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Singular quasilinear elliptic systems in $\mathbb{R}^N$

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Abstract
The existence of positive weak solutions to a singular quasilinear elliptic system in the whole space is established via suitable a priori estimates and Schauder's fixed point theorem.

Keywords Singular elliptic system · p-Laplacian · Schauder's fixed point theorem · A priori estimate

Mathematics Subject Classification 35J75 · 35J48 · 35J92

1 Introduction

In this paper, we consider the following system of quasilinear elliptic equations:

$$\begin{cases} -\Delta_{p_1} u = a_1(x) f(u, v) & \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v = a_2(x) g(u, v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geq 3$, $1 < p_i < N$, while Δ_{p_i} denotes the p_i-Laplace differential operator. Nonlinearities $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and fulfill the condition

($\mathrm{H}_{f,g}$) There exist $m_i, M_i > 0$, $i = 1, 2$, such that

$$m_1 s^{\alpha_1} \leq f(s, t) \leq M_1 s^{\alpha_1}(1 + t^{\beta_1}),$$
$$m_2 t^{\beta_2} \leq g(s, t) \leq M_2(1 + s^{\alpha_2}) t^{\beta_2}$$

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for all $s, t \in \mathbb{R}^+$, with $-1 < \alpha_1, \beta_2 < 0 < \alpha_2, \beta_1$,

$$\alpha_1 + \alpha_2 < p_1 - 1, \quad \beta_1 + \beta_2 < p_2 - 1, \tag{1.1}$$

as well as

$$\beta_1 < \frac{p_2^*}{p_1^*} \min\{p_1 - 1, p_1^* - p_1\}, \quad \alpha_2 < \frac{p_1^*}{p_2^*} \min\{p_2 - 1, p_2^* - p_2\}.$$

Here, p_i^* denotes the critical Sobolev exponent corresponding to p_i, namely $p_i^* := \frac{Np_i}{N-p_i}$. Coefficients $a_i : \mathbb{R}^N \to \mathbb{R}$ satisfy the assumption

(H_a) $a_i(x) > 0$ a.e. in $\mathbb{R}^N$ and $a_i \in L^1(\mathbb{R}^N) \cap L^{\zeta_i}(\mathbb{R}^N)$, where

$$\frac{1}{\zeta_1} \le 1 - \frac{p_1}{p_1^*} - \frac{\beta_1}{p_2^*}, \quad \frac{1}{\zeta_2} \le 1 - \frac{p_2}{p_2^*} - \frac{\alpha_2}{p_1^*}.$$

Let $\mathcal{D}^{1,p_i}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|w\|_{\mathcal{D}^{1,p_i}(\mathbb{R}^N)} := \|\nabla w\|_{L^{p_i}(\mathbb{R}^N)}.$$

Recall [12, Theorem 8.3] that

$$\mathcal{D}^{1,p_i}(\mathbb{R}^N) = \{w \in L^{p_i^*}(\mathbb{R}^N) : |\nabla w| \in L^{p_i}(\mathbb{R}^N)\}.$$

Moreover, if $w \in \mathcal{D}^{1,p_i}(\mathbb{R}^N)$, then w vanishes at infinity, i.e., the set $\{x \in \mathbb{R}^N : w(x) > k\}$ has finite measure for all $k > 0$; see [12, p. 201].

A pair $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ is called a (weak) solution to (P) provided $u, v > 0$ a.e. in $\mathbb{R}^N$ and

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u|^{p_1-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} a_1 f(u, v) \varphi \, dx, \\ \int_{\mathbb{R}^N} |\nabla v|^{p_2-2} \nabla v \nabla \psi \, dx = \int_{\mathbb{R}^N} a_2 g(u, v) \psi \, dx \end{cases}$$

for every $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$.

The most interesting aspect of the work probably lies in the fact that both f and g can exhibit singularities through $\mathbb{R}^N$, which, without loss of generality, are located at zero. Indeed, $-1 < \alpha_1, \beta_2 < 0$ by ($\mathrm{H}_{f,g}$). It represents a serious difficulty to overcome and is rarely handled in the literature.

As far as we know, singular systems in the whole space have been investigated only for $p := q := 2$, essentially exploiting the linearity of involved differential operators. In such a context, [3,4,17] treat the so-called Gierer–Meinhardt system, which arises from the mathematical modeling of important biochemical processes. Nevertheless, even in the semilinear case, (P) cannot be reduced to Gierer–Meinhardt's case once ($\mathrm{H}_{f,g}$) is assumed. The situation looks quite different when a bounded domain takes the place of $\mathbb{R}^N$: many singular systems fitting the framework of (P) have been studied, and meaningful contributions are already available [1,6–11,13–16].

Here, variational methods do not work, at least in a direct way, because the Euler function associated with problem (P) is not well defined. A similar comment holds for sub-super-solution techniques, which are usually employed in the case of bounded domains. Hence, we were naturally led to apply fixed point results. An a priori estimate in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ for solutions of (P) is first established (cf. Theorem 3.4) by a Moser's type iteration procedure and an adequate truncation, which, due to singular terms, require a specific treatment. We next perturb (P) by introducing a parameter $\varepsilon > 0$. This produces the family of regularized systems

$$\begin{cases} -\Delta_{p_1} u = a_1(x) f(u+\varepsilon, v) & \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v = a_2(x) g(u, v+\varepsilon) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P_ε}$$

whose study yields useful information on the original problem. In fact, the previous L^∞-boundedness still holds for solutions to (P_ε), regardless of ε. Thus, via Schauder's fixed point theorem, we get a solution $(u_\varepsilon, v_\varepsilon)$ lying inside a rectangle given by positive lower bounds, where ε does not appear, and positive upper bounds, that may instead depend on ε. Finally, letting $\varepsilon \to 0^+$ and using the $(\mathrm{S})_+$-property of the negative p-Laplacian in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ (see Lemma 3.3) yield a weak solution to (P); cf. Theorem 5.1.

The rest of this paper is organized as follows: Section 2 deals with preliminary results. An a priori estimate of solutions to (P) is proven in Sect. 3, while the next one treats system (P_ε). Section 5 contains our existence result for problem (P).

2 Preliminaries

Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set, let $t \in \mathbb{R}$, and let $w, z \in L^p(\mathbb{R}^N)$. We write $m(\Omega)$ for the Lebesgue measure of Ω, while $t^\pm := \max\{\pm t, 0\}$, $\Omega(w \le t) := \{x \in \Omega : w(x) \le t\}$, $\|w\|_p := \|w\|_{L^p(\mathbb{R}^N)}$. The meaning of $\Omega(w > t)$, etc. is analogous. By definition, $w \le z$ iff $w(x) \le z(x)$ a.e. in $\mathbb{R}^N$.

Given $1 \le q < p$, neither $L^p(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ nor the reverse embedding holds true. However, the situation looks better for functions belonging to $L^1(\mathbb{R}^N)$. Indeed (see also [2, p. 93]),

Proposition 2.1 *Suppose $p > 1$ and $w \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Then $w \in L^q(\mathbb{R}^N)$ whatever $q \in]1, p[$.*

Proof Thanks to Hölder's inequality, with exponents p/q and $p/(p-q)$, and Chebyshev's inequality, one has

$$\begin{aligned} \|w\|_q^q &= \int_{\mathbb{R}^N(|w|\le 1)} |w|^q \mathrm{d}x + \int_{\mathbb{R}^N(|w|>1)} |w|^q \mathrm{d}x \\ &\le \int_{\mathbb{R}^N(|w|\le 1)} |w| \, \mathrm{d}x + \left(\int_{\mathbb{R}^N(|w|>1)} |w|^p \mathrm{d}x \right)^{q/p} [m(\mathbb{R}^N(|w| > 1))]^{1-q/p} \\ &\le \int_{\mathbb{R}^N} |w| \, \mathrm{d}x + \left(\int_{\mathbb{R}^N} |w|^p \mathrm{d}x \right)^{q/p} \left(\int_{\mathbb{R}^N} |w|^p \mathrm{d}x \right)^{1-q/p} \\ &= \|w\|_1 + \|w\|_p^p. \end{aligned}$$

This completes the proof. □

The summability properties of a_i collected below will be exploited throughout the paper.

Remark 2.1 Let assumption (H_a) be fulfilled. Then, for any $i = 1, 2$,

(j_1) $a_i \in L^{(p_i^*)'}(\mathbb{R}^N)$.

(j$_2$) $a_i \in L^{\gamma_i}(\mathbb{R}^N)$, where $\gamma_i := 1/(1-t_i)$, with

$$t_1 := \frac{\alpha_1+1}{p_1^*} + \frac{\beta_1}{p_2^*}, \quad t_2 := \frac{\alpha_2}{p_1^*} + \frac{\beta_2+1}{p_2^*}.$$

(j$_3$) $a_i \in L^{\delta_i}(\mathbb{R}^N)$, for $\delta_i := 1/(1-s_i)$ and

$$s_1 := \frac{\alpha_1+1}{p_1^*}, \quad s_2 := \frac{\beta_2+1}{p_2^*}.$$

(j$_4$) $a_i \in L^{\xi_i}(\mathbb{R}^N)$, where $\xi_i \in]p_i^*/(p_i^*-p_i), \zeta_i[$.

To verify (j$_1$)–(j$_4$), we simply note that $\zeta_i > \max\{(p_i^*)', \gamma_i, \delta_i, \xi_i\}$ and apply Proposition 2.1.

Let us next show that the operator $-\Delta_p$ is of type $(\mathrm{S})_+$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Proposition 2.2 *If* $1 < p < N$ *and* $\{u_n\} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$ *satisfies*

$$u_n \rightharpoonup u \quad in \quad \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{2.1}$$

$$\limsup_{n\to\infty} \left\langle -\Delta_p u_n, u_n - u \right\rangle \le 0, \tag{2.2}$$

then $u_n \to u$ *in* $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Proof By monotonicity, one has

$$\left\langle -\Delta_p u_n - (-\Delta_p u), u_n - u \right\rangle \ge 0 \quad \forall n \in \mathbb{N},$$

which evidently entails

$$\liminf_{n\to\infty} \left\langle -\Delta_p u_n - (-\Delta_p u), u_n - u \right\rangle \ge 0.$$

Via (2.1)–(2.2), we then get

$$\limsup_{n\to\infty} \left\langle -\Delta_p u_n - (-\Delta_p u), u_n - u \right\rangle \le 0.$$

Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right) (\nabla u_n - \nabla u)\, \mathrm{d}x = 0. \tag{2.3}$$

Since [18, Lemma A.0.5] yields

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right) (\nabla u_n - \nabla u)\, \mathrm{d}x$$
$$\ge \begin{cases} C_p \int_{\mathbb{R}^N} \frac{|\nabla(u_n-u)|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\, \mathrm{d}x & \text{if } 1 < p < 2, \\ C_p \int_{\mathbb{R}^N} |\nabla(u_n-u)|^p\, \mathrm{d}x & \text{otherwise} \end{cases} \quad \forall n \in \mathbb{N},$$

the desired conclusion, namely

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(u_n - u)|^p\, \mathrm{d}x = 0,$$

directly follows from (2.3) once $p \geq 2$. If $1 < p < 2$, then Hölder's inequality and (2.1) lead to

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla(u_n-u)|^p\,\mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|\nabla(u_n-u)|^p}{(|\nabla u_n|+|\nabla u|)^{\frac{p(2-p)}{2}}}(|\nabla u_n|+|\nabla u|)^{\frac{p(2-p)}{2}}\mathrm{d}x\\ &\leq \left(\int_{\mathbb{R}^N}\frac{|\nabla(u_n-u)|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\,\mathrm{d}x\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^N}(|\nabla u_n|+|\nabla u|)^p\mathrm{d}x\right)^{\frac{2-p}{2}}\\ &\leq C\left(\int_{\mathbb{R}^N}\frac{|\nabla(u_n-u)|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\,\mathrm{d}x\right)^{\frac{p}{2}}, \quad n\in\mathbb{N},\end{aligned}$$

with appropriate $C > 0$. Now, the argument goes on as before. □

3 Boundedness of solutions

The main result of this section, Theorem 3.4 below, provides an $L^\infty(\mathbb{R}^N)$—a priori estimate for weak solutions to (P). Its proof will be performed into three steps.

Lemma 3.1 ($L^{p_i^*}(\mathbb{R}^N)$—uniform boundedness) *There exists $\rho > 0$ such that*

$$\max\left\{\|u\|_{p_1^*},\ \|v\|_{p_2^*}\right\} \leq \rho \tag{3.1}$$

for every $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ solving problem (P).

Proof Multiply both equations in (P) by u and v, respectively, integrate over $\mathbb{R}^N$, and use $(\mathrm{H}_{f,g})$ to arrive at

$$\begin{aligned}\|\nabla u\|_{p_1}^{p_1} &= \int_{\mathbb{R}^N} a_1 f(u,v)u\,\mathrm{d}x \leq M_1\int_{\mathbb{R}^N} a_1 u^{\alpha_1+1}(1+v^{\beta_1})\,\mathrm{d}x,\\ \|\nabla v\|_{p_2}^{p_2} &= \int_{\mathbb{R}^N} a_2 g(u,v)v\,\mathrm{d}x \leq M_2\int_{\mathbb{R}^N} a_2(1+u^{\alpha_2})v^{\beta_2+1}\,\mathrm{d}x.\end{aligned}$$

Through the embedding $\mathcal{D}^{1,p_i}(\mathbb{R}^N) \hookrightarrow L^{p_i^*}(\mathbb{R}^N)$, besides Hölder's inequality, we obtain

$$\begin{aligned}\|\nabla u\|_{p_1}^{p_1} &\leq M_1\left(\|a_1\|_{\delta_1}\|u\|_{p_1^*}^{\alpha_1+1} + \|a_1\|_{\gamma_1}\|u\|_{p_1^*}^{\alpha_1+1}\|v\|_{p_2^*}^{\beta_1}\right)\\ &\leq C_1\|\nabla u\|_{p_1}^{\alpha_1+1}\left(\|a_1\|_{\delta_1} + \|a_1\|_{\gamma_1}\|\nabla v\|_{p_2}^{\beta_1}\right);\end{aligned}$$

cf. also Remark 2.1. Likewise,

$$\|\nabla v\|_{p_2}^{p_2} \leq C_2\|\nabla v\|_{p_2}^{\beta_2+1}\left(\|a_2\|_{\delta_2} + \|a_2\|_{\gamma_2}\|\nabla u\|_{p_1}^{\alpha_2}\right).$$

Thus, a fortiori,

$$\begin{aligned}\|\nabla u\|_{p_1}^{p_1-1-\alpha_1} &\leq C_1\left(\|a_1\|_{\delta_1} + \|a_1\|_{\gamma_1}\|\nabla v\|_{p_2}^{\beta_1}\right),\\ \|\nabla v\|_{p_2}^{p_2-1-\beta_2} &\leq C_2\left(\|a_2\|_{\delta_2} + \|a_2\|_{\gamma_2}\|\nabla u\|_{p_1}^{\alpha_2}\right),\end{aligned} \tag{3.2}$$

which imply

$$\begin{aligned}&\|\nabla u\|_{p_1}^{p_1-1-\alpha_1} + \|\nabla v\|_{p_2}^{p_2-1-\beta_2}\\ &\quad\leq C_1\left(\|a_1\|_{\delta_1} + \|a_1\|_{\gamma_1}\|\nabla v\|_{p_2}^{\beta_1}\right) + C_2\left(\|a_2\|_{\delta_2} + \|a_2\|_{\gamma_2}\|\nabla u\|_{p_1}^{\alpha_2}\right).\end{aligned}$$

Rewriting this inequality as

$$\|\nabla u\|_{p_1}^{\alpha_2}\left(\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}-C_2\|a_2\|_{\gamma_2}\right)+\|\nabla v\|_{p_2}^{\beta_1}\left(\|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}-C_1\|a_1\|_{\gamma_1}\right)$$
$$\le C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2}, \tag{3.3}$$

four situations may occur. If

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}\le C_2\|a_2\|_{\gamma_2}, \quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}\le C_1\|a_1\|_{\gamma_1}$$

then (3.1) follows from (j_2) of Remark 2.1, conditions (1.1), and the embedding $\mathcal{D}^{1,p_i}(\mathbb{R}^N)\hookrightarrow L^{p_i^*}(\mathbb{R}^N)$. Assume next that

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}> C_2\|a_2\|_{\gamma_2}, \quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}> C_1\|a_1\|_{\gamma_1}. \tag{3.4}$$

Thanks to (3.3), one has

$$\|\nabla u\|_{p_1}^{\alpha_2}(\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}-C_2\|a_2\|_{\gamma_2})\le C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2},$$

whence, on account of (3.4),

$$\begin{aligned}\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}&\le\frac{C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2}}{\|\nabla u\|_{p_1}^{\alpha_2}}+C_2\|a_2\|_{\gamma_2}\\&\le\frac{C_1\|a_1\|_{\delta_1}+C_2\|a_2\|_{\delta_2}}{\|a_2\|_{\gamma_2}^{\frac{\alpha_2}{p_1-1-\alpha_1-\alpha_2}}}+C_2\|a_2\|_{\gamma_2}.\end{aligned}$$

A similar inequality holds true for v. So, (3.1) is achieved reasoning as before. Finally, if

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}\le C_2\|a_2\|_{\gamma_2}, \quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}> C_1\|a_1\|_{\gamma_1} \tag{3.5}$$

then (3.2) and (3.5) entail

$$\|\nabla v\|_{p_2}^{p_2-1-\beta_2}\le C_2\left[\|a_2\|_{\delta_2}+\|a_2\|_{\gamma_2}\left(C_2\|a_2\|_{\gamma_2}\right)^{\frac{\alpha_2}{p_1-1-\alpha_1-\alpha_2}}\right].$$

By (1.1) again, we thus get

$$\max\{\|\nabla u\|_{p_1},\|\nabla v\|_{p_2}\}\le C_3,$$

where $C_3>0$. This yields (3.1), because $\mathcal{D}^{1,p_i}(\mathbb{R}^N)\hookrightarrow L^{p_i^*}(\mathbb{R}^N)$. The last case, i.e.,

$$\|\nabla u\|_{p_1}^{p_1-1-\alpha_1-\alpha_2}> C_2\|a_2\|_{\gamma_2}, \quad \|\nabla v\|_{p_2}^{p_2-1-\beta_1-\beta_2}\le C_1\|a_1\|_{\gamma_1}$$

is analogous. □

To shorten notation, write

$$\mathcal{D}^{1,p_i}(\mathbb{R}^N)_+:=\{w\in\mathcal{D}^{1,p_i}(\mathbb{R}^N): w\ge 0 \text{ a.e. in } \mathbb{R}^N\}.$$

Lemma 3.2 (Truncation) *Let $(u,v)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)$ be a weak solution of (P). Then*

$$\int_{\mathbb{R}^N(u>1)}|\nabla u|^{p_1-2}\nabla u\nabla\varphi\,\mathrm{d}x\le M_1\int_{\mathbb{R}^N(u>1)}a_1(1+v^{\beta_1})\varphi\,\mathrm{d}x, \tag{3.6}$$

$$\int_{\mathbb{R}^N(v>1)}|\nabla v|^{p_2-2}\nabla v\nabla\psi\,\mathrm{d}x\le M_2\int_{\mathbb{R}^N(v>1)}a_2(1+u^{\alpha_2})\psi\,\mathrm{d}x \tag{3.7}$$

for all $(\varphi,\psi)\in\mathcal{D}^{1,p_1}(\mathbb{R}^N)_+\times\mathcal{D}^{1,p_2}(\mathbb{R}^N)_+$.

Proof Pick a C^1 cutoff function $\eta : \mathbb{R} \to [0, 1]$ such that

$$\eta(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t \ge 1, \end{cases} \quad \eta'(t) \ge 0 \quad \forall\, t \in [0, 1],$$

and, given $\delta > 0$, define $\eta_\delta(t) := \eta\left(\frac{t-1}{\delta}\right)$. If $w \in \mathcal{D}^{1,p_i}(\mathbb{R}^N)$, then

$$\eta_\delta \circ w \in \mathcal{D}^{1,p_i}(\mathbb{R}^N), \quad \nabla(\eta_\delta \circ w) = (\eta'_\delta \circ w)\nabla w, \tag{3.8}$$

as a standard verification shows.

Now, fix $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N)_+ \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)_+$. Multiply the first equation in (P) by $(\eta_\delta \circ u)\varphi$, integrate over $\mathbb{R}^N$ and use $(\mathrm{H}_{f,g})$ to achieve

$$\int_{\mathbb{R}^N} |\nabla u|^{p_1-2}\nabla u \nabla((\eta_\delta \circ u)\varphi)\,\mathrm{d}x \le M_1 \int_{\mathbb{R}^N} a_1 u^{\alpha_1}(1 + v^{\beta_1})(\eta_\delta \circ u)\varphi\,\mathrm{d}x.$$

By (3.8), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^{p_1-2}\nabla u \nabla((\eta_\delta \circ u)\varphi)\,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} |\nabla u|^{p_1}(\eta'_\delta \circ u)\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} (\eta_\delta \circ u)|\nabla u|^{p_1-2}\nabla u \nabla\varphi\,\mathrm{d}x, \end{aligned}$$

while $\eta'_\delta \circ u \ge 0$ in $\mathbb{R}^N$. Therefore,

$$\int_{\mathbb{R}^N} (\eta_\delta \circ u)|\nabla u|^{p_1-2}\nabla u \nabla\varphi\,\mathrm{d}x \le M_1 \int_{\mathbb{R}^N} a_1 u^{\alpha_1}(1 + v^{\beta_1})(\eta_\delta \circ u)\varphi\,\mathrm{d}x.$$

Letting $\delta \to 0^+$ produces (3.6). The proof of (3.7) is similar. □

Lemma 3.3 (Moser's iteration) *There exists $R > 0$ such that*

$$\max\{\|u\|_{L^\infty(\Omega_1)}, \|v\|_{L^\infty(\Omega_2)}\} \le R, \tag{3.9}$$

where

$$\Omega_1 := \mathbb{R}^N(u > 1) \quad \textit{and} \quad \Omega_2 := \mathbb{R}^N(v > 1),$$

for every $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ solving problem (P).

Proof Given $w \in L^p(\Omega_1)$, we shall write $\|w\|_p$ in place of $\|w\|_{L^p(\Omega_1)}$ when no confusion can arise. Observe that $m(\Omega_1) < +\infty$ and define, provided $M > 1$,

$$u_M(x) := \min\{u(x), M\}, \quad x \in \mathbb{R}^N.$$

Choosing $\varphi := u_M^{\kappa p_1+1}$, with $\kappa \ge 0$, in (3.6) gives

$$\begin{aligned} &(\kappa p_1 + 1) \int_{\Omega_1(u \le M)} u_M^{\kappa p_1} |\nabla u|^{p_1-2}\nabla u \nabla u_M\,\mathrm{d}x \\ &\quad \le M_1 \int_{\Omega_1} a_1(1 + v^{\beta_1}) u_M^{\kappa p_1+1}\,\mathrm{d}x. \end{aligned} \tag{3.10}$$

Through the Sobolev embedding theorem, one has

$$
\begin{aligned}
&(\kappa p_1+1)\int_{\Omega_1(u\le M)} u_M^{\kappa p_1}|\nabla u|^{p_1-2}\nabla u\nabla u_M\,\mathrm{d}x\\
&\quad=(\kappa p_1+1)\int_{\Omega_1(u\le M)}(|\nabla u|u^{\kappa})^{p_1}\mathrm{d}x=\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\int_{\Omega_1(u\le M)}|\nabla u^{\kappa+1}|^{p_1}\mathrm{d}x\\
&\quad=\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\int_{\Omega_1}|\nabla u_M^{\kappa+1}|^{p_1}\mathrm{d}x\ge C_1\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\|u_M^{\kappa+1}\|_{p_1^*}^{p_1}
\end{aligned}
$$

for appropriate $C_1>0$. By Remark 2.1, Hölder's inequality entails

$$
\begin{aligned}
\int_{\Omega_1} a_1(1+v^{\beta_1})u_M^{\kappa p_1+1}\mathrm{d}x &\le \int_{\Omega_1} a_1(1+v^{\beta_1})u^{\kappa p_1+1}\mathrm{d}x\\
&\le \left(\|a_1\|_{\xi_1}+\|a_1\|_{\zeta_1}\|v\|_{p_2^*}^{\beta_1}\right)\|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1+1}.
\end{aligned}
$$

Hence, (3.10) becomes

$$
\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\|u_M^{\kappa+1}\|_{p_1^*}^{p_1}\le C_2\left(\|a_1\|_{\xi_1}+\|a_1\|_{\zeta_1}\|v\|_{p_2^*}^{\beta_1}\right)\|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1+1}.
$$

Since $u(x)=\lim_{M\to\infty}u_M(x)$ a.e. in $\mathbb{R}^N$, using the Fatou lemma we get

$$
\frac{\kappa p_1+1}{(\kappa+1)^{p_1}}\|u\|_{(\kappa+1)p_1^*}^{(\kappa+1)p_1}\le C_2\left(\|a_1\|_{\xi_1}+\|a_1\|_{\zeta_1}\|v\|_{p_2^*}^{\beta_1}\right)\|u\|_{(\kappa p_1+1)\xi_1'}^{\kappa p_1+1},
$$

namely

$$
\|u\|_{(\kappa+1)p_1^*}\le C_3^{\eta(\kappa)}\sigma(\kappa)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa)}\|u\|_{(\kappa p_1+1)\xi_1'}^{\frac{\kappa p_1+1}{(\kappa+1)p_1}}, \tag{3.11}
$$

where $C_3>0$, while

$$
\eta(\kappa):=\frac{1}{(\kappa+1)p_1},\quad \sigma(\kappa):=\left[\frac{\kappa+1}{(\kappa p_1+1)^{1/p_1}}\right]^{\frac{1}{\kappa+1}}.
$$

Let us next verify that

$$
(\kappa+1)p_1^*>(\kappa p_1+1)\xi_1'\quad\forall\kappa\in\mathbb{R}_0^+,
$$

which clearly means

$$
\frac{1}{\xi_1}<1-\frac{\kappa p_1+1}{(\kappa+1)p_1^*},\quad \kappa\in\mathbb{R}_0^+. \tag{3.12}
$$

Indeed, the function $\kappa\mapsto\frac{\kappa p_1+1}{(\kappa+1)p_1^*}$ is increasing on $\mathbb{R}_0^+$ and tends to $\frac{p_1}{p_1^*}$ as $k\to\infty$. So, (3.12) holds true, because $\frac{1}{\xi_1}<1-\frac{p_1}{p_1^*}$; see Remark 2.1. Now, Moser's iteration can start. If there exists a sequence $\{\kappa_n\}\subseteq\mathbb{R}_0^+$ fulfilling

$$
\lim_{n\to\infty}\kappa_n=+\infty,\quad \|u\|_{(\kappa_n+1)p_1^*}\le 1\ \ \forall n\in\mathbb{N}
$$

then $\|u\|_{L^\infty(\Omega_1)}\le 1$. Otherwise, with appropriate $\kappa_0>0$, one has

$$
\|u\|_{(\kappa+1)p_1^*}>1\ \text{ for any }\ \kappa>\kappa_0,\quad\text{besides}\ \ \|u\|_{(\kappa_0+1)p_1^*}\le 1. \tag{3.13}
$$

Inequality (3.12) evidently forces $\frac{\kappa_0 p_1+1}{(\kappa_0+1)p_1^*} < \frac{1}{\xi_1'}$. Pick $\kappa_1 > \kappa_0$ such that $(\kappa_1 p_1 + 1)\xi_1' = (\kappa_0 + 1)p_1^*$, set $\kappa := \kappa_1$ in (3.11), and use (3.13) to arrive at

$$\begin{aligned}\|u\|_{(\kappa_1+1)p_1^*} &\le C_3^{\eta(\kappa_1)}\sigma(\kappa_1)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_1)}\|u\|_{(\kappa_0+1)p_1^*}^{\frac{\kappa_1 p_1+1}{(\kappa_1+1)p_1}} \\ &\le C_3^{\eta(\kappa_1)}\sigma(\kappa_1)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_1)}. \end{aligned} \tag{3.14}$$

Choose next $\kappa_2 > \kappa_0$ satisfying $(\kappa_2 p_1+1)\xi_1' = (\kappa_1+1)p_1^*$. From (3.11), written for $\kappa := \kappa_2$, as well as (3.13)–(3.14), it follows

$$\begin{aligned}\|u\|_{(\kappa_2+1)p_1^*} &\le C_3^{\eta(\kappa_2)}\sigma(\kappa_2)\left(1+\|v\|_{p_2^*}^{\beta_1}\right))^{\eta(\kappa_2)}\|u\|_{(\kappa_1+1)p_1^*}^{\frac{\kappa_2 p_1+1}{(\kappa_2+1)p_1}} \\ &\le C_3^{\eta(\kappa_2)}\sigma(\kappa_2)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_2)}\|u\|_{(\kappa_1+1)p_1^*} \\ &\le C_3^{\eta(\kappa_2)+\eta(\kappa_1)}\sigma(\kappa_2)\sigma(\kappa_1)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\eta(\kappa_2)+\eta(\kappa_1)}. \end{aligned}$$

By induction, we construct a sequence $\{\kappa_n\} \subseteq (\kappa_0, +\infty)$ enjoying the properties below:

$$(\kappa_n p_1 + 1)\xi_1' = (\kappa_{n-1}+1)p_1^*, \quad n \in \mathbb{N}; \tag{3.15}$$

$$\|u\|_{(k_n+1)p_1^*} \le C_3^{\sum_{i=1}^n \eta(\kappa_i)}\prod_{i=1}^n \sigma(\kappa_i)\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\sum_{i=1}^n \eta(\kappa_i)} \tag{3.16}$$

for all $n \in \mathbb{N}$. A simple computation based on (3.15) yields

$$\kappa_n + 1 = (\kappa_0+1)\left(\frac{p_1^*}{p_1\xi_1'}\right)^n + \frac{1}{p_1'}\sum_{i=0}^{n-1}\left(\frac{p_1^*}{p_1\xi_1'}\right)^i,$$

where $\frac{p_1^*}{p_1\xi_1'} > 1$ due to (j_4) of Remark 2.1. Hence,

$$\kappa_n + 1 \simeq C^*\left(\frac{p_1^*}{p_1\xi_1'}\right)^n \quad \text{as } n \to \infty, \tag{3.17}$$

with appropriate $C^* > 0$. Further, if $C_4 > 0$ satisfies

$$1 < \left[\frac{t+1}{(tp_1+1)^{1/p_1}}\right]^{\frac{1}{\sqrt{t+1}}} \le C_4, \quad t \in \mathbb{R}_0^+,$$

(cf. [5, p. 116]), then

$$\begin{aligned}\prod_{i=1}^n \sigma(\kappa_i) &= \prod_{i=1}^n \left[\frac{\kappa_i+1}{(\kappa_i p_1+1)^{1/p_1}}\right]^{\frac{1}{\kappa_i+1}} \\ &= \prod_{i=1}^n \left\{\left[\frac{\kappa_i+1}{(\kappa_i p_1+1)^{1/p_1}}\right]^{\frac{1}{\sqrt{\kappa_i+1}}}\right\}^{\frac{1}{\sqrt{\kappa_i+1}}} \le C_4^{\sum_{i=1}^n \frac{1}{\sqrt{\kappa_i+1}}}. \end{aligned}$$

Consequently, (3.16) becomes

$$\|u\|_{(k_n+1)p_1^*} \le C_3^{\sum_{i=1}^n \eta(\kappa_i)}C_4^{\sum_{i=1}^n \frac{1}{\sqrt{\kappa_i+1}}}\left(1+\|v\|_{p_2^*}^{\beta_1}\right)^{\sum_{i=1}^n \eta(\kappa_i)}.$$

Since, by (3.17), both $\kappa_n+1 \to +\infty$ and $\frac{1}{\kappa_n+1} \simeq \frac{1}{C^*}\left(\frac{p_1\xi_1'}{p_1^*}\right)^n$, while (3.1) entails $\|v\|_{p_2^*} \le \rho$, there exists a constant $C_5 > 0$ such that

$$\|u\|_{(\kappa_n+1)p_1^*} \le C_5 \quad \forall n \in \mathbb{N},$$

whence $\|u\|_{L^\infty(\Omega_1)} \le C_5$. Thus, in either case, $\|u\|_{L^\infty(\Omega_1)} \le R$, with $R := \max\{1, C_5\}$. A similar argument applies to v. □

Using (3.9), besides the definition of sets Ω_i, we immediately infer the following

Theorem 3.4 *Under assumptions* $(\mathrm{H}_{f,g})$ *and* (H_a), *one has*

$$\max\{\|u\|_\infty, \|v\|_\infty\} \le R \tag{3.18}$$

for every weak solution $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ *to problem* (P). *Here, R is given by Lemma* 3.3.

4 The regularized system

Assertion (j_1) of Remark 2.1 ensures that $a_i \in L^{(p_i^*)'}(\mathbb{R}^N)$. Therefore, thanks to Minty–Browder's theorem [2, Theorem V.16], the equation

$$-\Delta_{p_i} w_i = a_i(x) \quad \text{in} \quad \mathbb{R}^N \tag{4.1}$$

possesses a unique solution $w_i \in \mathcal{D}^{1,p_i}(\mathbb{R}^N)$, $i = 1, 2$. Moreover,

- $w_i > 0$, and
- $w_i \in L^\infty(\mathbb{R}^N)$.

Indeed, testing (4.1) with $\varphi := w_i^-$ yields $w_i \ge 0$, because $a_i > 0$ by (H_a). Through the strong maximum principle, we obtain

$$\operatorname{ess\,inf}_{B_r(x)} w_i > 0 \quad \text{for any } r > 0,\ x \in \mathbb{R}^N.$$

Hence, $w_i > 0$. Moser's iteration technique then produces $w_i \in L^\infty(\mathbb{R}^N)$.

Next, fix $\varepsilon \in]0, 1[$ and define

$$\begin{aligned}
(\underline{u}, \underline{v}) &= \left([m_1(R+1)^{\alpha_1}]^{\frac{1}{p_1-1}} w_1, [m_2(R+1)^{\beta_2}]^{\frac{1}{p_2-1}} w_2\right),\\
(\overline{u}_\varepsilon, \overline{v}_\varepsilon) &= \left([M_1\varepsilon^{\alpha_1}(1+R^{\beta_1})]^{\frac{1}{p_1-1}} w_1, [M_2\varepsilon^{\beta_2}(1+R^{\alpha_2})]^{\frac{1}{p_2-1}} w_2\right),
\end{aligned} \tag{4.2}$$

where $R > 0$ comes from Lemma 3.3, as well as

$$\mathcal{K}_\varepsilon := \left\{(z_1, z_2) \in L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N) : \underline{u} \le z_1 \le \overline{u}_\varepsilon\,,\ \underline{v} \le z_2 \le \overline{v}_\varepsilon\right\}.$$

Obviously, $\mathcal{K}_\varepsilon$ is bounded, convex, closed in $L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N)$. Given $(z_1, z_2) \in \mathcal{K}_\varepsilon$, write

$$\tilde{z}_i := \min\{z_i, R\}, \quad i = 1, 2. \tag{4.3}$$

Since, on account of (4.3), hypothesis $(\mathrm{H}_{f,g})$ entails

$$\begin{aligned}
a_1 m_1(R+1)^{\alpha_1} &\le a_1 f(\tilde{z}_1+\varepsilon, \tilde{z}_2) \le a_1 M_1 \varepsilon^{\alpha_1}(1+R^{\beta_1}),\\
a_2 m_2(R+1)^{\beta_2} &\le a_2 g(\tilde{z}_1, \tilde{z}_2+\varepsilon) \le a_2 M_2(1+R^{\alpha_2})\varepsilon^{\beta_2},
\end{aligned} \tag{4.4}$$

while, recalling Remark 2.1, $a_i \in L^{(p_i^*)'}(\mathbb{R}^N)$, the functions

$$x \mapsto a_1(x) f(\tilde{z}_1(x) + \varepsilon, \tilde{z}_2(x)), \quad x \mapsto a_2(x) g(\tilde{z}_1(x), \tilde{z}_2(x) + \varepsilon)$$

belong to $\mathcal{D}^{-1,p_1'}(\mathbb{R}^N)$ and $\mathcal{D}^{-1,p_2'}(\mathbb{R}^N)$, respectively. Consequently, by Minty–Browder's theorem again, there exists a unique weak solution $(u_\varepsilon, v_\varepsilon)$ of the problem

$$\begin{cases} -\Delta_{p_1} u = a_1(x) f(\tilde{z}_1(x) + \varepsilon, \tilde{z}_2(x)) & \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v = a_2(x) g(\tilde{z}_1(x), \tilde{z}_2(x) + \varepsilon) & \text{in } \mathbb{R}^N, \\ u_\varepsilon, v_\varepsilon > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.5}$$

Let $\mathcal{T} : \mathcal{K}_\varepsilon \to L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N)$ be defined by $\mathcal{T}(z_1, z_2) = (u_\varepsilon, v_\varepsilon)$ for every $(z_1, z_2) \in \mathcal{K}_\varepsilon$.

Lemma 4.1 *One has $\underline{u} \le u_\varepsilon \le \overline{u}_\varepsilon$ and $\underline{v} \le v_\varepsilon \le \overline{v}_\varepsilon$. So, in particular, $\mathcal{T}(\mathcal{K}_\varepsilon) \subseteq \mathcal{K}_\varepsilon$.*

Proof Via (4.2), (4.1), (4.5), and (4.4), we get

$$\begin{aligned} &\langle -\Delta_{p_1} \underline{u} - (-\Delta_{p_1} u_\varepsilon), (\underline{u} - u_\varepsilon)^+ \rangle \\ &= \langle -\Delta_{p_1} [m_1 (R+1)^{\alpha_1}]^{\frac{1}{p_1 - 1}} w_1 - (-\Delta_{p_1} u_\varepsilon), (\underline{u} - u_\varepsilon)^+ \rangle \\ &= \int_{\mathbb{R}^N} a_1 \left((m_1 (R+1)^{\alpha_1} - f(\tilde{z}_1 + \varepsilon, \tilde{z}_2) \right) (\underline{u} - u_\varepsilon)^+ \mathrm{d}x \le 0, \end{aligned}$$

while Lemma A.0.5 of [18] furnishes

$$\begin{aligned} &\langle -\Delta_{p_1} \underline{u} - (-\Delta_{p_1} u_\varepsilon), (\underline{u} - u_\varepsilon)^+ \rangle \\ &= \int_{\mathbb{R}^N} \left(|\nabla \underline{u}|^{p_1 - 2} \nabla \underline{u} - |\nabla u_\varepsilon|^{p_1 - 2} \nabla u_\varepsilon \right) \nabla (\underline{u} - u_\varepsilon)^+ \mathrm{d}x \ge 0. \end{aligned}$$

Now, arguing as in the proof of Proposition 2.2, one has $(\underline{u} - u_\varepsilon)^+ = 0$, i.e., $\underline{u} \le u_\varepsilon$. The remaining inequalities can be verified similarly. □

Lemma 4.2 *The operator $\mathcal{T}$ is continuous and compact.*

Proof Pick a sequence $\{(z_{1,n}, z_{2,n})\} \subseteq \mathcal{K}_\varepsilon$ such that

$$(z_{1,n}, z_{2,n}) \to (z_1, z_2) \quad \text{in} \quad L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N).$$

If $(u_n, v_n) := \mathcal{T}(z_{1,n}, z_{2,n})$ and $(u, v) := \mathcal{T}(z_1, z_2)$, then

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p_1 - 2} \nabla u_n \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} a_1 f(\tilde{z}_{1,n} + \varepsilon, \tilde{z}_{2,n}) \varphi \, \mathrm{d}x, \tag{4.6}$$

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p_2 - 2} \nabla v_n \nabla \psi \, \mathrm{d}x = \int_{\mathbb{R}^N} a_2 g(\tilde{z}_{1,n}, \tilde{z}_{2,n} + \varepsilon) \psi \, \mathrm{d}x, \tag{4.7}$$

$$\int_{\mathbb{R}^N} |\nabla u|^{p_1 - 2} \nabla u \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} a_1 f(\tilde{z}_1 + \varepsilon, \tilde{z}_2) \varphi \, \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} |\nabla v|^{p_2 - 2} \nabla v \nabla \psi \, \mathrm{d}x = \int_{\mathbb{R}^N} a_2 g(\tilde{z}_1, \tilde{z}_2 + \varepsilon) \psi \, \mathrm{d}x$$

for every $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$. Set $\varphi := u_n$ in (4.6). From (4.4), it follows after using Hölder's inequality,

$$\begin{aligned} \|\nabla u_n\|_{p_1}^{p_1} &= \int_{\mathbb{R}^N} a_1 f(\tilde{z}_{1,n} + \varepsilon, \tilde{z}_{2,n}) u_n \,\mathrm{d}x \\ &\le M_1 \int_{\mathbb{R}^N} a_1 \varepsilon^{\alpha_1} (1 + R^{\beta_1}) u_n \,\mathrm{d}x \le C_\varepsilon \int_{\mathbb{R}^N} a_1 u_n \,\mathrm{d}x \\ &\le C_\varepsilon \|a_1\|_{(p_1^*)'} \|u_n\|_{p_1^*} \le C_\varepsilon \|a_1\|_{(p_1^*)'} \|\nabla u_n\|_{p_1} \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $C_\varepsilon := M_1 \varepsilon^{\alpha_1}(1 + R^{\beta_1})$. This actually means that $\{u_n\}$ is bounded in $\mathcal{D}^{1,p_1}(\mathbb{R}^N)$, because $p_1 > 1$. By (4.7), an analogous conclusion holds for $\{v_n\}$. Along subsequences if necessary, we may thus assume

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N). \tag{4.8}$$

So, $\{(u_n, v_n)\}$ converges strongly in $L^{q_1}(B_{r_1}) \times L^{q_2}(B_{r_2})$ for any $r_i > 0$ and any $1 \le q_i \le p_i^*$, whence, up to subsequences again,

$$(u_n, v_n) \to (u, v) \text{ a.e. in } \mathbb{R}^N. \tag{4.9}$$

Now, combining Lemma 4.1 with Lebesgue's dominated convergence theorem, we obtain

$$(u_n, v_n) \to (u, v) \text{ in } L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N), \tag{4.10}$$

as desired. Let us finally verify that $\mathcal{T}(\mathcal{K}_\varepsilon)$ is relatively compact. If $(u_n, v_n) := \mathcal{T}(z_{1,n}, z_{2,n})$, $n \in \mathbb{N}$, then (4.6)–(4.7) can be written. Hence, the previous argument yields a pair $(u, v) \in L^{p_1^*}(\mathbb{R}^N) \times L^{p_2^*}(\mathbb{R}^N)$ fulfilling (4.10), possibly along a subsequence. This completes the proof. □

Thanks to Lemmas 4.1–4.2, Schauder's fixed point theorem applies, and there exists $(u_\varepsilon, v_\varepsilon) \in \mathcal{K}_\varepsilon$ such that $(u_\varepsilon, v_\varepsilon) = \mathcal{T}(u_\varepsilon, v_\varepsilon)$. Through Theorem 3.4, we next arrive at

Theorem 4.3 *Under hypotheses* $(\mathrm{H}_{f,g})$ *and* (H_a), *for every* $\varepsilon > 0$ *small, problem* (P_ε) *admits a solution* $(u_\varepsilon, v_\varepsilon) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ *complying with* (3.18).

5 Existence of solutions

We are now ready to establish the main result of this paper.

Theorem 5.1 *Let* $(\mathrm{H}_{f,g})$ *and* (H_a) *be satisfied. Then,* (P) *has a weak solution* $(u, v) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$, *which is essentially bounded.*

Proof Pick $\varepsilon := \frac{1}{n}$, with $n \in \mathbb{N}$ big enough. Theorem 4.3 gives a pair (u_n, v_n), where $u_n := u_{\frac{1}{n}}$ and $v_n := v_{\frac{1}{n}}$, such that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^{p_1-2} \nabla u_n \nabla \varphi \,\mathrm{d}x &= \int_{\mathbb{R}^N} a_1 f\left(u_n + \frac{1}{n}, v_n\right) \varphi \,\mathrm{d}x, \\ \int_{\mathbb{R}^N} |\nabla v_n|^{p_2-2} \nabla v_n \nabla \psi \,\mathrm{d}x &= \int_{\mathbb{R}^N} a_2 g\left(u_n, v_n + \frac{1}{n}\right) \psi \,\mathrm{d}x \end{aligned} \tag{5.1}$$

for every $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$, as well as (cf. Lemma 4.1)

$$0 < \underline{u} \le u_n \le R, \quad 0 < \underline{v} \le v_n \le R. \tag{5.2}$$

Thanks to ($H_{f,g}$), (5.2), and (H_a), choosing $\varphi := u_n$, $\psi := v_n$ in (5.1) easily entails

$$\|\nabla u_n\|_{p_1}^{p_1} \le M_1 \int_{\mathbb{R}^N} a_1 u_n^{\alpha_1+1}(1+v_n^{\beta_1})\mathrm{d}x \le M_1 R^{\alpha_1+1}(1+R^{\beta_1})\|a_1\|_1\,,$$

$$\|\nabla v_n\|_{p_2}^{p_2} \le M_2 \int_{\mathbb{R}^N} a_2 (1+u_n^{\alpha_2})v_n^{\beta_2+1}\mathrm{d}x \le M_2 (1+R^{\alpha_2})R^{\beta_2+1}\|a_2\|_1,$$

whence both $\{u_n\} \subseteq \mathcal{D}^{1,p_1}(\mathbb{R}^N)$ and $\{v_n\} \subseteq \mathcal{D}^{1,p_2}(\mathbb{R}^N)$ are bounded. Along subsequences if necessary, we thus have (4.8)–(4.9). Let us next show that

$$(u_n, v_n) \to (u, v) \quad \text{strongly in} \quad \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N). \tag{5.3}$$

Testing the first equation in (5.1) with $\varphi := u_n - u$ yields

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p_1-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x = \int_{\mathbb{R}^N} a_1 f\left(u_n + \frac{1}{n}, v_n\right)(u_n - u)\mathrm{d}x. \tag{5.4}$$

The right-hand side of (5.4) goes to zero as $n \to \infty$. Indeed, by ($H_{f,g}$), (5.2), and (H_a) again,

$$\left|a_1 f\left(u_n + \frac{1}{n}, v_n\right)(u_n - u)\right| \le 2M_1 R^{\alpha_1+1}(1+R^{\beta_1})a_1 \quad \forall n \in \mathbb{N},$$

so that, recalling (4.9), Lebesgue's dominated convergence theorem applies. Through (5.4), we obtain $\lim_{n\to\infty} \langle -\Delta_{p_1} u_n, u_n - u\rangle = 0$. Likewise, $\langle -\Delta_{p_2} v_n, v_n - v\rangle \to 0$ as $n \to \infty$, and (5.3) directly follows from Proposition 2.2. On account of (5.1), besides (5.3), the final step is to verify that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a_1 f\left(u_n + \frac{1}{n}, v_n\right)\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N} a_1 f(u, v)\varphi\,\mathrm{d}x, \tag{5.5}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a_2 g\left(u_n, v_n + \frac{1}{n}\right)\psi\,\mathrm{d}x = \int_{\mathbb{R}^N} a_2 g(u, v)\psi\,\mathrm{d}x \tag{5.6}$$

for all $(\varphi, \psi) \in \mathcal{D}^{1,p_1}(\mathbb{R}^N) \times \mathcal{D}^{1,p_2}(\mathbb{R}^N)$. If $\varphi \in \mathcal{D}^{1,p_1}(\mathbb{R}^N)$, then ($j_1$) in Remark 2.1 gives $a_1\varphi \in L^1(\mathbb{R}^N)$. Since, as before,

$$\left|a_1 f\left(u_n + \frac{1}{n}, v_n\right)\varphi\right| \le M_1 R^{\alpha_1+1}(1+R^{\beta_1})a_1|\varphi|, \quad n \in \mathbb{N},$$

assertion (5.5) stems from Lebesgue's dominated convergence theorem. The proof of (5.6) is similar at all. □

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