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A Regularity Criterion in Weak Spaces to Boussinesq Equations

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Abstract: In this paper, we study the regularity of weak solutions to the incompressible Boussinesq equations in $\mathbb{R}^3 \times (0, T)$. The main goal is to establish the regularity criterion in terms of one velocity component and the gradient of temperature in Lorentz spaces.

Keywords: Regularity results; Cauchy problem; Boussinesq equations; Lorentz spaces; Navier-Stokes equations; MHD equations; weak solutions

1. Introduction

In this paper, we consider the following Cauchy problem for the incompressible Boussinesq equations in $\mathbb{R}^3 \times (0, T)$

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u - \Delta u + \nabla \pi = \theta e_3, \\ \partial_t \theta - \Delta\theta + (u \cdot \nabla)\theta = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x),\ \theta(x, 0) = \theta_0(x), \end{cases} \tag{1}$$

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the unknown velocity vector, $\theta = (\theta_1(x, t), \theta_2(x, t), \theta_3(x, t))$ and $\pi = \pi(x, t)$ denote, respectively, the temperature and the hydrostatic pressure. While u_0 and θ_0 are the prescribed initial data for the velocity and temperature with properties $\nabla \cdot u_0 = 0$. Moreover, the term θe_3 represents buoyancy force on fluid motion.

We would like to point out that the system (1) at $\theta = 0$ reduces to the incompressible Navier-Stokes equations, which has been greatly analyzed. From the viewpoint of the model, therefore, Navier-Stokes flow is viewed as the flow of a simplified Boussinesq equation.

Besides their physical applications, the Boussinesq equations are also mathematically significant. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research, and many interesting results have been obtained (see, for example, [1–14] and references therein).

On the other hand, it is desirable to show the regularity of the weak solutions if some partial components of the velocity satisfy certain growth conditions. For the 3D Navier-Stokes equations, there are many results to show such regularity of weak solutions in terms of partial components of the velocity u (see, for example, References [15–23] and the references cited therein). It is obvious that, for the assumptions of all regularity criteria, every component of the velocity field must satisfy the same assumptions, and it does not make any difference between the components of the velocity field. As pointed out by Neustupa and Penel [24], it is interesting to know how to effect the regularity of the velocity field by the regularity of only one component of the velocity field. In particular, Zhou [25] showed that the solution is regular if one component of the velocity, for example, u_3 satisfies

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} \leq \frac{1}{2}, \quad 6 < q \leq \infty. \tag{2}$$

Condition (2) can be replaced respectively by the following:

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} \leq \frac{5}{8}, \quad \frac{24}{5} < q \leq \infty, \tag{3}$$

(see Kukavica and Ziane [26]). Later, Cao and Titi [16] showed the regularity of weak solution to the Navier-Stokes equations under the assumption

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} = \frac{2}{3}+\frac{2}{3q}, \quad q > \frac{7}{2}. \tag{4}$$

Motivated by the above work, Zhou and Pokorný [27] showed the following regularity condition

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} = \frac{3}{4}+\frac{1}{2q}, \quad q > \frac{10}{3}, \tag{5}$$

while the limiting case $u_3 \in L^\infty(0,T;L^{\frac{10}{3}}(\mathbb{R}^3))$ was covered in [21]. For many other result works, especially the regularity criteria involving only one velocity component, or its gradient, with no intention to be complete, one can consult [28,29] and references therein. However, the conditions (2)–(5) are quite strong compared with the condition of Serrin's regularity criterion :

$$u \in L^p(0,T;L^q(\mathbb{R}^3)) \quad \text{with } \frac{2}{p}+\frac{3}{q} \leq 1, \quad 3 < q \leq \infty, \tag{6}$$

and do not imply the invariance under the above scaling transformation. Therefore, it is of interest in showing regularity by imposing Serrin's condition (6) with respect to the one component of the velocity field.

Similar to the research of the 3D Navier-Stokes equations, the authors are interested in the regularity criterion of (1) by reducing to the components of u. There are many other or similar results on the hydrodynamical systems modeling the flow of nematic liquid crystal material, Boussinesq equations and MHD equations (see e.g., References [30,31] and the references therein).

Motivated by the reference mentioned above, the purpose of the present paper is to give a further observation on the global regularity of the solution for system (1) and to extend the regularity of weak solutions to the Boussinesq equations (1) in terms of one velocity component and the gradient of the temperature.

2. Notations and Main Result

Before stating our result, we introduce some notations and function spaces. These spaces can be found in many papers. For the functional space, $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space of real-valued functions with norm $\|\cdot\|_{L^p}$:

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^3} |f(x)|^p \, dx\right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\ ess \sup\limits_{x\in\mathbb{R}^3} |f(x)|, & \text{for } p = \infty. \end{cases}$$

On the other hand, the usual Sobolev space of order m is defined by

$$H^m(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \nabla^m u \in L^2(\mathbb{R}^3) \right\}$$

with the norm

$$\|u\|_{H^m} = \left(\|u\|_{L^2}^2 + \|\nabla^m u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

To prove Theorem 1, we use the theory of Lorentz spaces and introduce the following notations. We define the non-increasing rearrangement of f,

$$f^*(\lambda) = \inf\left\{t > 0 : m_f(t) \leq \lambda\right\}, \quad \text{for } \lambda > 0,$$

where f is a measurable function on $\mathbb{R}^3$ and $m_f(t)$ is the distribution function of f, which is defined by the Lebesgue measure of the set $\{x \in \mathbb{R}^3 : |f(x)| > t\}$. The Lorentz space $L^{p,q}((\mathbb{R}^3)$ is defined by

$$L^{p,q} = \left\{f : \mathbb{R}^3 \to \mathbb{R} \text{ measurable such that } \|f\|_{L^{p,q}} < \infty\right\} \text{ with } 1 \leq p < \infty$$

is equipped with the quasi-norm

$$\|f\|_{L^{p,q}} = \left(\frac{q}{p}\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, \text{ if } 1 < q < \infty.$$

Moreover, we define f^{**} by

$$f^{**}(\lambda) = \frac{1}{\lambda}\int_0^\lambda f^*(\lambda')d\lambda',$$

and Lorentz spaces $L^{p,\infty}(\mathbb{R}^3)$ by

$$L^{p,\infty}(\mathbb{R}^3) = \left\{f \in \mathcal{S}'(\mathbb{R}^3) : \|f\|_{L^{p,\infty}} < \infty\right\},$$

where

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda \geq 0}(\lambda^{\frac{1}{p}} f^{**}(\lambda)),$$

for $1 \leq p \leq \infty$. For details, we refer to References [32,33].

From the definition of the Lorentz space, we can obtain the following continuous embeddings :

$$L^p(\mathbb{R}^3) = L^{p,p}(\mathbb{R}^3) \hookrightarrow L^{p,q}(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3), \quad 1 \leq p \leq q < \infty.$$

In order to prove Theorem 1, we recall the Hölder inequality in the Lorentz spaces (see, e.g., O'Neil [34]).

Lemma 1. *Let $f \in L^{p_2,q_2}(\mathbb{R}^3)$ and $g \in L^{p_3,q_3}(\mathbb{R}^3)$ with $1 \leq p_2, p_3 \leq \infty$, $1 \leq q_2, q_3 \leq \infty$. Then $fg \in L^{p_1,q_1}(\mathbb{R}^3)$ with*

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}, \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$$

and the Hölder inequality of Lorentz spaces

$$\|fg\|_{L^{p_1,q_1}} \leq C\,\|f\|_{L^{p_2,q_2}}\,\|g\|_{L^{p_3,q_3}},$$

holds true for a positive constant C.

The following result plays an important role in the proof of our theorem, the so-called Gagliardo-Nirenberg inequality in Lorentz spaces, its proof can be found in Reference [35].

Lemma 2. *Let $f \in L^{p,q}(\mathbb{R}^3)$ with $1 \leq p, q, p_4, q_4, p_5, q_5 \leq \infty$. Then the Gagliardo-Nirenberg inequality of Lorentz spaces*

$$\|f\|_{L^{p,q}} \leq C\,\|f\|_{L^{p_4,q_4}}^{\theta}\,\|f\|_{L^{p_5,q_5}}^{1-\theta}$$

holds for a positive constant C and

$$\frac{1}{p} = \frac{\theta}{p_4} + \frac{1-\theta}{p_5}, \quad \frac{1}{q} = \frac{\theta}{q_4} + \frac{1-\theta}{q_5}, \quad \theta \in (0,1).$$

Now we give the definition of the weak solution.

Definition 1. *Let $T > 0$, $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions. A measurable function $(u(x,t), \theta(x,t))$ is called a weak solution to the Boussinesq equations (1) on $[0,T]$ if the following conditions hold:*

1. $(u(x,t), \theta(x,t)) \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$;
2. *system (1) is satisfied in the sense of distributions;*
3. *the energy inequality, that is,*

$$\|u(\cdot,t)\|_{L^2}^2 + \|\theta(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau + 2\int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 \, d\tau \le \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 .$$

By a strong solution, we mean that a weak solution u of the Navier-Stokes equations (1) satisfies

$$(u(x,t), \theta(x,t)) \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^2(\mathbb{R}^3)).$$

It is well known that the strong solution is regular and unique.

Our main result is stated as follows:

Theorem 1. *Let $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that (u, θ) is a weak solution to system (1). If u_3 and $\nabla\theta$ satisfies the following conditions*

$$\begin{cases} u_3 \in L^{\frac{30\alpha}{7\alpha-45}}(0,T; L^{\alpha,\infty}(\mathbb{R}^3)), \ \ \text{with} \ \ \frac{45}{7} \le \alpha \le \infty, \\ \nabla\theta \in L^{\frac{2\beta}{2\beta-3}}(0,T; L^{\beta,\infty}(\mathbb{R}^3)), \ \ \text{with} \ \ \frac{3}{2} < \beta \le \infty, \end{cases} \tag{7}$$

then the solution (u, θ) is regular on $(0, T]$.

Remark 1. *If $\theta = 0$, it is clear that theorem 1 improves the earlier results of References [21,27] for 3D Navier-Stokes equations and extend the regularity criterion (5) from Lebesgue space L^α to Lorentz space $L^{\alpha,\infty}$.*

Remark 2. *This result proves a new regularity criterion for weak solutions to the Cauchy problem of the 3D Boussinesq equations via one component of the velocity field and the gradient of the temperature in the framework of the Lorentz spaces. This result reveals that the one component of the velocity field plays a dominant role in regularity theory of the Boussinesq equations.*

3. Proof of the Main Result

In this section, under the assumptions of the Theorem 1, we prove our main result. Before proving our result, we recall the following multiplicative Sobolev imbedding inequality in the whole space $\mathbb{R}^3$ (see, for example Reference [16]) :

$$\|f\|_{L^6} \le C \|\nabla_h f\|_{L^2}^{\frac{2}{3}} \|\partial_3 f\|_{L^2}^{\frac{1}{3}}, \tag{8}$$

where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient operator. We now give the proof of our main theorem.

Proof. To prove our result, it suffices to show that for any fixed $T > T^*$, there holds

$$\sup_{0\le t\le T^*}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) \le C_T,$$

where T^*, which denotes the maximal existence time of a strong solution and C_T is an absolute constant which only depends on T, u_0 and θ_0.

The method of our proof is based on two major parts. The first one establishes the bounds of $(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h\theta(t)\|_{L^2}^2)$, while the second gives the bounds of the H^1-norm of velocity u and temperature θ in terms of the results of part one.

Taking the inner product of $(1)_1$ with $-\Delta_h u$, $(1)_2$ with $-\Delta_h\theta$ in $L^2(\mathbb{R}^3)$, respectively, then adding the three resulting equations together, we obtain, after integrating by parts, that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h\theta\|_{L^2}^2) + \|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h\theta\|_{L^2}^2 \\
=\;& \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta_h u dx + \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta_h\theta - \int_{\mathbb{R}^3}(\theta e_3)\cdot\Delta_h u dx \\
=\;& I_1 + I_2 + I_3,
\end{aligned} \tag{9}$$

where $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ is the horizontal Laplacian. For the notational simplicity, we set

$$\begin{aligned}
\mathcal{L}^2(t) &= \sup_{\tau\in[\Gamma,t]}(\|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h\theta\|_{L^2}^2) + \int_\Gamma^t(\|\nabla\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla\nabla_h\theta\|_{L^2}^2)d\tau, \\
\mathcal{J}^2(t) &= \sup_{\tau\in[\Gamma,t]}(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla\theta(\tau)\|_{L^2}^2) + \int_\Gamma^t(\|\Delta u(\tau)\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2)d\tau,
\end{aligned}$$

for $t \in [\Gamma, T^*)$. In view of (7), we choose $\epsilon, \eta > 0$ to be precisely determined subsequently and then select $\Gamma < T^*$ sufficiently close to T^* such that for all $\Gamma \le t < T^*$,

$$\int_\Gamma^t(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla\theta(\tau)\|_{L^2}^2)d\tau \le \epsilon \ll 1, \tag{10}$$

and

$$\int_\Gamma^t \|\nabla\theta(\tau)\|_{L^\beta}^{\frac{2\beta}{2\beta-3}} d\tau \le \eta \ll 1. \tag{11}$$

Integrating by parts and using the divergence-free condition, it is clear that (see e.g., Reference [23])

$$I_1 = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta_h u dx \le \int_{\mathbb{R}^3}|u_3|\,|\nabla u|\,|\nabla\nabla_h u|\,dx. \tag{12}$$

By appealing to Lemma 1, (8), and the Young inequality, it follows that

$$\begin{aligned}
I_1 &\le C\|u_3\|_{L^{\alpha,\infty}}\|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2},2}}\|\nabla\nabla_h u\|_{L^2} \\
&\le C\|u_3\|_{L^{\alpha,\infty}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\|\nabla u\|_{L^6}^{\frac{3}{\alpha}}\|\nabla\nabla_h u\|_{L^2} \\
&\le C\|u_3\|_{L^{\alpha,\infty}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}}\|\Delta u\|_{L^2}^{\frac{1}{\alpha}}\|\nabla\nabla_h u\|_{L^2}^{1+\frac{2}{\alpha}} \\
&\le C\|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_{L^2}^{2-\frac{2}{\alpha-2}}\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} + \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2,
\end{aligned}$$

where we have used the following Gagliardo-Nirenberg inequality in Lorentz spaces :

$$\|\phi\|_{L^{\frac{2s}{s-2},2}} \le C\|\phi\|_{L^2}^{1-\frac{3}{s}}\|\nabla\phi\|_{L^6}^{\frac{3}{s}} \quad \text{with} \quad 3 < s \le \infty.$$

To estimate the term I_2 of (9), first observe that by applying integration by parts and $\nabla \cdot u = 0$, we derive

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^3} (u \cdot \nabla)\theta \cdot \Delta_h \theta dx = \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_i \partial_i \theta_j \partial_{kk}^2 \theta_j dx \\ &= -\sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i \theta_j \partial_k \theta_j dx - \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_i \partial_k \partial_i \theta_j \partial_k \theta_j dx \\ &= -\sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i \theta_j \partial_k \theta_j dx, \end{aligned}$$

where we have used

$$\begin{aligned} -\sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_i \partial_k \partial_i \theta_j \partial_k \theta_j dx &= \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_i u_i (\partial_k \theta_j)^2 dx + \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_i \partial_i \partial_k \theta_j \partial_k \theta_j dx \\ &= \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_i \partial_i \partial_k \theta_j \partial_k \theta_j dx, \end{aligned}$$

so that

$$\sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_i \partial_k \partial_i \theta_j \partial_k \theta_j dx = 0.$$

It follows from Hölder's inequality, (8) and Young's inequality that

$$\begin{aligned} I_2 &= -\sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i \theta_j \partial_k \theta_j dx \leq \int_{\mathbb{R}^3} |\nabla_h u| \, |\nabla \theta| \, |\nabla_h \theta| \, dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla_h u|^2 + |\nabla_h \theta|^2 \right) |\nabla \theta| \, dx \\ &\leq C \|\nabla \theta\|_{L^{\beta,\infty}} \left(\|\nabla_h u\|_{L^{\frac{2\beta}{\beta-2},2}} \|\nabla_h u\|_{L^2} + \|\nabla_h \theta\|_{L^{\frac{2\beta}{\beta-2},2}} \|\nabla_h \theta\|_{L^2} \right) \\ &\leq C \|\nabla \theta\|_{L^{\beta,\infty}} \left(\|\nabla_h u\|_{L^2}^{2-\frac{3}{\beta}} \|\nabla \nabla_h u\|_{L^2}^{\frac{3}{s}} + \|\nabla_h \theta\|_{L^2}^{2-\frac{3}{\beta}} \|\nabla \nabla_h \theta\|_{L^2}^{\frac{3}{s}} \right) \\ &\leq C \|\nabla \theta\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} \left(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \theta\|_{L^2}^2 \right) + \frac{1}{2} \|\nabla \nabla_h \theta\|_{L^2}^2 + \frac{1}{4} \|\nabla \nabla_h u\|_{L^2}^2 . \end{aligned}$$

Finally, we we want to estimate I_3. It follows from integration by parts and Cauchy inequality that

$$\begin{aligned} I_3 &= -\int_{\mathbb{R}^3} (\theta e_3) \cdot \Delta_h u dx = \int_{\mathbb{R}^3} \nabla_h (\theta e_3) \cdot \nabla_h u dx \\ &\leq 2 \|\nabla_h u\|_{L^2}^2 + 2 \|\nabla_h \theta\|_{L^2}^2 . \end{aligned}$$

Inserting all the estimates into (12), Gronwall's type argument using

$$1 \leq \sup_{\lambda \in [\Gamma, \tau]} \exp \left(c \int_{\lambda}^{\tau} \|\nabla \theta(\varphi)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} d\varphi \right) \lesssim \exp \left(c \int_{0}^{T^*} \|\nabla \theta(\varphi)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} d\varphi \right) \lesssim 1,$$

due to (7) leads to, for every $\tau \in [\Gamma, t]$

$$\begin{aligned} \mathcal{L}^2(t) &\leq C + C \int_{\Gamma}^{t} \|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{2-\frac{2}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} d\tau + C \int_{\Gamma}^{t} \|\nabla \theta\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}} \left(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \theta\|_{L^2}^2 \right) d\tau \\ &= C + \mathcal{I}_1(t) + \mathcal{I}_2(t). \end{aligned} \tag{13}$$

Next, we analyze the right-hand side of (13) one by one. First, due to (10) and the definition of $\mathcal{J}^2$, we have

$$\begin{aligned}
\mathcal{I}_1(t) &\leq C\left(\sup_{\tau\in[\Gamma,t]}\|\nabla u(\tau)\|_{L^2}^{\frac{3}{2}-\frac{2}{\alpha-2}}\right)\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{2\alpha}{\alpha-2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}\|\Delta u(\tau)\|_{L^2}^{\frac{2}{\alpha-2}}\,d\tau \\
&\leq C\mathcal{J}^{\frac{3}{2}-\frac{2}{\alpha-2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,d\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}\left(\int_\Gamma^t \|\nabla u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{4}}\left(\int_\Gamma^t \|\Delta u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{\alpha-2}} \\
&\leq C\mathcal{J}^{\frac{3}{2}-\frac{2}{\alpha-2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,d\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{2}{\alpha-2}}(t) \\
&= C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,d\tau\right)^{\frac{3}{4}-\frac{1}{\alpha-2}}.
\end{aligned}$$

Finally, we deal with the term $\mathcal{I}_2(t)$. Applying Hölder and Young inequalities, one has

$$\begin{aligned}
\mathcal{I}_2(t) &\leq C\sup_{\tau\in[\Gamma,t]}(\|\nabla_h u(\tau)\|_{L^2}^2+\|\nabla_h\theta(\tau)\|_{L^2}^2)\int_\Gamma^t \|\nabla\theta(\tau)\|_{L^{\beta,\infty}}^{\frac{2\beta}{2\beta-3}}\,d\tau \\
&\leq C\eta\mathcal{L}^2(t).
\end{aligned}$$

Hence, choosing η small enough such that $C\eta<1$ and inserting the above estimates of $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ into (13), we derive that for all $\Gamma\leq t<T^*$:

$$\begin{aligned}
\mathcal{L}^2(t) &\leq C+C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t \|u_3(\tau)\|_{L^{\alpha,\infty}}^{\frac{8\alpha}{3\alpha-10}}\,d\tau\right)^{\frac{3\alpha-10}{4(\alpha-2)}} \\
&\leq C+C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t)\left(\int_\Gamma^t 1+\|u_3(\tau)\|_{L^{\alpha}}^{\frac{30\alpha}{7\alpha-45}}\,d\tau\right)^{\frac{3\alpha-10}{4(\alpha-2)}},
\end{aligned}$$

which leads to

$$\mathcal{L}^2(t)\leq C+C\epsilon^{\frac{1}{4}}\mathcal{J}^{\frac{3}{2}}(t). \tag{14}$$

Now, we will establish the bounds of H^1-norm of the velocity magnetic field and micro-rotational velocity. In order to do it, taking the inner product of $(1)_1$ with $-\Delta u$, $(1)_2$ with $-\Delta b$ and $(1)_3$ with $-\Delta\theta$ in $L^2(\mathbb{R}^3)$, respectively. Then, integration by parts gives the following identity:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2 \\
=\ &\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx+\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx-\int_{\mathbb{R}^3}(\theta e_3)\cdot\Delta u dx.
\end{aligned}$$

Integrating by parts and using the divergence-free condition, one can easily deduce that (see e.g., Reference [27])

$$\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx\leq C\int_{\mathbb{R}^3}|\nabla_h u|\,|\nabla u|^2\,dx.$$

We now treat the $\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx$-term. By integration by parts, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx &= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k\theta_j\cdot\partial_i\theta_j dx-\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_k\cdot\partial_k\theta_j\cdot\partial_3\theta_j dx\\
&= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k\theta_j\cdot\partial_i\theta_j dx-\sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_k\cdot\partial_k\theta_j\cdot\partial_3\theta_j dx\\
&\quad -\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_3\cdot\partial_3\theta_j\cdot\partial_3\theta_j dx\\
&= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k\theta_j\cdot\partial_i\theta_j dx-\sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_k\cdot\partial_k\theta_j\cdot\partial_3\theta_j dx\\
&\quad +\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_i u_i\cdot\partial_3\theta_j\cdot\partial_3\theta_j dx\\
&= R_1+R_2+R_3. \qquad (15)
\end{aligned}
$$

Therefore, we have

$$|R_1+R_3|\le\int_{\mathbb{R}^3}|\nabla_h u|\,|\nabla\theta|^2\,dx,$$

and

$$|R_2|\le\int_{\mathbb{R}^3}|\nabla u|\,|\nabla\theta|\,|\nabla_h\theta|\,dx\le\frac{1}{2}\int_{\mathbb{R}^3}|\nabla_h\theta|\,(|\nabla u|^2+|\nabla\theta|^2)dx,$$

where the last inequality is obtained by using Cauchy inequality.

Putting all the inequalities above into (15) yields

$$\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx\le\int_{\mathbb{R}^3}|\nabla_h u|\,|\nabla\theta|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla_h\theta|\,(|\nabla u|^2+|\nabla\theta|^2)dx.$$

Finally, we deal with the term $-\int_{\mathbb{R}^3}(\theta e_3)\cdot\Delta u dx$. By integration by parts and Cauchy inequality, we have

$$-\int_{\mathbb{R}^3}(\theta e_3)\cdot\Delta u dx\le\frac{1}{2}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2).$$

Combining the above estimates, by Hölder's inequality, Nirenberg-Gagliardo's interpolation inequality and (8), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\\
\le\;& C\int_{\mathbb{R}^3}(1+|\nabla_h u|+|\nabla_h\theta|)(|\nabla u|^2+|\nabla\theta|^2)dx\\
\le\;& C(1+\|\nabla_h u\|_{L^2}+\|\nabla_h\theta\|_{L^2})(\|\nabla u\|_{L^4}^2+\|\nabla\theta\|_{L^4}^2)\\
\le\;& C(1+\|\nabla_h u\|_{L^2}+\|\nabla_h\theta\|_{L^2})(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^6}^{\frac{3}{2}}+\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^6}^{\frac{3}{2}})\\
\le\;& C(1+\|\nabla_h u\|_{L^2}+\|\nabla_h\theta\|_{L^2})(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}+\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}^{\frac{1}{2}}).
\end{aligned}
$$

Integrating this last inequality in time, we deduce that for all $\tau \in [\Gamma, t]$

$$\begin{aligned}
\mathcal{J}^2(t) \leq\ & 1 + \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla \theta(\Gamma)\|_{L^2}^2 + C \sup_{\tau \in [\Gamma, t]} (\|\nabla_h u(\tau)\|_{L^2} + \|\nabla_h \theta(\tau)\|_{L^2}) \\
& \times \left(\int_\Gamma^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{1}{4}} \left(\int_\Gamma^t \|\nabla \nabla_h u(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_\Gamma^t \|\Delta u(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{1}{4}} \\
& + C \sup_{\tau \in [\Gamma, t]} (\|\nabla_h u(\tau)\|_{L^2} + \|\nabla_h \theta(\tau)\|_{L^2}) \\
& \times \left(\int_\Gamma^t \|\nabla \theta(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{1}{4}} \left(\int_\Gamma^t \|\nabla \nabla_h \theta(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_\Gamma^t \|\Delta \theta(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{1}{4}} \\
\leq\ & 1 + \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla \theta(\Gamma)\|_{L^2}^2 + 2C\mathcal{L}(t)\epsilon^{\frac{1}{4}} \mathcal{L}(t) \mathcal{J}^{\frac{1}{2}}(t) \\
=\ & 1 + \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla \theta(\Gamma)\|_{L^2}^2 + C\epsilon^{\frac{1}{4}} \mathcal{L}^2(t) \mathcal{J}^{\frac{1}{2}}(t).
\end{aligned} \tag{16}$$

Inserting (14) into (16) and taking ϵ small enough, then it is easy to see that for all $\Gamma \leq t < T^*$, there holds

$$\mathcal{J}^2(t) \leq 1 + \|\nabla u(\Gamma)\|_{L^2}^2 + \|\nabla \theta(\Gamma)\|_{L^2}^2 + C\epsilon^{\frac{1}{4}} \mathcal{J}^{\frac{1}{2}}(t) + C\epsilon^{\frac{1}{2}} \mathcal{J}^2(t) < \infty,$$

which proves

$$\sup_{\Gamma \leq t < T^*} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2) < +\infty.$$

This implies that $(u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3))$. Thus, according to the regularity results in [2], (u, θ) is smooth on $[0, T]$. Then we complete the proof of Theorem 1. $\square$

4. Conclusions

It should be noted that the condition (7) is somewhat stronger than in Reference [6], since it is wort emphasizing that there are no assumptions on the two components velocity field (u_1, u_2). In other words, our result demonstrates that the two components velocity field (u_1, u_2) plays a less dominant role than the one component velocity field does in the regularity theory of solutions to the Boussinesq equations. In a certain sense, our result is consistent with the numerical simulations of Alzmann et al. in Reference [36].

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