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A regularity criterion of the 3D MHD equations involving one velocity and one current density component in Lorentz space

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Abstract. In this paper, we study the regularity criterion of weak solutions to the three-dimensional (3D) MHD equations. It is proved that the solution (u, b) becomes regular provided that one velocity and one current density component of the solution satisfy

$$u_3 \in L^{\frac{30\alpha}{7\alpha-45}}\left(0, T; L^{\alpha,\infty}\left(\mathbb{R}^3\right)\right) \quad \text{with } \frac{45}{7} \leq \alpha \leq \infty, \tag{0.1}$$

and

$$j_3 \in L^{\frac{2\beta}{2\beta-3}}\left(0, T; L^{\beta,\infty}\left(\mathbb{R}^3\right)\right) \quad \text{with } \frac{3}{2} \leq \beta \leq \infty, \tag{0.2}$$

which generalize some known results.

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1. Introduction

This paper deals with the well-known problem of the regularity of the solutions for the 3D magneto-hydrodynamical (MHD) system

$$\begin{cases} \partial_t u + u \cdot \nabla u - (b \cdot \nabla) b - \Delta u + \nabla \pi = 0, \\ \partial_t b + u \cdot \nabla b - b \cdot \nabla u - \Delta b = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \ b(x, 0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity field, $b = (b_1, b_2, b_3)$ is the magnetic field and π is the scalar pressure, while u_0 and b_0 are the corresponding initial data satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distribution.

Since Duvaut-Lions [9] and Sermange-Temam [25] constructed the so-called well-known weak solution $(u, b)(x, t)$ of the incompressible MHD equation for arbitrary $(u_0, b_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0(x) = \nabla \cdot b_0(x) = 0$ in last century, the problem on the uniqueness and regularity of the weak solutions is one of the most challenging problems of the mathematical community. Hence, many researchers have developed different regularity criteria for the 3D MHD equations under assumption of certain growth conditions on the velocity or on the magnetic field (see, e.g., [3–5,7,8,10,14–16,27–30,37,38] and the references therein).

Recent years, the problem of so-called regularity criteria via one components was investigated for the MHD equations by some researchers (see [1,11–13,17–21,23,32,33,36] and the references therein). In particular, in [31], Yamazaki established the following regularity criterion by involving one velocity and one current density component, which shows that a weak solution (u, b) is smooth on a time interval $(0, T]$ if

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p} + \frac{3}{q} \leq \frac{1}{3} + \frac{1}{2q}, \ \frac{15}{2} < q \leq \infty, \tag{1.2}$$

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and

$$j_3 \in L^{p'}(0,T;L^{q'}(\mathbb{R}^3)) \quad \text{with } \frac{2}{p'}+\frac{3}{q'} \le 2,\ \frac{3}{2} < q' \le \infty,$$

where j_3 is the third component of the current density $j = \nabla \times b = (j_1, j_2, j_3)$. Later, Zhang [34] improved the regularity criterion (1.2) to the following conditions

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} = \frac{4}{9}-\frac{1}{3q},\ \frac{15}{2} \le q \le \infty, \tag{1.3}$$

and

$$j_3 \in L^{p'}(0,T;L^{q'}(\mathbb{R}^3)) \quad \text{with } \frac{2}{p'}+\frac{3}{q'} \le 2,\ \frac{3}{2} < q' \le \infty.$$

Very recently, this result (1.3) is further refined by Zhang [35] to prove the regularity criterion as long as the following conditions

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} = \frac{4}{9},\ \frac{27}{4} \le q \le \infty, \tag{1.4}$$

are satisfied.

Motivated by the papers [31,34,35], the purpose of the present paper is to refine (1.4) and to extend the above regularity criterion to the Lorentz space $L^{\alpha,\infty}$ which is larger than L^{α} . More precisely, our main result now read as follows.

Theorem 1.1. *Suppose $T > 0$, $(u_0, b_0) \in L^2(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distributions. Assume that (u, b) is a weak solution of the 3D MHD equations (1.1) on $(0, T)$. If u_3 and j_3 satisfy the following growth conditions*

$$\int_0^T \left(\|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{30\alpha}{7\alpha-45}} + \|j_3(\tau)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} \right) \mathrm{d}\tau < \infty, \tag{1.5}$$

where $\frac{45}{7} \le \alpha \le \infty$ and $\frac{3}{2} < \beta \le \infty$, then the weak solution (u, b) is regular on $(0; T]$.

Remark 1.1. Theorem 1.1 extends the previous results on Navier–Stokes equations due to the fact that the MHD equations with $b(x,t) = 0$ reduce the Navier–Stokes equations. According to the embedding relation $L^{\alpha} \subseteq L^{\alpha,\infty}$, it is easy to see that our result of Theorem 1.1 is an improvement in the recent works by Yamazaki [31] and Zhang [34,35].

2. Preliminaries

Throughout this paper, we use the following usual notations. $L^p(\mathbb{R}^3)$ denotes the Lebesgue space associated with norm

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^3} |f(x)|^p \, dx \right)^{\frac{1}{p}}, & \text{for } 1 \le p < \infty, \\ ess \sup_{x \in \mathbb{R}^3} |f(x)|, & \text{for } p = \infty. \end{cases}$$

$H^k(\mathbb{R}^3)$ denotes the Hilbert space $\left\{ u \in L^2(\mathbb{R}^3) : \left\| \nabla^k u \right\|_{L^2} < \infty \right\}$. Let $(X, \mathcal{M}, \mu)$ be a non-atomic measurable space. For a complex- or real-valued $\mu-$measurable function $f(x)$ defined on X, its distributional function is defined by

$$f_*(\sigma) = \mu \left\{ x \in X : f(x) > \sigma \right\}, \quad \text{for } \sigma > 0,$$

which is non-increasing and continuous from the right. Furthermore, its non-increasing rearrangement f^* is defined by

$$f^*(t) = \inf \left\{ s > 0 : f_*(s) \le t \right\}, \quad \text{for } t > 0,$$

which is also non-increasing and continuous from the right and has the same distributional function as $f(x)$.

The Lorentz space $L^{p,q}$ on $(X, \mathcal{M}, \mu)$ is the collection of all real- or complex-valued $\mu-$measurable functions $f(x)$ defined on X such that $\|f\|_{L^{p,q}} < \infty$, where

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\frac{q}{p}\int\limits_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, & \text{if } 1 \le p < \infty,\ 1 < q < \infty \\ \sup\limits_{t>0}(t^{\frac{1}{p}} f^*(t)), & \text{if } 1 \le p < \infty,\ q = \infty. \end{cases}$$

Moreover,

$$\|f\|_{L^{p,\infty}} = \sup_{t>0}(t^{\frac{1}{p}} f^*(t)) = \sup_{\sigma>0}\sigma(f_*(\sigma))^{\frac{1}{p}}$$

for any $f \in L^{p,\infty}$. For details, we refer to [2] and [26].

The space definition implies the following continuous embeddings:

$$L^p(\mathbb{R}^3) = L^{p,p}(\mathbb{R}^3) \hookrightarrow L^{p,q}(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3), \quad 1 \le p \le q < \infty.$$

In order to prove Theorem 1.1, we need the following Hölder inequality in Lorentz spaces (see, e.g., O'Neil [24] and [22]).

Lemma 2.1. ([24], Theorems 3.4 and 3.5) *Let $f \in L^{p_2,q_2}(\mathbb{R}^3)$ and $g \in L^{p_3,q_3}(\mathbb{R}^3)$ with $1 \le p_2, p_3 \le \infty$, $1 \le q_2, q_3 \le \infty$. Then, $fg \in L^{p_1,q_1}(\mathbb{R}^3)$ with*

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}, \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$$

and the Hölder inequality of Lorentz spaces

$$\|fg\|_{L^{p_1,q_1}} \le C \|f\|_{L^{p_2,q_2}} \|g\|_{L^{p_3,q_3}},$$

holds true for a positive constant C.

We also recall Gagliardo–Nirenberg inequality in Lorentz spaces which plays an important role in the proofs of Theorem 1.1.

Lemma 2.2. *Let $f \in L^{p,q}(\mathbb{R}^3)$ with $1 \le p, q, p_4, q_4, p_5, q_5 \le \infty$. Then, the Gagliardo–Nirenberg inequality of Lorentz spaces*

$$\|f\|_{L^{p,q}} \le C \|f\|^{\theta}_{L^{p_4,q_4}} \|f\|^{1-\theta}_{L^{p_5,q_5}}$$

holds for a positive constant C and

$$\frac{1}{p} = \frac{\theta}{p_4} + \frac{1-\theta}{p_5}, \quad \frac{1}{q} = \frac{\theta}{q_4} + \frac{1-\theta}{q_5}, \quad \theta \in (0,1).$$

3. Proof of main result

In this section, under the assumptions of Theorem 1.1, we prove our main result. Before proving our result, we recall the following multiplicative Sobolev imbedding inequality in the whole space $\mathbb{R}^3$ (see, for example, [6]) :

$$\|f\|_{L^6} \le C \|\nabla_h f\|^{\frac{2}{3}}_{L^2} \|\partial_3 f\|^{\frac{1}{3}}_{L^2}, \tag{3.1}$$

where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient operator. We are now given the proof of our main theorem.

Proof. To prove our result, it suffices to show that for any fixed $T > T^*$, there holds

$$\sup_{0\le t\le T^*} \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \le C_T,$$

where T^*, which denotes the maximal existence time of a strong solution and C_T is an absolute constant which only depends on T, u_0 and b_0.

The method of our proof is the standard energy estimates as in [31]. We will base on two major parts. The first one establishes the bounds of $(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2)$, while the second gives the bounds of the H^1-norm of velocity u and magnetic field b in terms of the results of part one.

For this purpose, we multiply the first and second equations of (1.1) by $-\Delta_h u$ and $-\Delta_h b$, respectively, and integrate them over $\mathbb{R}^3$ with respect to the spatial variable. Then, integration by parts gives the following identity:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + \|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2\\
&\quad= \int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot\Delta_h u\mathrm{d}x - \int_{\mathbb{R}^3} (b\cdot\nabla)b\cdot\Delta_h u\mathrm{d}x\\
&\qquad+ \int_{\mathbb{R}^3} (u\cdot\nabla)b\cdot\Delta_h b\mathrm{d}x - \int_{\mathbb{R}^3} (b\cdot\nabla)u\cdot\Delta_h b\mathrm{d}x\\
&\quad= \mathrm{RHS},
\end{aligned}$$

where $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ is the horizontal Laplacian. For simplicity of exposition, we denote

$$\mathcal{L}^2(t) = \sup_{\tau\in[\Gamma,t]} (\|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h b(\tau)\|_{L^2}^2) + \int_\Gamma^t (\|\nabla\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla\nabla_h b(\tau)\|_{L^2}^2)\mathrm{d}\tau,$$

$$\mathcal{J}^2(t) = \sup_{\tau\in[\Gamma,t]} (\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2) + \int_\Gamma^t (\|\Delta u(\tau)\|_{L^2}^2 + \|\Delta b(\tau)\|_{L^2}^2)\mathrm{d}\tau,$$

for $t \in [\Gamma, T^*)$. We choose $\epsilon, \eta > 0$ to be precisely determined subsequently and then select $\Gamma < T^*$ sufficiently close to T^* such that for all $\Gamma \le t < T^*$,

$$\int_\Gamma^t (\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2)\mathrm{d}\tau \le \epsilon \ll 1 \text{ and } \int_\Gamma^t \|j_3(\tau)\|_{L^\beta}^{\frac{2\beta}{2\beta-3}}\,\mathrm{d}\tau \le \eta \ll 1. \tag{3.2}$$

Applying the divergence-free condition, $\nabla\cdot u = \nabla\cdot b = 0$, we find that RHS can be estimated as

$$\begin{aligned}
\mathrm{RHS} &\le \int_{\mathbb{R}^3} |u_3|\,|\nabla u|\,|\nabla\nabla_h u|\,\mathrm{d}x + \int_{\mathbb{R}^3} |u_3|\,|\nabla b|\,|\nabla\nabla_h b|\,\mathrm{d}x + \int_{\mathbb{R}^3} |b_3|\,|\nabla u|\,|\nabla\nabla_h b|\,\mathrm{d}x\\
&\quad+ \int_{\mathbb{R}^3} |b_3|\,|\nabla b|\,|\nabla\nabla_h u|\,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla_h u|\,|\nabla_h b|\,|j_3|\,\mathrm{d}x\\
&= L_1 + L_2 + L_3 + L_4 + L_5,
\end{aligned} \tag{3.3}$$

where the last inequality was proved in [31] (see, Proposition 3.1 in [31] for details).

With the use of Lemma 2.1, (3.1), and the Young inequality, we derive the estimate of the first term L_1 of (3.3) as follows :

$$\begin{aligned}
L_1 &\le C\,\|u_3\|_{L^{\alpha,\infty}}\,\|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2},2}}\,\|\nabla\nabla_h u\|_{L^2}\\
&\le C\,\|u_3\|_{L^{\alpha,\infty}}\,\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\,\|\nabla u\|_{L^6}^{\frac{3}{\alpha}}\,\|\nabla\nabla_h u\|_{L^2}
\end{aligned}$$

$$\leq C \|u_3\|_{L^{\alpha,\infty}} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\Delta u\|_{L^2}^{\frac{1}{\alpha}} \|\nabla\nabla_h u\|_{L^2}^{1+\frac{2}{\alpha}}$$
$$\leq C \|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{2-\frac{2}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} + \frac{1}{8} \|\nabla\nabla_h u\|_{L^2}^2 ,$$

where we have used the following Gagliardo–Nirenberg inequality in Lorentz spaces:

$$\|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2},2}} \leq C \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla u\|_{L^6}^{\frac{3}{\alpha}} .$$

Similarly, employing the Hölder inequality and the Gagliardo–Nirenberg inequality gives that for L_1,

$$L_2 \leq C \|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}} \|\nabla b\|_{L^2}^{2-\frac{2}{\alpha-2}} \|\Delta b\|_{L^2}^{\frac{2}{\alpha-2}} + \frac{1}{8} \|\nabla\nabla_h b\|_{L^2}^2 .$$

We now estimate L_3,

$$L_3 \leq \|b_3\|_{L^{10}} \|\nabla u\|_{L^{\frac{5}{2}}} \|\nabla\nabla_h b\|_{L^2} \leq C \|b_3\|_{L^{10}} \|\nabla u\|_{L^2}^{\frac{7}{10}} \|\nabla u\|_{L^6}^{\frac{3}{10}} \|\nabla\nabla_h b\|_{L^2}$$
$$\leq C \|b_3\|_{L^{\frac{10}{3}}} \|\nabla u\|_{L^2}^{\frac{7}{10}} \|\nabla\nabla_h u\|_{L^2}^{\frac{1}{5}} \|\Delta u\|_{L^2}^{\frac{1}{10}} \|\nabla\nabla_h b\|_{L^2}$$
$$\leq C \|b_3\|_{L^{10}}^{\frac{5}{2}} \|\nabla u\|_{L^2}^{\frac{7}{4}} \|\Delta u\|_{L^2}^{\frac{1}{4}} + \frac{1}{8}(\|\nabla\nabla_h b\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2),$$

where we have used the fact$\|\nabla u\|_{L^{\frac{5}{2}}} \leq C \|\nabla u\|_{L^2}^{\frac{7}{10}} \|\nabla u\|_{L^6}^{\frac{3}{10}}$.

Likewise,

$$L_4 \leq C \|b_3\|_{L^{10}}^{\frac{5}{2}} \|\nabla b\|_{L^2}^{\frac{7}{4}} \|\Delta b\|_{L^2}^{\frac{1}{4}} + \frac{1}{8}(\|\nabla\nabla_h b\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2).$$

For L_5, by applying the Hölder inequality, the Gagliardo–Nirenberg inequality and the Young inequality, one shows that

$$L_5 = \int_{\mathbb{R}^3} |\nabla_h u| \, |\nabla_h b| \, |j_3| \, \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla_h u|^2 + |\nabla_h b|^2 \right) |j_3| \, \mathrm{d}x$$
$$\leq C \|j_3\|_{L^{\beta,\infty}} \left(\|\nabla_h u\|_{L^{\frac{2\beta}{\beta-2},2}} \|\nabla_h u\|_{L^2} + \|\nabla_h u\|_{L^{\frac{2\beta}{\beta-2},2}} \|\nabla_h b\|_{L^2} \right)$$
$$\leq C \|j_3\|_{L^{\beta,\infty}} \left(\|\nabla_h u\|_{L^2}^{2-\frac{3}{\beta}} \|\nabla\nabla_h u\|_{L^2}^{\frac{3}{s}} + \|\nabla_h b\|_{L^2}^{2-\frac{3}{\beta}} \|\nabla\nabla_h b\|_{L^2}^{\frac{3}{s}} \right)$$
$$\leq C \|j_3\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + \frac{1}{8}(\|\nabla\nabla_h b\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2).$$

Inserting all the estimates into (3.3), Gronwall's type argument using

$$1 \leq \sup_{\lambda\in[\Gamma,\tau]} \exp\left(c \int_{\lambda}^{\tau} \|j_3(\varphi)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} \, d\varphi \right) \lesssim \exp\left(c \int_{0}^{T^*} \|j_3(\varphi)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} \, d\varphi \right) \lesssim 1,$$

due to (1.5) leads to, for every $\tau \in [\Gamma, t]$

$$\mathcal{L}^2(t) \leq C + C \int_{\Gamma}^{t} \|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}} \left(\|\nabla u\|_{L^2}^{2-\frac{2}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} + \|\nabla b\|_{L^2}^{2-\frac{2}{\alpha-2}} \|\Delta b\|_{L^2}^{\frac{2}{\alpha-2}} \right) \mathrm{d}\tau$$
$$+C \int_{\Gamma}^{t} \|b_3\|_{L^{10}}^{\frac{5}{2}} \left(\|\nabla u\|_{L^2}^{\frac{7}{4}} \|\Delta u\|_{L^2}^{\frac{1}{4}} + \|\nabla b\|_{L^2}^{\frac{7}{4}} \|\Delta b\|_{L^2}^{\frac{1}{4}} \right) \mathrm{d}\tau$$
$$+C \int_{\Gamma}^{t} \|j_3\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} \left(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2 \right) \mathrm{d}\tau$$
$$= C + \mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{I}_3(t). \tag{3.4}$$

Next, we analyze the right-hand side of (3.4) one by one. First, due to (3.2) and the definition of $\mathcal{J}^2$, we have

$$
\begin{aligned}
\mathcal{I}_1(t) &\leq C\left(\sup_{\tau\in[\Gamma,t]}\|\nabla u(\tau)\|_{L^2}^{\frac{3}{2}-\frac{2}{\alpha-2}}\right)\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}\|\Delta u(\tau)\|_{L^2}^{\frac{2}{\alpha-2}}\,\mathrm{d}\tau\\
&\quad+C\left(\sup_{\tau\in[\Gamma,t]}\|\nabla b(\tau)\|_{L^2}^{\frac{3}{2}-\frac{2}{\alpha-2}}\right)\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}}\|\nabla b(\tau)\|_{L^2}^{\frac{1}{2}}\|\Delta b(\tau)\|_{L^2}^{\frac{2}{\alpha-2}}\,\mathrm{d}\tau\\
&\leq C\mathcal{J}^{\frac{3}{2}-\frac{2}{\alpha-2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}\left(\int_\Gamma^t \|\nabla u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{4}}\left(\int_\Gamma^t \|\Delta u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{\alpha-2}}\\
&\quad+C\mathcal{J}^{\frac{3}{2}-\frac{2}{\alpha-2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}\left(\int_\Gamma^t \|\nabla b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{4}}\left(\int_\Gamma^t \|\Delta b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{\alpha-2}}\\
&\leq C\mathcal{J}^{\frac{3}{2}-\frac{2}{\alpha-2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{2}{\alpha-2}}(t)\\
&= C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}.
\end{aligned}
$$

Now, we estimate the term $\mathcal{I}_2(t)$ as

$$
\begin{aligned}
\mathcal{I}_2(t) &\leq C\left(\sup_{\tau\in[\Gamma,t]}\|b_3(\tau)\|_{L^{10}}^{\frac{5}{2}}\right)\int_\Gamma^t \|\nabla u(\tau)\|_{L^2}^{\frac{7}{4}}\|\Delta u(\tau)\|_{L^2}^{\frac{1}{4}}\,\mathrm{d}\tau\\
&\quad+\left(\sup_{\tau\in[\Gamma,t]}\|b_3(\tau)\|_{L^{10}}^{\frac{5}{2}}\right)\int_\Gamma^t \|\nabla b(\tau)\|_{L^2}^{\frac{7}{4}}\|\Delta b(\tau)\|_{L^2}^{\frac{1}{4}}\,\mathrm{d}\tau\\
&\leq\left(\sup_{\tau\in[\Gamma,t]}\|b_3(\tau)\|_{L^{10}}^{\frac{5}{2}}\right)\left(\int_\Gamma^t \|\nabla u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{7}{8}}\left(\int_\Gamma^t \|\Delta u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{8}}\\
&\quad+\left(\sup_{\tau\in[\Gamma,t]}\|b_3(\tau)\|_{L^{10}}^{\frac{5}{2}}\right)\left(\int_\Gamma^t \|\nabla b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{7}{8}}\left(\int_\Gamma^t \|\Delta b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{8}}\\
&\leq C\left(\sup_{\tau\in[\Gamma,t]}\|b_3(\tau)\|_{L^{10}}^{\frac{5}{2}}\right)\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t).
\end{aligned}
$$

For $\mathcal{I}_3(t)$, applying Hölder's and Young's inequalities, we get

$$
\begin{aligned}
\mathcal{I}_3(t) &\leq C\sup_{\tau\in[\Gamma,t]}(\|\nabla_h u(\tau)\|_{L^2}^2+\|\nabla_h b(\tau)\|_{L^2}^2)\int_\Gamma^t \|j_3(\tau)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}}\,\mathrm{d}\tau\\
&\leq C\eta\mathcal{L}^2(t).
\end{aligned}
$$

Therefore, combining the estimates of $\mathcal{I}_1(t), \mathcal{I}_2(t)$ and $\mathcal{I}_3(t)$ together with (3.4) and taking η small enough, it is easy to see that for all $\Gamma \leq t < T^*$:

$$\mathcal{L}^2(t) \leq C + C\epsilon^{\frac{1}{4}} \mathcal{J}^{\frac{3}{2}}(t) \left(\int_{\Gamma}^{t} \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}} \, \mathrm{d}\tau \right)^{\frac{3\alpha-10}{4(\alpha-2)}} + C \left(\sup_{\tau\in[\Gamma,t]} \|b_3(\tau)\|_{L^{10}}^{\frac{5}{2}} \right) \epsilon^{\frac{7}{8}} \mathcal{J}^{\frac{1}{4}}(t) \tag{3.5}$$

Now, we will establish the bounds of L^{10}-norm of the magnetic field b_3. In order to do it, we recall the third equation of the magnetic field:

$$\partial_t b_3 - \Delta b_3 + (u \cdot \nabla) b_3 = (b \cdot \nabla) u_3,$$

and multiply this equation by $|b_3|^8 b_3$, integrating by parts, using incompressibility conditions to obtain

$$\begin{aligned} \frac{1}{10} \frac{d}{dt} \int_{\mathbb{R}^3} |b_3|^{10} \, \mathrm{d}x + \frac{9}{25} \int_{\mathbb{R}^3} \left|\nabla(b_3^5)\right|^2 \mathrm{d}x &= \int_{\mathbb{R}^3} (b \cdot \nabla u_3)(|b_3|^8 b_3) \mathrm{d}x \\ &= -9 \int_{\mathbb{R}^3} b \cdot |b_3|^4 \left(|b_3|^4 \nabla b_3\right) u_3 \mathrm{d}x \\ &\leq \frac{9}{5} \int_{\mathbb{R}^3} |b| \left(|b_3|^5\right)^{\frac{4}{5}} |u_3| \left|\nabla(b_3^5)\right| \mathrm{d}x = I. \end{aligned} \tag{3.6}$$

Using the Hölder, Young inequalities and interpolation, the estimates of I are given by

$$\begin{aligned} I &\leq \frac{9}{5} \|b\|_{L^6} \left\| u_3 \left(|b_3|^5\right)^{\frac{4}{5}} \right\|_{L^3} \left\|\nabla(b_3^5)\right\|_{L^2} \\ &\leq C \|\nabla_h b\|_{L^2}^{\frac{2}{3}} \|\nabla b\|_{L^2}^{\frac{1}{3}} \|u_3\|_{L^{\alpha,\infty}} \left\| \left(|b_3|^5\right)^{\frac{4}{5}} \right\|_{L^{\frac{3\alpha}{\alpha-3},3}} \left\|\nabla(b_3^5)\right\|_{L^2} \\ &\leq C \|\nabla_h b\|_{L^2}^{\frac{2}{3}} \|\nabla b\|_{L^2}^{\frac{1}{3}} \|u_3\|_{L^{\alpha,\infty}} \left\| \left(|b_3|^5\right)^{\frac{4}{5}} \right\|_{L^{\frac{3\alpha}{\alpha-3},2}} \left\|\nabla(b_3^5)\right\|_{L^2} \\ &\leq C \|\nabla_h b\|_{L^2}^{\frac{2}{3}} \|\nabla b\|_{L^2}^{\frac{1}{3}} \|u_3\|_{L^{\alpha,\infty}} \left(\left\|b_3^5\right\|_{L^2}^{\frac{3(\alpha-5)}{4\alpha}} \left\|\nabla(b_3^5)\right\|_{L^2}^{\frac{15+\alpha}{4\alpha}} \right)^{\frac{4}{5}} \left\|\nabla(b_3^5)\right\|_{L^2} \\ &\leq C \|\nabla_h b\|_{L^2}^{\frac{20\alpha}{3(4\alpha-15)}} \|\nabla b\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}} \|u_3\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}} \left\|b_3^5\right\|_{L^2}^{\frac{6(\alpha-5)}{4\alpha-15}} + \frac{9}{5} \left\|\nabla(b_3^5)\right\|_{L^2}^2 \end{aligned}$$

Putting I in (3.6), we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} |b_3|^{10} \, \mathrm{d}x \leq C \|\nabla_h b\|_{L^2}^{\frac{20\alpha}{3(4\alpha-15)}} \|\nabla b\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}} \|u_3\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}} \|b_3\|_{L^{10}}^{\frac{30(\alpha-5)}{4\alpha-15}}.$$

Dividing by $\|b_3\|_{L^{10}}^{\frac{30(\alpha-5)}{4\alpha-15}}$, we arrive at

$$\frac{d}{dt} \|b_3\|_{L^{10}}^{\frac{10\alpha}{4\alpha-15}} \leq C \|\nabla_h b\|_{L^2}^{\frac{20\alpha}{3(7\alpha-15)}} \|\nabla b\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}} \|u_3\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}}.$$

Integrating over interval $[\Gamma, \tau)$, it follows that

$$\|b_3(\tau)\|_{L^{10}} \leq \left[\|b_3(\Gamma)\|_{L^{10}}^{\frac{10\alpha}{4\alpha-15}} + C \int_{\Gamma}^{\tau} \|\nabla_h b(\lambda)\|_{L^2}^{\frac{20\alpha}{3(4\alpha-15)}} \|\nabla b(\lambda)\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}} \|u_3(\lambda)\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}} \, d\lambda \right]^{\frac{4\alpha-15}{10\alpha}}, \tag{3.7}$$

for all $\tau \in [\Gamma, t)$. It follows from (3.7) and (3.5) that

$$
\begin{aligned}
\mathcal{L}^2(t) \leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3\alpha-10}{4(\alpha-2)}} \\
& + C\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t)\sup_{\tau\in[\Gamma,t]}\left[\|b_3(\Gamma)\|_{L^{10}}^{\frac{10\alpha}{7\alpha-15}} + C\int_\Gamma^\tau \|\nabla_h b(\lambda)\|_{L^2}^{\frac{20\alpha}{3(4\alpha-15)}}\|\nabla b(\lambda)\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}}\|u_3(\lambda)\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}}\,d\lambda\right]^{\frac{4\alpha-15}{4\alpha}} \\
\leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3\alpha-10}{4(\alpha-2)}} \\
& + C\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t)\left[\|b_3(\Gamma)\|_{L^{10}}^{\frac{10\alpha}{4\alpha-15}} + C\int_\Gamma^t \|\nabla_h b(\tau)\|_{L^2}^{\frac{20\alpha}{3(4\alpha-15)}}\|\nabla b(\tau)\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}}\|u_3(\tau)\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}}\,\mathrm{d}\tau\right]^{\frac{4\alpha-15}{4\alpha}} \\
\leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}\,\mathrm{d}\tau\right)^{\frac{3\alpha-10}{4(\alpha-2)}} + C\,\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\,\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t) \\
& + C\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t)\sup_{\tau\in[\Gamma,t]}\|\nabla_h b(\tau)\|_{L^2}^{\frac{5}{3}}\left[\int_\Gamma^t \|\nabla b(\tau)\|_{L^2}^{\frac{10\alpha}{3(4\alpha-15)}}\|u_3(\tau)\|_{L^\alpha}^{\frac{10\alpha}{4\alpha-15}}\,\mathrm{d}\tau\right]^{\frac{4\alpha-15}{4\alpha}} \\
\leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t 1 + \|u_3(\tau)\|_{L^\alpha}^{\frac{30\alpha}{7\alpha-45}}\,\mathrm{d}\tau\right)^{\frac{3\alpha-10}{4(\alpha-2)}} + C\,\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\,\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t) \\
& + C\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t)\mathcal{L}^{\frac{5}{3}}(t)\left[\left(\int_\Gamma^t \|\nabla b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{5\alpha}{3(4\alpha-15)}}\left(\int_\Gamma^t \|u_3(\tau)\|_{L^\alpha}^{\frac{30\alpha}{7\alpha-45}}\,\mathrm{d}\tau\right)^{1-\frac{5\alpha}{3(4\alpha-15)}}\right]^{\frac{4\alpha-15}{4\alpha}} \\
\leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t) + C\,\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\,\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t) + C\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t)\mathcal{L}^{\frac{5}{3}}(t)\left[\epsilon + \int_\Gamma^t \|u_3(\tau)\|_{L^\alpha}^{\frac{30\alpha}{7\alpha-45}}\,\mathrm{d}\tau\right]^{\frac{4\alpha-15}{4\alpha}} \\
\leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t) + C\,\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\,\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t) + C\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t)\mathcal{L}^{\frac{5}{3}}(t)\left[1 + \int_\Gamma^t \|u_3(\tau)\|_{L^\alpha}^{\frac{30\alpha}{7\alpha-45}}\,\mathrm{d}\tau\right]^{\frac{4\alpha-15}{4\alpha}} \\
\leq\ & C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t) + C\,\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\,\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t) + C\epsilon^{\frac{21}{4}}\mathcal{J}^{\frac{3}{2}}(t) + \frac{5}{6}\mathcal{L}^2(t)
\end{aligned}
$$

which leads to

$$
\mathcal{L}^2(t) \leq C + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t) + C\,\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\,\epsilon^{\frac{7}{8}}\mathcal{J}^{\frac{1}{4}}(t) + C\epsilon^{\frac{21}{4}}\mathcal{J}^{\frac{3}{2}}(t). \tag{3.8}
$$

Now, we will establish the bounds of H^1-norm of the velocity and magnetic field. In order to do it, we multiply the first and second equations of (1.1) by $-\Delta u$ and $-\Delta b$, respectively, and integrate them over $\mathbb{R}^3$ with respect to the spatial variable. Then, integration by parts gives the following identity:

$$
\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2
$$

$$
\begin{aligned}
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\mathrm{d}x + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\mathrm{d}x - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i b_j\partial_k b_j\mathrm{d}x \\
&\quad + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

Applying the divergence-free condition, $\nabla\cdot u = \nabla\cdot b = 0$, by using the Hölder inequality, the interpolation inequality and (3.1), it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \\
&\le C\int_{\mathbb{R}^3}(|\nabla_h u| + |\nabla_h b|)(|\nabla u|^2 + |\nabla b|^2)\mathrm{d}x \\
&\le C(\|\nabla_h u\|_{L^2} + \|\nabla_h b\|_{L^2})(\|\nabla u\|_{L^4}^2 + \|\nabla b\|_{L^4}^2) \\
&\le C(\|\nabla_h u\|_{L^2} + \|\nabla_h b\|_{L^2})(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^6}^{\frac{3}{2}} + \|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^6}^{\frac{3}{2}}) \\
&\le C(\|\nabla_h u\|_{L^2} + \|\nabla_h b\|_{L^2})(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}} + \|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla b\|_{L^2}\|\Delta b\|_{L^2}^{\frac{1}{2}}).
\end{aligned}
$$

Integrating this last inequality in time, we deduce that for all $\tau\in[\Gamma,t]$

$$
\begin{aligned}
\mathcal{J}^2(t) &\le \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla b(\Gamma)\|_{L^2}^2 + C\sup_{\tau\in[\Gamma,t]}(\|\nabla_h u(\tau)\|_{L^2} + \|\nabla_h b(\tau)\|_{L^2}) \\
&\quad\times\left(\int_\Gamma^t\|\nabla u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{4}}\left(\int_\Gamma^t\|\nabla\nabla_h u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_\Gamma^t\|\Delta u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{4}} \\
&\quad + C\sup_{\tau\in[\Gamma,t]}(\|\nabla_h u(\tau)\|_{L^2} + \|\nabla_h b(\tau)\|_{L^2}) \\
&\quad\times\left(\int_\Gamma^t\|\nabla b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{4}}\left(\int_\Gamma^t\|\nabla\nabla_h b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_\Gamma^t\|\Delta b(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{4}} \\
&\le \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla b(\Gamma)\|_{L^2}^2 + 2C\mathcal{L}(t)\epsilon^{\frac{1}{4}}\mathcal{L}(t)\mathcal{J}^{\frac{1}{2}}(t) \\
&= \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla b(\Gamma)\|_{L^2}^2 + C\epsilon^{\frac{1}{4}}\mathcal{L}^2(t)\mathcal{J}^{\frac{1}{2}}(t). \qquad (3.10)
\end{aligned}
$$

Inserting (3.8) into (3.10) and taking ϵ small enough, then it is easy to see that for all $\Gamma\le t<T^*$, there holds

$$
\begin{aligned}
\mathcal{J}^2(t) &\le \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla b(\Gamma)\|_{L^2}^2 + C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{1}{2}}(t) + C\epsilon^{\frac{1}{2}}\mathcal{J}^2(t) \\
&\quad + C\|b_3(\Gamma)\|_{L^{10}}^{\frac{5}{2}}\epsilon^{\frac{9}{8}}\mathcal{J}^{\frac{3}{4}}(t) + C\epsilon^{\frac{11}{2}}\mathcal{J}^2(t) \\
&< \infty,
\end{aligned}
$$

which proves

$$
\sup_{\Gamma\le t<T^*}\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 < \infty.
$$

This completes the proof of Theorem 1.1. □

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References

[1] Benbernou, S., Gala, S., Ragusa, M.A.: On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO space. Math. Methods Appl. Sci. **37**, 2320–2325 (2013)
[2] Bergh, J., Löfström, J.: Interpolation Spaces. Springer, New York (1976)
[3] Chen, X., Gala, S., Guo, Z.: A new regularity criterion in terms of the direction of the velocity for the MHD equations. Acta Appl. Math. **113**, 207–213 (2011)
[4] Chen, X., Guo, Z., Zhu, M.: A new regularity criterion for the 3D MHD equations involving partial components. Acta Appl. Math. **134**, 161–171 (2014)
[5] Chen, Q.L., Miao, C.X., Zhang, Z.F.: On the regularity criterion of weak solution for the 3D viscous magnetohydrodynamics equations. Commun. Math. Phys. **284**, 919–930 (2008)
[6] Cao, C., Titi, E.S.: Regularity criteria for the three dimensional Navier–Stokes equations. Indiana Univ. Math. J. **57**, 2643–2661 (2008)
[7] Cao, C., Wu, J.: Two regularity criteria for the 3D MHD equations. J. Differ. Equ. **248**, 2263–2274 (2010)
[8] Dong, B.Q., Jia, Y., Zhang, W.: An improved regularity criterion of three-dimensional magnetohydrodynamic equations. Nonlinear Anal. RWA **13**, 1159–1169 (2012)
[9] Duvaut, G., Lions, J.: Inéquations en thermoélasticit é et magnétohydrodynamique. Arch. Ration. Mech. Anal. **46**, 241–279 (1972)
[10] Gala, S.: Extension criterion on regularity for weak solutions to the 3D MHD equations. Math. Methods Appl. Sci. **33**, 1496–1503 (2010)
[11] Gala, S., Ragusa, M.A.: A new regularity criterion for the 3D incompressible MHD equations via partial derivatives. J. Math. Anal. Appl. **481**(2), 123497 (2020)
[12] Gala, S., Ragusa, M.A.: On the regularity criterion of weak solutions for the 3D MHD equations. Z. Angew. Math. Phys. **68**, 140 (2017)
[13] Gala, S., Ragusa, M.A.: A logarithmic regularity criterion for the two-dimensional MHD equations. J. Math. Anal. Appl. **444**(2), 1752–1758 (2016)
[14] He, C., Xin, Z.: On the regularity of weak solutions to the magnetohydrodynamic equations. J. Differ. Equ. **213**, 235–254 (2005)
[15] He, C., Wang, Y.: Remark on the regularity for weak solutions to the magnetohydrodynamic equations. Math. Methods Appl. Sci. **31**, 1667–1684 (2008)
[16] Ji, E., Lee, J.: Some regularity criteria for the 3D incompressible magnetohydrodynamics. J. Math. Anal. Appl. **369**, 317–322 (2010)
[17] Jia, X., Zhou, Y.: Regularity criteria for the 3D MHD equations involving partial components. Nonlinear Anal. RWA **13**, 410–418 (2012)
[18] Jia, X., Zhou, Y.: Regularity criteria for the 3D MHD equations via partial derivatives. Kinet. Relat. Models **5**, 505–516 (2012)
[19] Jia, X., Zhou, Y.: Regularity criteria for the 3D MHD equations via partial derivatives II. Kinet. Relat. Models **7**, 291–304 (2014)
[20] Jia, X., Zhou, Y.: Ladyzhenskaya–Prodi–Serrin type regularity criteria for the 3D incompressible MHD equations in terms of 33 mixture matrices. Nonlinearity **28**, 3289–3307 (2015)
[21] Jia, X., Zhou, Y.: On regularity criteria for the 3D incompressible MHD equations involving one velocity component. J. Math. Fluid Mech. **18**, 187–206 (2016)
[22] Lorentz, G.G.: Some new functional spaces. Ann. Math. **51**, 37–55 (1950)
[23] Ni, L., Guo, Z., Zhou, Y.: Some new regularity criteria for the 3D MHD equations. J. Math. Anal. Appl. **396**, 108–118 (2012)
[24] O'Neil, R.: Convolution operators and $L^{p,q}$ spaces. Duke Math. J. **30**, 129–142 (1963)

[25] Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. **36**, 635–664 (1983)
[26] Triebel, H.: Theory of Function Spaces. Birkhäuser, Basel (1983)
[27] Wu, J.: Viscous and inviscid magnetohydrodynamics equations. J. Anal. Math. **73**, 251–265 (1997)
[28] Wu, J.: Bounds and new approaches for the 3D MHD equations. J. Nonlinear Sci. **12**, 395–413 (2002)
[29] Wu, J.: Regularity results for weak solutions of the 3D MHD equations. Discrete Contin. Dyn. Syst. **10**, 543–556 (2004)
[30] Zhou, Y.: Remarks on regularities for the 3D MHD equations. Discrete Contin. Dyn. Syst. **12**, 881–886 (2005)
[31] Yamazaki, K.: Regularity criteria of MHD system involving one velocity and one current density component. J. Math. Fluid Mech. **16**, 551–570 (2014)
[32] Yamazaki, K.: Regularity criteria of the three-dimensional MHD system involving one velocity and one vorticity component. Nonlinear Anal. **135**, 78–83 (2016)
[33] Yamazaki, K.: Remarks on the regularity criteria of the three-dimensional magnetohydrodynamics system in terms of two velocity field components. J. Math. Phys. **55**, 031505 (2014)
[34] Zhang, Z.: Regularity criteria for the 3D MHD equations involving one current density and the gradient of one velocity component. Nonlinear Anal. **115**, 41–49 (2015)
[35] Zhang, Z.: Remarks on the global regularity criteria for the 3D MHD equations via two components. Z. Angew. Math. Phys. **66**, 977–987 (2015)
[36] Zhang, Z.: Re ned regularity criteria for the MHD system involving only two components of the solution. Appl. Anal. **96**, 2130–2139 (2017)
[37] Zhou, Y., Gala, S.: Regularity criteria for the solutions to the 3D MHD equations in the multiplier space. Z. Angew. Math. Phys. **61**, 193–199 (2010)
[38] Zhou, Y., Gala, S.: A new regularity criterion for weak solutions to the viscous MHD equations in terms of the vorticity field. Nonlinear Anal. **72**, 3643–3648 (2010)

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