robin problems, respectively, have been investigated in the scalar case (i.e., for a single differential equation) and with special reactions, which are non-negative, sub-linear, and split as the sum of a singular term plus a convective one. Here, these restrictions are removed at all. Adding potential terms, truncating nonlinearities, and using trapping region techniques allow to solve an auxiliary system with frozen gradients (cf. Lemma 3.3). Through arguments partially patterned after those in [20,14] we next achieve a solution of (P); see Theorem 3.8. It should be noted that problem (P) includes both non-cooperative and non-competitive systems, because no monotonicity is assumed.

Unlike [20,14], where a sub-solution permits to avoid the singularity of reactions and the differential operators are maximal monotone, we also need a super-solution. In fact, Poincaré's inequality is not available now, and potential terms do not appear, neither in equations nor in boundary conditions. Consequently, Schaefer's fixed point theorem, on which [20,14] basically rely, seems to be inapplicable here. The analysis of such problems looks harder and there are only few works on this subject; see [28] for a critical point theory approach. A further benefit of having a super-solution is the possibility to truncate nonlinearities with bad behavior, as a super-critical one (cf., e.g., [15]). Although the argument is elementary, constructing a super-solution could be very difficult. In our case, we succeed provided reactions grow super-linearly near the origin.

Another hopefully interesting aspect of this work comes from Theorems 4.2 and 4.3, where a whole sequence of solutions is obtained without assuming any symmetry condition or parametric control on the reactions. In fact, recall that a symmetric source often produces infinitely many critical points of the energy functional associated with the differential problem. A classical reference is [33], while [17] contains more recent results; concerning applications, see for instance [15,21]. Some variational principles (cf., e.g., [2]) yield the same conclusion for parametric problems, once the parameter belongs to a suitable interval. Theorems 4.2–4.3 below are obtained by first constructing monotone sequences of sub-super-solution pairs in the $C^1(\overline{\Omega})^2$-cone of positive functions, and then using Theorem 3.8 in each order interval.

The literature on elliptic problems with convection terms looks by now daily increasing; let us mention the very recent papers [22,29,30] for equations and [1,27] concerning singular systems, as well as their bibliographies. The monographs [4,12] represent general references on singular problems, while cooperative and competitive structures are discussed in [25] and [26], respectively. Dirichlet systems have been thoroughly investigated, mainly via variational techniques [23], sub-super-solution and truncation methods [5], or fixed point theorems [16]. To the best of our knowledge, much less attention received Neumann problems; actually, we can only cite [7,10].

The paper is organized as follows. Section 2 deals with preliminaries. The abstract existence result concerning (P) is proved in Section 3. The last section contains some meaningful special cases, obtained after explicitly constructing sub-super-solution pairs.

2. Preliminaries

Let Y, Z be two metric spaces and let $\mathcal{S}$ be a multifunction from Y into Z (briefly, $\mathcal{S} : Y \to 2^Z$). We say that $\mathcal{S}$ is lower semicontinuous when for every $y \in Y$, $\{y_n\}_n \subseteq Y$ converging to y, and $z \in \mathcal{S}(y)$ there exists $\{z_n\}_n \subseteq Z$ with the following properties: $z_n \to z$ in Z; $z_n \in \mathcal{S}(y_n)$ for all $n \in \mathbb{N}$.

Let Y, Z be two Banach spaces. An operator $T : Y \to Z$ is called compact if it maps bounded sets into relatively compact sets. An analogous definition holds for multifunctions. We denote by $Y \hookrightarrow Z$ the continuous embedding of Y into Z; if the embedding is compact, then we write $Y \overset{c}{\hookrightarrow} Z$.

Henceforth, for $0 < \alpha < 1 < r < +\infty$, Ω as in the Introduction, and $z : \overline{\Omega} \to \mathbb{R}$, the following notation will be adopted:

$$\|z\|_{L^r(\Omega)} := \left(\int_\Omega |z(x)|^r \mathrm{d}x \right)^{\frac{1}{r}}; \quad \|z\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |z(x)|;$$

$$\|z\|_{W^{1,r}(\Omega)} := (\|z\|^r_{L^r(\Omega)} + \|\nabla z\|^r_{L^r(\Omega)})^{\frac{1}{r}}; \; \|z\|_{C^1(\overline{\Omega})} := \|z\|_{L^\infty(\Omega)} + \|\nabla z\|_{L^\infty(\Omega)};$$

$$[\nabla z]_{C^{0,\alpha}(\Omega)} := \sup_{x \neq y} \frac{|\nabla z(x) - \nabla z(y)|}{|x-y|^\alpha}; \quad \|z\|_{C^{1,\alpha}(\overline{\Omega})} := \|z\|_{C^1(\overline{\Omega})} + [\nabla z]_{C^{0,\alpha}(\Omega)}.$$

Moreover,

$$W^{1,r}_b(\Omega) := W^{1,r}(\Omega) \cap L^\infty(\Omega).$$

Recall that $C^{1,\alpha}(\overline{\Omega}) \overset{c}{\hookrightarrow} C^1(\overline{\Omega})$, according to Ascoli-Arzelà's theorem [18, Theorem 1.5.3].

If Z is a real function space on Ω and $v, w \in Z$, then $v \le w$ means $v(x) \le w(x)$ for almost every $x \in \Omega$, while

$$v^+ := \max\{0, v\}, \quad [v, w] := \{z \in Z : v \le z \le w\}, \quad Z_+ := \{z \in Z : 0 \le z\}.$$

Let $Z^2 := Z \times Z$ and let $(v_1, v_2), (w_1, w_2) \in Z^2$. By definition, one has

$$(v_1, v_2) \le (w_1, w_2) \iff v_1 \le w_1 \text{ and } v_2 \le w_2.$$

If $\| \cdot \|_Z$ is a norm on Z, then we put $B_Z(\rho) := \{z \in Z : \|z\|_Z \le \rho\}$, $\rho > 0$, as well as

$$\|(z_1, z_2)\|_{Z^2} := \|z_1\|_Z + \|z_2\|_Z \quad \forall (z_1, z_2) \in Z^2.$$

3. An existence result

Recall that $f, g : \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ satisfy Carathéodory's conditions. Pick any $w := (w_1, w_2) \in C^1(\overline{\Omega})^2$ and consider problem (P) with 'frozen' gradients, i.e.,

$$\begin{cases} -\Delta_p u = h_1(x, u, v) & \text{in } \Omega, \\ -\Delta_q v = h_2(x, u, v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_{\rm w}$}$$

where

$$\begin{aligned} h_1(x, s, t) &:= f(x, s, t, \nabla w_1(x), \nabla w_2(x)), \\ h_2(x, s, t) &:= g(x, s, t, \nabla w_1(x), \nabla w_2(x)). \end{aligned} \tag{3.1}$$

The assumption below will be posited.

(H) There exist $\varepsilon > 0$, $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in W_b^{1,p}(\Omega) \times W_b^{1,q}(\Omega)$ such that

$$(\varepsilon, \varepsilon) \le (\underline{u}, \underline{v}) \le (\overline{u}, \overline{v}).$$

Moreover, if $K := C^1(\overline{\Omega})^2 \cap ([\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}])$, then:

(i) For appropriate $\rho, C > 0$ one has

$$|f(\cdot, u, v, \nabla w)| \le \rho, \quad |g(\cdot, u, v, \nabla w)| \le \rho$$

whenever $(u, v, w) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \times D$, where

$$D := \{w \in K : \|\nabla w\|_{L^\infty(\Omega)^2} \le C\}. \tag{3.2}$$

(ii) For every fixed $w \in D$ the pair $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ is a sub-super-solution to problem (P$_{\rm w}$), namely

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, dx & \le \int_\Omega h_1(\cdot, \underline{u}, v) \varphi \, dx, \\ \int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \psi \, dx & \le \int_\Omega h_2(\cdot, u, \underline{v}) \psi \, dx, \end{cases}$$
$$\begin{cases} \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi \, dx & \ge \int_\Omega h_1(\cdot, \overline{u}, v) \varphi \, dx, \\ \int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v} \nabla \psi \, dx & \ge \int_\Omega h_2(\cdot, u, \overline{v}) \psi \, dx \end{cases} \tag{3.3}$$

whenever $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, $(\varphi, \psi) \in W_b^{1,p}(\Omega)_+ \times W_b^{1,q}(\Omega)_+$.

Now, given $(u, v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, we define

$$T_p(u)(x) := \begin{cases} \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \end{cases} \qquad x \in \Omega,$$

$$T_q(v)(x) := \begin{cases} \underline{v}(x) & \text{if } v(x) < \underline{v}(x), \\ v(x) & \text{if } \underline{v}(x) \le v(x) \le \overline{v}(x), \\ \overline{v}(x) & \text{if } v(x) > \overline{v}(x). \end{cases} \qquad x \in \Omega.$$

Lemma 2.89 of [4] ensures that the operators $T_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ and $T_q : W^{1,q}(\Omega) \to W^{1,q}(\Omega)$ are continuous.

Truncating reactions allows to neglect the singular behavior in zero as well as possible supercritical growths at infinity. Hence, we add a potential term in both sides, which makes the differential operator strictly monotone, and truncate the right-hand one, thus coming to the problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = k_1(x,u,v) & \text{in } \Omega, \\ -\Delta_q v + |v|^{q-2}v = k_2(x,u,v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{\mathrm{P}}_{\mathrm{w}}$}$$

where

$$\begin{aligned} k_1(\cdot,u,v) &:= h_1(\cdot, T_p(u), T_q(v)) + |T_p(u)|^{p-2}T_p(u), \\ k_2(\cdot,u,v) &:= h_2(\cdot, T_p(u), T_q(v)) + |T_q(v)|^{q-2}T_q(v). \end{aligned} \tag{3.4}$$

Solutions of ($\tilde{\mathrm{P}}_{\mathrm{w}}$) will be sought by freezing reactions again. Accordingly, bear in mind (3.2), and, for every fixed $(u,v,w) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega) \times D$, consider the variational problem

$$\begin{cases} -\Delta_p \hat{u} + |\hat{u}|^{p-2}\hat{u} = k_1(x,u(x),v(x)) & \text{in } \Omega, \\ -\Delta_q \hat{v} + |\hat{v}|^{q-2}\hat{v} = k_2(x,u(x),v(x)) & \text{in } \Omega, \\ \frac{\partial \hat{u}}{\partial \nu} = \frac{\partial \hat{v}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$}$$

Remark 3.1. Hypothesis (H)(i) evidently forces

$$k_1(\cdot,u,v), k_2(\cdot,u,v) \in L^\infty(\Omega).$$

So, through Moser's iteration technique [11, Theorem 6.2.6], we see that any solution $(\hat{u},\hat{v})$ of ($\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$) turns out essentially bounded. Lieberman's regularity theory up to the boundary [19] (see also [24, Theorem 8.10]), yields $\alpha \in (0,1)$ and $R > 0$ (depending only on p, q, Ω, ρ) such that

$$(\hat{u},\hat{v}) \in B_{C^{1,\alpha}(\overline{\Omega})^2}(R) \subseteq B_{C^1(\overline{\Omega})^2}(R).$$

Lemma 3.2. *Let* (H)(i) *be satisfied and let* $(u,v,w) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega) \times D$. *Then problem* ($\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$) *possesses a unique solution* $(\hat{u},\hat{v}) \in B_{C^{1,\alpha}(\overline{\Omega})^2}(R)$.

Proof. The energy functionals associated with equations in ($\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$), i.e.,

$$\Psi_1(z) := \|z\|^p_{W^{1,p}(\Omega)} - \int_\Omega k_1(\cdot, u, v) z \,\mathrm{d}x, \quad z \in W^{1,p}(\Omega),$$

$$\Psi_2(z) := \|z\|^q_{W^{1,q}(\Omega)} - \int_\Omega k_2(\cdot, u, v) z \,\mathrm{d}x, \quad z \in W^{1,q}(\Omega),$$

are weakly lower semi-continuous, strictly convex, and coercive, because $p, q > 1$. By Weierstrass-Tonelli's theorem, they have a unique global minimizer, say $\hat{u} \in W^{1,p}(\Omega)$ for Ψ_1 and $\hat{v} \in W^{1,q}(\Omega)$ for Ψ_2. Obviously, $(\hat{u}, \hat{v})$ is a weak solution to ($\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$). In fact, the nonlinear Green's formula [6, Theorem 3] entails $\frac{\partial \hat{u}}{\partial \nu} = \frac{\partial \hat{v}}{\partial \nu} = 0$ on $\partial\Omega$. Now the conclusion stems from Remark 3.1. □

Next, pick $w \in D$. For every $(u, v) \in B_{C^1(\overline{\Omega})^2}(R)$ we set

$$\Phi(u, v) := (\hat{u}, \hat{v}), \tag{3.5}$$

where $(\hat{u}, \hat{v})$ is as in Lemma 3.2. Since

$$B_{C^{1,\alpha}(\overline{\Omega})^2}(R) \overset{c}{\hookrightarrow} B_{C^1(\overline{\Omega})^2}(R), \tag{3.6}$$

the operator $\Phi : B_{C^1(\overline{\Omega})^2}(R) \to B_{C^1(\overline{\Omega})^2}(R)$ defined by (3.5) is compact. It will play a basic role to prove the following

Lemma 3.3. *If* (H) *holds and $w \in D$, then* ($\mathrm{P_w}$) *admits solutions in K.*

Proof. We claim that Φ is continuous. In fact, let $\{(u_n, v_n)\}_n \subseteq B_{C^1(\overline{\Omega})^2}(R)$ satisfy $(u_n, v_n) \to (u, v)$ in $C^1(\overline{\Omega})^2$ and let $(\hat{u}_n, \hat{v}_n) := \Phi(u_n, v_n)$, $n \in \mathbb{N}$. The compactness of Φ forces $(\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v})$ in $C^1(\overline{\Omega})^2$, where a sub-sequence is considered when necessary. On the other hand, each $(\hat{u}_n, \hat{v}_n)$ solves ($\tilde{\mathrm{P}}_{(\mathrm{u_n,v_n,w})}$), whence $(\hat{u}, \hat{v})$ turns out a solution to ($\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$), as we easily see once $n \to \infty$ in ($\tilde{\mathrm{P}}_{(\mathrm{u_n,v_n,w})}$). By uniqueness one has $(\hat{u}, \hat{v}) = \Phi(u, v)$, thus showing the continuity of Φ.

Now, Schauder's fixed point theorem gives $(u, v) \in B_{C^1(\overline{\Omega})^2}(R)$ such that $(u, v) = \Phi(u, v)$, namely (u, v) solves ($\tilde{\mathrm{P}}_{\mathrm{w}}$). Testing with $((\underline{u} - u)^+, (\underline{v} - v)^+)$ produces

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(\underline{u}-u)^+\mathrm{d}x+\int_\Omega |u|^{p-2}u(\underline{u}-u)^+\mathrm{d}x\\&\qquad=\int_\Omega k_1(\cdot,u,v)(\underline{u}-u)^+\mathrm{d}x,\\&\int_\Omega |\nabla v|^{q-2}\nabla v\nabla(\underline{v}-v)^+\mathrm{d}x+\int_\Omega |v|^{q-2}v(\underline{v}-v)^+\mathrm{d}x\\&\qquad=\int_\Omega k_2(\cdot,u,v)(\underline{v}-v)^+\mathrm{d}x.\end{aligned}\tag{3.7}$$

Through (H)(ii), written for $(T_p(u),T_q(v))$ in place of (u,v), we get

$$\begin{aligned}&\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla(\underline{u}-u)^+\mathrm{d}x+\int_\Omega |\underline{u}|^{p-2}\underline{u}(\underline{u}-u)^+\mathrm{d}x\\&\qquad\le\int_\Omega k_1(x,\underline{u},v)(\underline{u}-u)^+\mathrm{d}x,\\&\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}\nabla(\underline{v}-v)^+\mathrm{d}x+\int_\Omega |\underline{v}|^{q-2}\underline{v}(\underline{v}-v)^+\mathrm{d}x\\&\qquad\le\int_\Omega k_2(x,u,\underline{v})(\underline{v}-v)^+\mathrm{d}x.\end{aligned}\tag{3.8}$$

Subtracting (3.7) from (3.8) leads to

$$\begin{aligned}&\int_\Omega (|\nabla \underline{u}|^{p-2}\nabla \underline{u}-|\nabla u|^{p-2}\nabla u)\nabla(\underline{u}-u)^+\mathrm{d}x\\&\qquad+\int_\Omega (|\underline{u}|^{p-2}\underline{u}-|u|^{p-2}u)(\underline{u}-u)^+\mathrm{d}x\le 0,\\&\int_\Omega (|\nabla \underline{v}|^{q-2}\nabla \underline{v}-|\nabla v|^{q-2}\nabla v)\nabla(\underline{v}-v)^+\mathrm{d}x\\&\qquad+\int_\Omega (|\underline{v}|^{q-2}\underline{v}-|v|^{q-2}v)(\underline{v}-v)^+\mathrm{d}x\le 0.\end{aligned}$$

By strict monotonicity of the operator $z\mapsto -\Delta_r z+|z|^{r-2}z$ (cf., e.g., [31, Lemma A.0.5]), this entails both $(\underline{u}-u)^+=0$ and $(\underline{v}-v)^+=0$ in Ω. So, $(\underline{u},\underline{v})\le(u,v)$. An analogous argument yields $(u,v)\le(\overline{u},\overline{v})$. The proof ends bearing in mind (3.4). □

Define, for every $w \in D$,

$$\mathcal{S}(w) := \{(u, v) \in K : (u, v) \text{ is a solution to } (\mathrm{P_w})\}.$$

The above lemma ensures that the multifunction $\mathcal{S} : D \to 2^K$ takes nonempty values. Moreover,

Lemma 3.4. *Let* (H) *be fulfilled. Then* $\mathcal{S}$ *is compact.*

Proof. Since $(u, v) \in K$, problem $(\tilde{\mathrm{P}}_{\mathrm{w}})$ coincides with $(\mathrm{P_w})$ for any $w \in D$. Thus, the compactness of $\mathcal{S}$ is a consequence of Remark 3.1 and (3.6). □

Lemma 3.5. *Under* (H)*, the multifunction* $\mathcal{S}$ *is lower semicontinuous.*

Proof. Although the reasoning is similar to that in [14, Lemma 3.7], we will sketch it, because here super-solutions play a role. Pick $\{w_n\}_n \subseteq D$ satisfying

$$\lim_{n\to\infty} w_n = w \ \text{ in } \ C^1(\overline{\Omega})^2 \tag{3.9}$$

and choose any

$$(u^0, v^0) \in \mathcal{S}(w). \tag{3.10}$$

Consider the family of two-index problems, say $m, n \in \mathbb{N}$,

$$\begin{cases} -\Delta_p u_n^m + |u_n^m|^{p-2} u_n^m = r_n^m(x) & \text{in } \Omega, \\ -\Delta_q v_n^m + |v_n^m|^{q-2} v_n^m = s_n^m(x) & \text{in } \Omega, \\ \frac{\partial u_n^m}{\partial \nu} = \frac{\partial v_n^m}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P_n^m}$}$$

where

$$\begin{aligned} r_n^m(x) &= f(x, u_n^{m-1}(x), v_n^{m-1}(x), \nabla w_n(x)) + (u_n^{m-1}(x))^{p-1}, \\ s_n^m(x) &= g(x, u_n^{m-1}(x), v_n^{m-1}(x), \nabla w_n(x)) + (v_n^{m-1}(x))^{q-1}, \\ (u_n^0, v_n^0) &= (u^0, v^0), \end{aligned}$$

for all m, n. We construct a double sequence $\{(u_n^m, v_n^m)\}_{m,n}$ by fixing $n \in \mathbb{N}$ and using Weierstrass-Tonelli's theorem to solve $(\mathrm{P_n^m})$ inductively on m. In fact, $(\mathrm{P_{w_n}})$ coincides with $(\tilde{\mathrm{P}}_{\mathrm{w_n}})$ since $(u_n^{m-1}, v_n^{m-1}) \in K$, thus the argument exploited in the proof of Lemma 3.2 works. Observe next that

$$(u_n^m, v_n^m) \in B_{C^{1,\alpha}(\overline{\Omega})^2}(R) \cap K \quad \forall m \in \mathbb{N};$$

cf. Lemma 3.3. Hence, from (3.6) it follows

$$\lim_{m\to\infty} (u_n^m, v_n^m) = (u_n, v_n) \ \text{ in } \ C^1(\overline{\Omega})^2 \tag{3.11}$$

up to sub-sequences. Letting $m \to \infty$ in $(\mathrm{P_n^m})$ we readily obtain

$$(u_n, v_n) \in \mathcal{S}(w_n), \quad n \in \mathbb{N}. \tag{3.12}$$

Keep now $m \in \mathbb{N}$ fixed and reason similarly, to arrive at

$$\lim_{n\to\infty} (u_n^m, v_n^m) = (u^m, v^m) \text{ in } C^1(\overline{\Omega})^2. \tag{3.13}$$

One actually has

$$(u^m, v^m) = (u^0, v^0) \text{ for all } m. \tag{3.14}$$

In fact, through ($\mathrm{P}_\mathrm{n}^\mathrm{m}$), (3.9), and (3.13), we deduce that (u^m, v^m) solves the problem

$$\begin{cases} -\Delta_p u^m + |u^m|^{p-2}u^m = r^m(x) & \text{in } \Omega, \\ -\Delta_q v^m + |v^m|^{q-2}v^m = s^m(x) & \text{in } \Omega, \\ \frac{\partial u^m}{\partial \nu} = \frac{\partial v^m}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$r^m(x) = f(x, u^{m-1}(x), v^{m-1}(x), \nabla w(x)) + (u^{m-1}(x))^{p-1},$$
$$s^m(x) = g(x, u^{m-1}(x), v^{m-1}(x), \nabla w(x)) + (v^{m-1}(x))^{q-1},$$

which possesses a unique solution. Recalling (3.10), an induction procedure on m yields (3.14). Finally, the double limit lemma, when combined with (3.11), (3.13), and (3.14), entails

$$\lim_{n\to\infty} (u_n, v_n) = \lim_{n\to\infty} \lim_{m\to\infty} (u_n^m, v_n^m) = \lim_{m\to\infty} (u^m, v^m) = (u^0, v^0).$$

On account of (3.9), (3.10), and (3.12), this completes the proof. □

Via a standard argument (cf., e.g., [14, Lemmas 3.8–3.9]), chiefly based on Lemma 3.4, we can verify the following

Lemma 3.6. *Suppose* (H) *to be satisfied. Then, for every $w \in D$, the set $\mathcal{S}(w)$ admits minimum.*

So, it makes sense to define

$$\Gamma(w) := \min \mathcal{S}(w), \quad w \in D. \tag{3.15}$$

Obviously, $\Gamma : D \to K$ turns out a selection of $\mathcal{S}$. Moreover,

Lemma 3.7. *Let* (H) *be fulfilled. Then the map Γ is compact and continuous.*

Proof. Compactness directly stems from Lemma 3.4, once we realize that $\Gamma(A) \subseteq \mathcal{S}(A)$ for any $A \subseteq D$. Pick $\{w_n\}_n \subseteq D$ with $w_n \to w$ in $C^1(\overline{\Omega})^2$. Along a sub-sequence if necessary, one has

$$\lim_{n\to\infty} \|\Gamma(w_n) - (u^*, v^*)\|_{C^1(\overline{\Omega})^2} = 0 \tag{3.16}$$

for some $(u^*, v^*) \in C^1(\overline{\Omega})^2$, because Γ is compact. We claim that $(u^*, v^*) = \Gamma(w)$. In fact, letting $n \to \infty$ in $(\mathrm{P}_{\mathrm{w_n}})$ provides

$$(u^*, v^*) \in \mathcal{S}(w). \tag{3.17}$$

Thanks to Lemma 3.5, there exists $\{(u_n, v_n)\}_n \subseteq D$ such that

$$(u_n, v_n) \in \mathcal{S}(w_n) \quad \forall n \in \mathbb{N}, \tag{3.18}$$

$$\lim_{n\to\infty} \|(u_n, v_n) - \Gamma(w)\|_{C^1(\overline{\Omega})^2} = 0. \tag{3.19}$$

The minimality of Γ, together with (3.17), (3.16), (3.18), and (3.19), yield

$$\Gamma(w) \le (u^*, v^*) = \lim_{n\to\infty} \Gamma(w_n) \le \lim_{n\to\infty} (u_n, v_n) = \Gamma(w),$$

whence $\Gamma(w_n) \to \Gamma(w)$, as desired. □

By [9, Theorem 3.1], any solution $(u, v) \in K$ to $(\mathrm{P_w})$ satisfies the gradient estimates

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega)} &\le \eta_1 \|f(\cdot, u, v, \nabla w)\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}, \\ \|\nabla v\|_{L^\infty(\Omega)} &\le \eta_2 \|g(\cdot, u, v, \nabla w)\|_{L^\infty(\Omega)}^{\frac{1}{q-1}}, \end{aligned} \tag{3.20}$$

where $\eta_1, \eta_2 > 0$ denote suitable constants. Evidently, there is no loss of generality in assuming $\eta_1, \eta_2 \ge 1$.

Our main result requires a further condition on the reaction terms, which however complies with various meaningful cases; see Theorems 4.2–4.3 below. Hereafter, we suppose that

$$\rho \le \min\left\{ \left(\frac{C}{\eta_1}\right)^{p-1}, \left(\frac{C}{\eta_2}\right)^{q-1} \right\}, \tag{3.21}$$

where ρ, C come from (H), while η_1, η_2 are as in (3.20).

Theorem 3.8. *If* (H) *and* (3.21) *hold, then problem* (P) *possesses a solution belonging to* $C^{1,\alpha}(\overline{\Omega})^2 \cap K$.

Proof. Let Γ be given by (3.15). Condition (3.21) and (3.20) guarantee that $\Gamma(D) \subseteq D$. Thus, on account of Lemma 3.7, Schauder's fixed point theorem can be applied, which entails $(u, v) = \Gamma(u, v)$ for some $(u, v) \in D$. Through (3.1) and Remark 3.1 we easily verify that (u, v) satisfies the conclusion. □

4. Infinitely many solutions

Let $e_r \in C^{1,\alpha}(\overline{\Omega})_+$, $r = p, q$, be the unique solution to the problem

$$\begin{cases} -\Delta_r e_r = 1 & \text{in } \Omega, \\ e_r = 0 & \text{on } \partial\Omega, \end{cases}$$

let $L_r := \|e_r\|_{L^\infty(\Omega)}$, and let $\Lambda_r > L_r$. The following sequences of sub-super-solution pairs, which depend on a positive constant C_n, will be employed:

$$\begin{aligned} (\underline{u}_n, \underline{v}_n) &:= (C_n(\Lambda_p - e_p), C_n(\Lambda_q - e_q)), \quad n \in \mathbb{N}; \\ (\overline{u}_n, \overline{v}_n) &:= (C_n(\Lambda_p + e_p), C_n(\Lambda_q + e_q)), \quad n \in \mathbb{N}. \end{aligned} \tag{4.1}$$

By the Boundary Point Lemma [32, Theorem 5.5.1] one has

$$\max\left\{\frac{\partial \overline{u}_n}{\partial \nu}, \frac{\partial \overline{v}_n}{\partial \nu}\right\} < 0 < \max\left\{\frac{\partial \underline{u}_n}{\partial \nu}, \frac{\partial \underline{v}_n}{\partial \nu}\right\} \quad \text{on } \partial\Omega, \tag{4.2}$$

while the choice of Λ_r (recall also that $e_r \geq 0$) produces

$$(C_n(\Lambda_p - L_p), C_n(\Lambda_q - L_q)) \leq (\underline{u}_n, \underline{v}_n) \leq (\overline{u}_n, \overline{v}_n). \tag{4.3}$$

4.1. The sub-linear case

We make the hypotheses below.

(F_1) There exist $\alpha_1 < 0 < \beta_1$ satisfying $\alpha_1 + \beta_1 < p - 1$, $\gamma_1, \delta_1 \in [0, p-1)$, and $a_1, b_1, c_1 \in L^\infty(\Omega)$ such that

$$|f(x, s, t, \xi_1, \xi_2)| \leq a_1(x) s^{\alpha_1} t^{\beta_1} + b_1(x)(|\xi_1|^{\gamma_1} + |\xi_2|^{\delta_1}) + c_1(x)$$

for all $(x, s, t, \xi_1, \xi_2) \in \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N}$.

(G_1) There exist $\beta_2 < 0 < \alpha_2$ satisfying $\alpha_2 + \beta_2 < q - 1$, $\gamma_2, \delta_2 \in [0, q-1)$, and $a_2, b_2, c_2 \in L^\infty(\Omega)$ such that

$$|g(x, s, t, \xi_1, \xi_2)| \leq a_2(x) s^{\alpha_2} t^{\beta_2} + b_2(x)(|\xi_1|^{\gamma_2} + |\xi_2|^{\delta_2}) + c_2(x)$$

for all $(x, s, t, \xi_1, \xi_2) \in \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N}$.

Incidentally, similar conditions already appear in [3].

Remark 4.1. One can take $\gamma_1, \delta_1 \in [0, p-1]$ provided $\|b_1\|_{L^\infty(\Omega)} < \frac{1}{2}\eta_1^{-(p-1)}$, with η_1 given by (3.20). An analogous comment applies to γ_2, δ_2.

Theorem 4.2. *Let* (F_1)–(G_1) *be satisfied. Then problem* (P) *admits a sequence of solutions* $\{(u_n, v_n)\}_n \subseteq C^1(\overline{\Omega})^2$ *such that* $(u_n, v_n) < (u_{n+1}, v_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = +\infty$ *uniformly in* $\overline{\Omega}$.

Proof. Define

$$K_n := C^1(\overline{\Omega})^2 \cap \big([\underline{u}_n, \overline{u}_n] \times [\underline{v}_n, \overline{v}_n]\big)$$

as well as

$$D_n := \{w \in K_n : \|\nabla w\|_{L^\infty(\Omega)^2} \le C_n\},$$

where $C_n > 0$ comes from (4.1). If $(u, v, w) \in K_n \times D_n$, then through ($F_1$) we obtain

$$\begin{aligned} &|f(\cdot,u,v,\nabla w)| \\ &\quad \le \|a_1\|_{L^\infty(\Omega)} \underline{u}_n^{\alpha_1} \overline{v}_n^{\beta_1} + \|b_1\|_{L^\infty(\Omega)} (C_n^{\gamma_1} + C_n^{\delta_1}) + \|c_1\|_{L^\infty(\Omega)} \\ &\quad \le \|a_1\|_{L^\infty(\Omega)} C_n^{\alpha_1+\beta_1} (\Lambda_p - L_p)^{\alpha_1} (\Lambda_q + L_q)^{\beta_1} \\ &\qquad\quad + \|b_1\|_{L^\infty(\Omega)} (C_n^{\gamma_1} + C_n^{\delta_1}) + \|c_1\|_{L^\infty(\Omega)} \le \left(\frac{C_n}{\eta_1}\right)^{p-1}, \end{aligned} \tag{4.4}$$

once $C_n > C^*$, with $C^* > 0$ large enough. Likewise, (G_1) yields

$$|g(\cdot, u, v, \nabla w)| \le \left(\frac{C_n}{\eta_2}\right)^{q-1}. \tag{4.5}$$

Hence, assumption (3.21) of Theorem 3.8 holds for $K := K_n$, $D := D_n$, $C := C_n$. Observe next that

$$-\Delta_p \underline{u}_n = -C_n^{p-1} \le -\frac{C_n^{p-1}}{\eta_1^{p-1}} \le f(\cdot, u, v, \nabla w) \le \frac{C_n^{p-1}}{\eta_1^{p-1}} \le C_n^{p-1} = -\Delta_p \overline{u}_n$$

thanks to (4.1), the inequality $\eta_1 \ge 1$, and (4.4). Similarly, from (4.5) it follows

$$-\Delta_q \underline{v}_n \le g(\cdot, u, v, \nabla w) \le -\Delta_q \overline{v}_n.$$

Now, integrating by parts and using (4.2) we easily achieve (3.3). So, also due to (4.3), hypothesis (H) in Theorem 3.8 is fulfilled. Thus, for every $n \in \mathbb{N}$, problem (P) possesses a solution $(u_n, v_n) \in K_n$. If $C_1 > C^*$ and

$$C_{n+1} > \max\left\{\frac{\Lambda_p + L_p}{\Lambda_p - L_p}, \frac{\Lambda_q + L_q}{\Lambda_q - L_q}\right\} C_n, \quad n \in \mathbb{N},$$

then $\overline{u}_n < \underline{u}_{n+1}$ as well as $\overline{v}_n < \underline{v}_{n+1}$, which entails $(u_n, v_n) < (u_{n+1}, v_{n+1})$. The proof ends by noting that $C_n \to +\infty$, whence $\lim_{n\to\infty} \underline{u}_n = \lim_{n\to\infty} \underline{v}_n = +\infty$ uniformly in $\overline{\Omega}$. □

4.2. The super-linear case

The conditions below will be posited.

(F_2) There exist $\alpha_1 < 0 < \beta_1$ satisfying $\alpha_1 + \beta_1 > p - 1$, $\gamma_1, \delta_1 \in (p-1, +\infty)$, and $a_1, b_1 \in L^\infty(\Omega)$ such that

$$|f(x,s,t,\xi_1,\xi_2)| \le a_1(x)s^{\alpha_1}t^{\beta_1} + b_1(x)(|\xi_1|^{\gamma_1} + |\xi_2|^{\delta_1})$$

for all $(x,s,t,\xi_1,\xi_2) \in \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$.

(G_2) There exist $\beta_2 < 0 < \alpha_2$ satisfying $\alpha_2 + \beta_2 > q - 1$, $\gamma_2, \delta_2 \in (q-1, +\infty)$, and $a_2, b_2 \in L^\infty(\Omega)$ such that

$$|g(x,s,t,\xi_1,\xi_2)| \le a_2(x)s^{\alpha_2}t^{\beta_2} + b_2(x)(|\xi_1|^{\gamma_2} + |\xi_2|^{\delta_2})$$

for all $(x,s,t,\xi_1,\xi_2) \in \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$.

Remark 4.1 can be adapted to (F_2)–(G_2).

Theorem 4.3. *Under assumptions* (F_2)–(G_2), *problem* (P) *has a sequence of solutions* $\{(u_n, v_n)\}_n \subseteq C^1(\overline{\Omega})^2$ *such that* $(u_{n+1}, v_{n+1}) < (u_n, v_n)$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$ *uniformly in* $\overline{\Omega}$.

Proof. The argument is patterned after that of Theorem 4.2, because (4.4), written for $c_1 \equiv 0$, and (4.5) hold whenever $C_n < C_*$, with C_* sufficiently small. So, if $C_1 < C_*$ and

$$C_{n+1} < \min\left\{\frac{\Lambda_p - L_p}{\Lambda_p + L_p}, \frac{\Lambda_q - L_q}{\Lambda_q + L_q}\right\} C_n, \quad n \in \mathbb{N},$$

then the conclusion follows at once. □

Remark 4.4. Conditions (F_i) and (G_i), $i = 1, 2$, above have been formulated on the whole $\Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$ just for the sake of simplicity. In fact, consider, e.g., Theorem 4.3. Since C_* is small enough, it suffices to request (F_2) in $(0,\delta]^2 \times B_{\mathbb{R}^N}(\delta)^2$, because

$$C_* < \frac{\delta}{\max\{\Lambda_p + L_p, \Lambda_q + L_q, 1\}} \implies \max\{\overline{u}_1, \overline{v}_1, |\nabla w_1|, |\nabla w_2|\} < \delta$$

for any $(w_1, w_2) \in D_n$, $n \in \mathbb{N}$, and the same arguments work. So, we can actually treat reactions f, g with arbitrary behavior, provided they exhibit a super-linear growth near the origin. Evidently, a 'dual' comment holds for Theorem 4.2.

Let us finally make two examples of nonlinearities f, g fulfilling (F_1)–(G_1) and (F_2)–(G_2), respectively, settled according to Remark 4.4.

Example 4.5. Define, for every $(x, s, t, \xi_1, \xi_2) \in \Omega \times [1, +\infty)^2 \times \mathbb{R}^{2N}$,

$$f(x, s, t, \xi_1, \xi_2) := \sigma_1(x)\, e^{\frac{1}{st}} \left(|\xi_1|^{\frac{p-1}{2}} + |\xi_2|^{\frac{p-1}{2}} \right),$$
$$g(x, s, t, \xi_1, \xi_2) := \sigma_2(x) \left(\frac{s^q}{t^2} - |\xi_1|^{\frac{q-1}{4}} |\xi_2|^{\frac{q-1}{4}} \right),$$

where $\sigma_1, \sigma_2 \in L^\infty(\Omega)$. A simple computation shows that (F_1)–(G_1) are true provided $a_1 \equiv c_1 \equiv c_2 \equiv 0$, $b_1 \equiv e\sigma_1$, $a_2 \equiv \sigma_2$, $b_2 \equiv \frac{1}{2}\sigma_2$, $\gamma_1 = \delta_1 = \frac{p-1}{2}$, $\alpha_2 = q$, $\beta_2 = -2$, and $\gamma_2 = \delta_2 = \frac{q-1}{2}$.

Example 4.6. Set, for every $(x, s, t, \xi_1, \xi_2) \in \Omega \times (0, 1]^2 \times B_{\mathbb{R}^N}(\frac{1}{2})^2$,

$$f(x, s, t, \xi_1, \xi_2) := \theta_1(x) \left[\frac{t^{p+1}}{s} \sin\left(\frac{1}{s}\right) + t^{p^*}(e^s - 1) + e^{|\xi_1|^p + |\xi_2|^p} - 1 \right],$$
$$g(x, s, t, \xi_1, \xi_2) := \theta_2(x) e^{|\xi_1| + |\xi_2|} s^q \left(\log t + t^{q^*} \right),$$

where $\theta_1, \theta_2 \in L^\infty(\Omega)$. It is not difficult to see that (F_2)–(G_2) hold with $a_1 \equiv e\theta_1$, $b_1 \equiv 2\theta_1$, $a_2 \equiv 2e\theta_2$, $b_2 \equiv 0$, $\alpha_1 = p + 1$, $\beta_1 = -1$, $\gamma_1 = \delta_1 = p$, $\alpha_2 = q$, and $\beta_2 = -\frac{1}{2}$.

Acknowledgment

This work is performed within PTR 2018–2020 - Linea di intervento 2: 'Metodi Variazionali ed Equazioni Differenziali' of the University of Catania and partly funded by Research project of MIUR (Italian Ministry of Education, University and Research) Prin 2017 'Nonlinear Differential Problems via Variational, Topological and Set-valued Methods' (Grant Number 2017AYM8XW).

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