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AN EXISTENCE THEOREM FOR GENERALIZED QUASI-VARIATIONAL INEQUALITIES INVOLVING THE GRASSMANNIAN MANIFOLD WITH AN APPLICATION

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Abstract. We present an existence theorem for a class of generalized quasi-variational problem involving Grassmannian manifolds. This class is directly inspired by a general equilibrium problem with time, uncertainty and incomplete financial market with real assets. The problem of the existence of this equilibrium cannot be analyzed using standard techniques employed in similar models. Then, we show how the concept of equilibrium is strictly related to the concept of Grassmannian manifolds. Finally, we present a variational inequality problem, whose solutions are equilibria of the proposed model.

1. Introduction

We present a generalized quasi-variational inequality problem involving Grassmannian spaces, i.e., the families of given dimension vector subspaces of a (finite dimensional) real vector space, endowed with both a topological and smooth abstract manifold structure.

The theory of variational inequalities was introduced in the seventies by Fichera (1964) and Stampacchia (1964), as an innovative and effective method to solve equilibrium problems arising in mathematical physics. Some years later, Bensoussan *et al.* (1973) introduced the quasi-variational inequalities as an important generalization of the variational inequalities. Starting from 1985, this theory was applied to numerous equilibrium problems arising from the applied world as traffic equilibrium problem, spatial price equilibrium problem, oligopolistic market equilibrium problem, general equilibrium problem (see Dafermos (1980), Nagurney (1993) and bibliography therein, De Luca and Maugeri (1989), Maugeri (1987), Jofré *et al.* (2007)). Subsequently, the question of how to introduce time in the variational inequality framework has been investigated (see Daniele *et al.* (1999), Daniele (2006) and bibliography therein). Due to its powerful capacity of application, nowadays variational inequality theory represents an excellent tools in the analysis of unsolved problems in real-life situations. (see *e.g.* Barbagallo *et al.* (2012), Colajanni *et al.* (2018), Donato *et al.* (2018), Donato *et al.* (2020), Donato *et al.* (2014), Scrimali and Mirabella (2018)).

Dedicated to Professor Antonino Maugeri on the occasion of his 75th birthday with best wishes

It is important to stress that a crucial assumption in the variational inequality theory is the convexity of the main involved sets. On the other hand, Grassmannian manifolds are clearly not convex,[1] a fact that makes mathematically interesting the result we get. Indeed, using a fixed point theorem involving Grassmannian manifolds presented by Bich and Cornet (2004), we prove a variational inequality version of that result which is novel in the literature.

Moreover, a general equilibrium economic model with time, uncertainty and financial market with real assets is proposed as a framework to apply the above results. The first paper on that topic is the one by Duffie and Shafer (1985) which are the first authors to introduce Grassmannian in the above mentioned models. After that path-breaking contribution, several other proofs of existence have been provided in the literature. While that paper uses a degree/homotopy argument, Husseini *et al.* (1990) use a fixed point argument on Grassmannian manifolds that generalizes a result by Dierker (1974); on the other hand, Bich and Cornet (2004) provide a more standard fixed point approach (see also Villanacci *et al.* (2002)).

The paper is organized as follows. In Section **2**, definitions and main results on Grassmannian manifolds are presented. In Section **3**, the result on variational inequalities dealing with Grassmannians is presented. Finally, in Section **4** we present the set-up and the main features of the economic model referred to above.

2. Some results on Grassmannian manifolds

In this section some basic results on Grassmannian manifolds are recalled. To reach that goal, we use the classical and standard reference of Milnor and Stasheff (1974). Some results are also taken from Bich and Cornet (2004). Milnor and Stasheff (1974) uses an atlas different from ours. The proof that our proposed triple is indeed an atlas and that $\mathscr{G}_{A,S}$ satisfies all properties we need in our analysis is not presented in a complete manner in any published (or unpublished) work we know the existence of.

We proceed as follows. We first present the definition of Grassmannian set $\mathscr{G}_{A,S}$; then we endow that set with a topology and show that it is a Hausdorff topological space. Then, we recall the definition of abstract C^∞ manifold and show that $\mathscr{G}_{A,S}$ satisfies all the requirements of that definition; finally, we show that $\mathscr{G}_{A,S}$ has some "nice" properties which are crucial in our analysis (in particular, it is a metrizable topological space).

Definition 1. *Given $A,S \in \mathbb{N}$ with $A \leq S$, we denote by $\mathscr{G}_{A,S}$ the set of all A dimensional vector subspaces of $\mathbb{R}^S$. $\mathscr{G}_{A,s}$ is called a Grassmannian set.*

Definition 2. *An A-frame in $\mathbb{R}^S$ is a collection of A linearly independent vectors in $\mathbb{R}^S$. We denote by $V_A\left(\mathbb{R}^S\right)$ the family of all A-frames. $V_A\left(\mathbb{R}^S\right)$ is called Stiefel manifold.*

We can identify $V_A\left(\mathbb{R}^S\right)$ with the family $\mathbb{M}^*(S,A)$ of full rank $S \times A$ matrices.

Proposition 3. *$V_A\left(\mathbb{R}^S\right)$ is $\mathscr{C}^\infty$ diffeomorphic to an open subset of $\mathbb{R}^{SA}$ (with the Euclidean topology).*

[1] Indeed, it is enough to observe that "a convex combination of the horizontal and vertical axis in the plane is equal to the plane itself".

Proof. Given $M \in \mathbb{M}(S,A)$, let $\mathscr{F}$ be the family of all square submatrices of M. Define $F : \mathbb{M}(S,A) \to \mathbb{R}$ such that $F(M) = \sum_{M^* \in \mathscr{F}} |\det M^*|$. Then F is continuous and $\mathbb{M}^*(S,A) = F^{-1}(\mathbb{R}\backslash\{0\})$ is therefore open. □

We now endow $\mathscr{G}_{A,S}$ with a topology using the Euclidean topology of $V_A\left(\mathbb{R}^S\right)$.

Definition 4. *Given the function* $q : V_A\left(\mathbb{R}^S\right) \to \mathscr{G}_{A,S}$ *such that*

$$q(x_1,...,x_A) := \operatorname{span}(x_1,...,x_A),$$

we say that a set $U \subseteq \mathscr{G}_{A,S}$ *is open if and only if* $q^{-1}(U)$ *is open in* $V_A\left(\mathbb{R}^S\right)$. *Let* $\mathscr{T}$ *be the family of so defined open sets in* $\mathscr{G}_{A,S}$.

Proposition 5. $(\mathscr{G}_{A,S}, \mathscr{T})$ *is a topological space.*

Proof. Clearly $\varnothing = q^{-1}(\varnothing)$ and $V_A\left(\mathbb{R}^S\right) = q^{-1}(\mathscr{G}_{A,S})$. Furthermore, take a family $\left\{S_\gamma : \gamma \in \Gamma\right\}$ of $\mathscr{G}_{A,S}$ - open sets. Then, for any $\gamma \in \Gamma$, $q^{-1}\left(S_\gamma\right)$ is $V_A\left(\mathbb{R}^S\right)$ - open. We get our desired result observing that $q^{-1}\left(\cup_{\gamma\in\Gamma} S_\gamma\right) = \cup_{\gamma\in\Gamma} q^{-1}\left(S_\gamma\right)$ is open and $q^{-1}\left(\cap_{\gamma\in\Gamma} S_\gamma\right) = \cap_{\gamma\in\Gamma} q^{-1}\left(S_\gamma\right)$ is open if $|\Gamma| \in \mathbb{N}$. □

Remark 6. *By definition 4, q is a continuous function. Moreover, it is easy to show that q is onto: for any* $L \in \mathscr{G}_{A,S}$, *take a basis* $\mathscr{B}$ *of L; then,* $q(\mathscr{B}) = L$.

We can also give the following alternative description of $\mathscr{G}_{A,S}$.

Definition 7. *We denote by* $V_A^0\left(\mathbb{R}^S\right)$ *the family of all orthonormal A-frames.*

We can then endow $\mathscr{G}_{A,S}$ with a topology using $V_A^0\left(\mathbb{R}^S\right)$.

Definition 8. *Given the function* $q_0 : V_A^0\left(\mathbb{R}^S\right) \to \mathscr{G}_{A,S}$ *such that*

$$q_0(x_1^0,...,x_A^0) := \operatorname{span}\left(x_1^0,...,x_A^0\right),$$

we say that a set $U \subseteq \mathscr{G}_{A,S}$ *is open if and only if* $q_0^{-1}(U)$ *is open in* $V_A^0\left(\mathbb{R}^S\right)$. *Let* $\mathscr{T}_0$ *be the family of so defined open sets in* $\mathscr{G}_{A,S}$.

We now want to show that $\mathscr{T} = \mathscr{T}_0$. To accomplish that goal we need some preliminary results.

Definition 9. *Define the inclusion map from* $V_A^0\left(\mathbb{R}^S\right)$ *to* $V_A\left(\mathbb{R}^S\right)$ *as*
$in : V_A^0\left(\mathbb{R}^S\right) \to V_A\left(\mathbb{R}^S\right)$ *such that* $in(x_1^0,...,x_A^0) = (x_1^0,...,x_A^0)$
and the Gram-Schmidt function
$g : V_A\left(\mathbb{R}^S\right) \to V_A^0\left(\mathbb{R}^S\right)$ *such that* $g(x_1,...,x_A) = (x_1^0,...,x_A^0)$,
where $\left(x_1^0,...,x_A^0\right)$ *is obtained using the Gram-Schmidt orthonormalization process.*

Remark 10. *From standard linear algebra (see for example Section 6.6, page 212 of Lipschutz (1991),), the function g is continuous. Moreover,* $(x_1,...,x_A) \in V_A\left(\mathbb{R}^S\right)$ *and* $(x_1^0,...,x_A^0) \in V_A^0\left(\mathbb{R}^S\right)$ *are a basis of the same vector space.*

Proposition 11. *The following diagram commutes*[2].

$$\begin{array}{ccccc} V_A^0\left(\mathbb{R}^S\right) & \xrightarrow{in} & V_A\left(\mathbb{R}^S\right) & \xrightarrow{g} & V_A^0\left(\mathbb{R}^S\right) \\ q_0\downarrow & & q\downarrow & & q_0\downarrow \\ \mathscr{G}_{A,S} & \xrightarrow{id} & \mathscr{G}_{A,S} & \xrightarrow{id} & \mathscr{G}_{A,S} \end{array}$$

Proof. The result follows from the definition of $q_0 := q_{|V_A^0(\mathbb{R}^S)}$, the definition of g and Remark 10. In particular, $q = q_0 \circ g$, which is a consequence of the basic fact that the span of a basis and the span of the orthonormal basis obtain from it, using Gram-Schmidt, do coincide (see also Remark 2 page 213 of Lipschutz (1991)). □

Remark 12. *By definition 8, q_0 is a continuous function. Moreover, q_0 is onto, because of the following simple argument. Since q is onto, for any $L \in \mathscr{G}_{A,S}$, there exists $(x_1,...,x_A) \in V_A\left(\mathbb{R}^S\right)$ such that* $\operatorname{span}(x_1,...,x_A) = L$. *From Remark 10, there exists $\left(x_1^0,...,x_A^0\right) \in V_A^0\left(\mathbb{R}^S\right)$ such that* $\operatorname{span}\left(x_1^0,...,x_A^0\right) = \operatorname{span}(x_1,...,x_A) = L$, *i.e., $q_0\left(x_1^0,...,x_A^0\right) = L$, as desired.*

Proposition 13. $\mathscr{T} = \mathscr{T}_0$.

Proof. Indeed, $S \in \mathscr{T}_0 \Leftrightarrow q_0^{-1}(S)$ is open in $V_A^0\left(\mathbb{R}^S\right) \Leftrightarrow^3 q^{-1}(S) = g^{-1}\left(q_0^{-1}(S)\right)$ is open in $V_A\left(\mathbb{R}^S\right) \Leftrightarrow S \in \mathscr{T}$. □

Lemma 14. *(Checcucci* et al. *(1968), Proposition 8.1) Let $(Y,\mathscr{T}')$ be a topological space, given $f : \mathscr{G}_{A,S} \to (Y,\mathscr{T}')$ and $q : V_A(\mathbb{R}^S) \to \mathscr{G}_{A,S}$ (continuous function), then f is continuous $\Leftrightarrow f \circ q$ is continuous.*

Proof.

$$V_A\left(\mathbb{R}^S\right) \xrightarrow{q} \mathscr{G}_{A,S} \xrightarrow{f} (Y,\mathscr{T}'), \qquad V_A\left(\mathbb{R}^S\right) \xrightarrow{f\circ q} (Y,\mathscr{T}')$$

[⇒] From Remark 6, q is continuous. Then $f \circ q$ is continuous because composition of continuous functions.

[⇐] We want to show that if A is Y- open, then $f^{-1}(A)$ is $\mathscr{G}_{A,S}$ - open. Indeed, since A is open and $f \circ q$ is continuous, then $(f \circ q)(A) = q^{-1}\left(f^{-1}(A)\right)$ is open in $V_A\left(\mathbb{R}^S\right)$. From Definition 4, $f^{-1}(A)$ is $\mathscr{G}_{A,S}$ - open, as desired. □

We need some preliminary results.

Proposition 15. *(Ostaszewski (1990), page 41) For any real matrix $M \in \mathbb{M}(m,n)$,*[4]

$$\operatorname{Im} M^T = (\ker M)^{\perp}.$$

Proposition 16. *If $L \in \mathscr{G}_{A,S}$, then $L^{\perp} \in \mathscr{G}_{S-A,S}$.*

Proof. The desired result is an immediate consequence of the following basic linear algebra results (see, for example, Theorem 6, page 436, of Nicholson (1990)). For any $L \in \mathscr{G}_{A,S}$, $\mathbb{R}^S = L \oplus L^{\perp}$ and $\mathbb{R}^S = \dim L + \dim L^{\perp}$. □

[2] A commutative diagram is a collection of functions $\{f_i : A_i \to B_i : I = 1,...,n\}$ in which all function compositions starting from the same set and ending with the same set give the same result.

[3] It follows from the fact that g is continuous.

[4] Below, we identify a matrix $M \in \mathbb{M}(m,n)$ with $l_M \in \mathscr{L}(\mathbb{R}^n,\mathbb{R}^m)$.

Corollary 17. *Given $M \in \mathbb{M}(m,n)$, then*

$$\mathbb{R}^n = \mathrm{Im} M^T \oplus \ker M.$$

Proposition 18. *Let M and $M' \in \mathbb{M}^*(S-A,A)$ be given. Then,* $\ker M = \ker M'$ *if and only if there exists $B \in \mathbb{M}^*(S-A,S-A)$ such that $M' = BM$.*

Proof. [⇒] Define $L := \ker M = \ker M'$. Then,

$$L^\perp = (\ker M)^\perp = \left(\ker M'\right)^\perp = \mathrm{Im} M^T = \mathrm{Im}\left(M'\right)^T.$$

Since $M, M' \in \mathbb{M}^*(S-A,S)$, then $M^T, (M')^T \in \mathbb{M}^*(S,S-A)$ and their $S-A$ columns are a basis of $L^\perp$. Then, the rows of M and M' are a basis of $L^\perp$. Defined

$$M = \begin{bmatrix} m_1 \\ \dots \\ m_{S-A} \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} m'_1 \\ \dots \\ m'_{S-A} \end{bmatrix},$$

by definition of basis each row m_i can be written as a linear combination of the rows $\left(m'_1, \dots, m'_{S-A}\right)$ through well chosen, uniquely determined vectors $b_i \in \mathbb{R}^{S-A}$. In other words,

$$\begin{bmatrix} m_1 \\ \dots \\ m_{S-A} \end{bmatrix} = \begin{bmatrix} b_1 \cdot M' \\ \dots \\ b_{S-A} \cdot M' \end{bmatrix},$$

and then defined

$$B = \begin{bmatrix} b_1 \\ \dots \\ b_{S-A} \end{bmatrix},$$

we get $M = BM'$. We are then left with showing that B is invertible. Indeed,

$$S-A = \mathrm{rank} M = \mathrm{rank}\ BM' \leq \min\left\{\mathrm{rank}\ B, \mathrm{rank}\ M'\right\}.$$

If our claim were false we would have $S-A \leq \min\{\mathrm{rank}\ B, \mathrm{rank}\ M'\} < S-A$, a contradiction.

[⇐]

We first show that $\ker M \subseteq \ker M'$.

$x \in \ker M \Leftrightarrow Mx = 0 \Rightarrow BMx = 0 \overset{BM=M'}{\Leftrightarrow} M'x = 0 \Leftrightarrow x \in \ker M'$.

We now show that $\ker M' \subseteq \ker M$.

$x \in \ker M' \Leftrightarrow M'x = 0 \overset{BM=M'}{\Leftrightarrow} BMx = 0 \overset{B \text{ invertible}}{\Rightarrow} Mx = 0.$ □

Proposition 19. *Let $Y, Y' \in M^*(S,A)$ be given. Then*

$$\mathrm{Im} Y = \mathrm{Im} Y' \Leftrightarrow \text{ there exists } C \in \mathbb{M}^*(A,A) \text{ such that } Y' = YC.$$

Proof. [⇒]

From, Proposition 15, we have that

$$(\ker Y)^\perp = \mathrm{Im} Y^T,$$

and therefore,

$$\ker Y^T = (\mathrm{Im} Y)^\perp = \left(\mathrm{Im} Y'\right)^\perp = \ker(Y')^T.$$

Then, since $Y, Y' \in M^*(S,A)$, from Proposition 18,

$$\text{there exists } \widehat{B} \in \mathbb{M}^*(A,A) \text{ such that } Y'^T = \widehat{B} Y^T.$$

and

$$Y' = Y \widehat{B}^T,$$

and it is then enough to take $C = \widehat{B}^T$.

[$\Leftarrow$]

We first show that $\operatorname{Im} Y \subseteq \operatorname{Im} Y'$. $w \in \operatorname{Im} Y \Rightarrow \exists z \in \mathbb{R}^A$ such that $w = Yz = Y'C^{-1}z \Rightarrow \exists z' = C^{-1}z \in \mathbb{R}^A$ such that $w = Y'z' \Leftrightarrow w \in \operatorname{Im} Y'$.

We now show that $\operatorname{Im} Y' \subseteq \operatorname{Im} Y$. $w' \in \operatorname{Im} Y' \Rightarrow \exists z' \in \mathbb{R}^A$ such that $w' = Y'z' = YCz' \Rightarrow \exists z = Cz' \in \mathbb{R}^A$ such that $w' = Yz \Leftrightarrow w' \in \operatorname{Im} Y$. □

In order to prove that $\mathscr{G}_{A,S}$ is a Hausdorff space, we need to introduce the following preliminary results (for details on the results we do not provide proofs of, see Munkres (1975)).

Definition 20. *A topological space $(X, \mathscr{T})$ is a T_1 space if*

$$x, y \in X,\ x \neq y \Rightarrow \exists\ G_x, G_y \in \mathscr{T} \text{ such that } x \in G_x,\ y \notin G_x \text{ and } x \notin G_y,\ y \in G_y,$$

where the open sets G_x and G_y are not necessarily disjoint.

Proposition 21. *Let a topological space $(X, \mathscr{T})$ be given.*

$$(X, \mathscr{T}) \text{ is } T_1 \ \Leftrightarrow\ \forall\, x \in X,\ \{x\} \text{ is closed}.$$

Definition 22. *A topological space $(X, \mathscr{T})$ is a T_2 or Hausdorff space if*

$$x, y \in X,\ x \neq y \ \Rightarrow\ \exists G_x, G_y \in \mathscr{T} \text{ such that } x \in G_x, y \in G_y \text{ and } G_x \cap G_y = \varnothing.$$

Remark 23. *If $(X, \mathscr{T})$ is T_2, then $(X, \mathscr{T})$ is T_1.*

Definition 24. *A topological space $(X, \mathscr{T})$ is a regular space if*

(1) *it is T_1 (Munkres 1975);*
(2) *F closed, $x \notin F \Rightarrow \exists\ G_x,\ G_F \in \mathscr{T}$ such that $x \in G_x,\ F \subseteq G_F$ and $G_x \cap G_F = \varnothing$.*

Definition 25. *A topological space $(X, \mathscr{T})$ is a normal space if*

(1) *it is T_1 (Munkres 1975);*
(2) *F_1 and F_2 are disjoint closed sets $\Rightarrow \exists\ G_1,\ G_2 \in \mathscr{T} :\ F_1 \subseteq G_2,\ F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$.*

Remark 26. *If a space is normal, then it is regular.*

Definition 27. *Let (X, d) be a metric space. A set U in X is open with respect to d or it is (X, d) open if for any $x \in U$, there exists $\delta > 0$ such that $B_{(X,d)}(x, \delta) := \{y \in X : d(y, x) < \delta\} \subseteq U$. The topology induced by the metric d on X is the collection of sets in X which are (X, d) open.*

Definition 28. *Let X be a topological space. X is said to be metrizable if there exists a metric d on the set X that induces the topology of X.*

Definition 29. *Let X and Y be arbitrary nonempty sets. Then, the family of functions $\mathscr{S} := \{f_i : X \to Y\}_{i \in I}$ is said to separate points (in X) if $\forall\ a,\ b \in X$ such that $a \neq b$, $\exists\ i \in I$ such that $f_i(a) \neq f_i(b)$.*

Proposition 30. *Let $(X, \mathscr{T})$ be a topological space. If $\mathscr{C}^0(X, \mathbb{R})$ separates points, then $(X, \mathscr{T})$ is Hausdorff.*

To prove that $\mathscr{G}_{A,S}$ is Hausdorff, we apply Proposition 30, i.e., we find a continuous function $f : \mathscr{G}_{A,S} \to \mathbb{R}$ such that for any $L_1, L_2 \in \mathscr{G}_{A,S}$ with $L_1 \neq L_2$, we have $f(L_1) \neq f(L_2)$. Observe that f may depend on L_1 and L_2. The desired function f is defined as follows. Taken $v \in \mathbb{R}^S$, $f_v : \mathscr{G}_{A,S} \to \mathbb{R}$, $f_v(L) = d(v, L)$, where $d(v, L)$ is the distance of v from L. To prove the desired results we present some Lemmas (see Section 3.6, pages 55-58 of Luenberger (1969).)

Given $z_1, ..., z_m$ vectors in $\mathbb{R}^S$, define the continuous function

$$g : \left(\mathbb{R}^S\right)^m \to \mathbb{R}, \quad g(z_1, ..., z_m) = \det\left[\langle z_i, z_j \rangle_S\right]_{m \times m}.$$

Proposition 31. *(Proposition 1, page 56 of Luenberger (1969)) $g(z_1, ..., z_m) \neq 0$ if and only if $z_1, ..., z_m$ are linearly independent.*

Proposition 32. *(Theorem 1, page 57 of Luenberger (1969)) Let $y_1, ..., y_A$ be linearly independent vectors in $\mathbb{R}^S$. The distance δ between $x \in \mathbb{R}^S$ and the vector space* span $(y_1, ..., y_A) \in \mathscr{G}_{S,A}$ *is such that*

$$\delta^2 = \frac{g(y_1, ..., y_A, x)}{g(y_1, ..., y_A)}$$

Remark 33. *Let $\left(\mathbb{R}^{SA}\right)^* := \left\{y_1, ..., y_A \in \left(\mathbb{R}^S\right)^A : y_1, ..., y_A \text{ are linearly independent}\right\}$. The following function*

$f : \left(\mathbb{R}^{SA}\right)^* \times \mathbb{R}^S \to \mathbb{R}$ *such that* $f(y_1, ..., y_A, x) = \left(\frac{g(y_1, ..., y_A, x)}{g(y_1, ..., y_A)}\right)^{\frac{1}{2}}$

is continuous. Furthermore, if $x \in$ span $(y_1, ..., y_A)$, *then $f(y_1, ..., y_A, x) = 0$, simply because x is a linear combination of $y_1, ..., y_A$ and then $(y_1, ..., y_A, x)$ are linearly dependent.*

Proposition 34. *$\mathscr{G}_{A,S}$ is a Hausdorff topological space.*

Proof. Let $L_1, L_2 \in \mathscr{G}_{A,S}$ such that $L_1 \neq L_2$ be given. For any $v \in \mathbb{R}^S$, define

$f_v : \mathscr{G}_{A,S} \to \mathbb{R}$ such that $f_v(L) = d(v, L)$.

Then, from Lemma 14, f_v is continuous iff

$$f_v \circ q : V_A\left(\mathbb{R}^S\right) \to \mathbb{R}, \quad (f_v \circ q)(x_1, ..., x_A) = d(\text{span}(x_1, , ..., x_A), v) = \left(\frac{g\left(\widehat{\widehat{y}}_1, ..., \widehat{\widehat{y}}_A, x\right)}{g\left(\widehat{\widehat{y}}_1, ..., \widehat{\widehat{y}}_A\right)}\right)^{\frac{1}{2}}$$

is continuous, which is the case for what said above.

Now, take $x \in L_1$ such that $x \notin L_2$. Then $f_x(L_1) = 0$. Observe that if C is closed and $x \notin C$, then $d(x, C) > 0$. Then since L_2 is a finite dimensional vectors space and therefore it is closed, we have that $f_x(L_2) > 0$, as desired. □

We are now ready to show that $\mathscr{G}_{A,S}$ is a C^∞ manifold. For the redear's convenience we present below that definition.

Definition 35. *A topological space M is a said to be an m-dimensional topological manifold if there exists a collection of triples $(\phi_i, U_i, V_i)_{i\in I}$ such that*

(1) *$\{V_i\}_{i\in I}$ is an open covering of M, i.e., for any $i \in I$, V_i is open in M and $M \subseteq \cup_{i\in I} V_i$;*
(2) *for each $i \in I$, U_i is an open subset of $\mathbb{R}^m$;*
(3) *for each $i \in I$, $\phi_i : U_i \to V_i$ is a homeomorphism.*

Every triple (ϕ_i, U_i, V_i) is called a local parametrization of M and the set of local parametrizations is called a system of local parametrizations. Conversely, (ψ_i, V_i, U_i), where $\psi_i \equiv \phi_i^{-1}$, is called a chart of M and $\{(\psi_i, V_i, U_i)\}_{i\in I}$ is called an atlas of M.

Definition 36. *Let M be an m-dimensional topological manifold. Consider an atlas $\{(\psi_i, V_i, U_i)\}_{i\in I}$ of M, and let $r \in \mathbb{N} \cup \{+\infty\}$. M is said to be an m-dimensional C^r manifold if, for every $i, j \in I$ such that $V_i \cap V_j \neq \emptyset$,*

$$\psi_j \circ \left(\psi_i^{-1}\right)_{|\psi_i(V_i\cap V_j)} : \psi_i(V_i \cap V_j) \subseteq \mathbb{R}^m \to \psi_j(V_i \cap V_j) \subseteq \mathbb{R}^m \tag{1}$$

is a C^r diffeomorphism.

The family $\{(\psi_i, V_i, U_i)\}_{i\in I}$ is called a C^r atlas of M and every triple (ψ_i, V_i, U_i) is a C^r chart. The corresponding parametrizations (ϕ_i, U_i, V_i) are called C^r local parametrizations.

The proof that $\mathscr{G}_{A,S}$ is a $\mathscr{C}^\infty$ abstract manifold requires some preliminary work and it is finally presented in Proposition 46.

Observe that $\overline{L} \in \mathscr{G}_{A,S}$, $\left(\overline{L}^{\perp}\right)^A$, i.e., the Cartesian product A times of $\overline{L}^{\perp}$, is isomorphic to $\mathbb{R}^{(S-A)A}$.

An atlas for $\mathscr{G}_{A,S}$ can be constructed as follows. Given an orthonormal basis $(\overline{f}^1, ..., \overline{f}^A)$ of $\overline{L}$, we can define

$$\varphi_{\overline{L}} : \left(\overline{L}^{\perp}\right)^A \to \mathscr{G}_{A,S},$$

$$\varphi_{\overline{L}}\left(u^1, ..., u^A\right) = \operatorname{span}\left(\overline{f}^1 + u^1, ..., \overline{f}^A + u^A\right).$$

Observe that the above function is well defined, i.e., $\varphi_{\overline{L}}\left(u^1, ..., u^A\right) \in \mathscr{G}_{A,S}$, as verified below.

Lemma 37. *Given an orthonormal basis $(\overline{f}^1, ..., \overline{f}^A)$ of $\overline{L} \in \mathscr{G}_{A,S}$ and $\left(u^1, ..., u^A\right) \in \left(\overline{L}^{\perp}\right)^A$, then*

$$\operatorname{span}\left((\overline{f}^1 + u^1, ..., \overline{f}^A + u^A)\right) \in \mathscr{G}_{A,S}.$$

Proof. To get the desired result, it suffices to show that $(\overline{f}^1 + u^1, ..., \overline{f}^A + u^A)$ is linearly independent, i.e., that $\sum_{a=1}^A \beta_a \left(\overline{f}^a + u^a\right) = 0$ implies that for any $a \in \{1, .., A\}$, $\beta_a = 0$. Indeed, $\sum_{a=1}^A \beta_a \left(\overline{f}^a + u^a\right) = 0$ implies that $\sum_{a=1}^A \beta_a \overline{f}^a \cdot \overline{f}^1 = -\sum_{a=1}^A \beta_a u^a \cdot \overline{f}^1$. Moreover, $\sum_{a=1}^A \beta_a \overline{f}^a \cdot f^1 = \beta_1$, because $(\overline{f}^1, ..., \overline{f}^A)$ is an orthonormal basis, and $-\sum_{a=1}^A \beta_a u^a \cdot \overline{f}^1 = 0$, because $\overline{f}^1 \in \overline{L}$ and $\left(u^1, ..., u^A\right) \in \left(\overline{L}^{\perp}\right)^A$. Hence $\beta_1 = 0$; similar procedure can be used to show that $\beta_2 = ... = \beta_A = 0$, as desired. □

Proposition 38. *The function $\varphi_{\overline{L}}$ is one-to-one and continuous.*

Proof. To prove that $\varphi_{\overline{L}}$ is one-to-one, we want to show that if

$$\varphi_{\overline{L}}\left(u^1,...,u^A\right) := \text{span}\left((\overline{f}^1+u^1,...,\overline{f}^A+u^A)\right)$$

$$= \text{span}\left((\overline{f}^1+v^1,...,\overline{f}^A+v^A)\right) := \varphi_{\overline{L}}\left(v^1,...,v^A\right)$$

then $\left(u^1,...,u^A\right) = \left(v^1,...,v^A\right)$.

Since for any $a \in \{1,...,A\}$, we have that $\overline{f}^a + u^a \in \text{span}\left((\overline{f}^1+v^1,...,\overline{f}^A+v^A)\right)$, then there exists $(\lambda_{ai})_{i=1}^A \in \mathbb{R}^A$ such that

$$\overline{f}^a + u^a = \sum_{i=1}^{A} \lambda_{ai}\left(\overline{f}^i + v^i\right) \tag{2}$$

Multiplying by $\overline{f}^a$, we get $\langle \overline{f}^a, \overline{f}^a\rangle + \langle u^a, \overline{f}^a\rangle = \langle \sum_{i=1}^A \lambda_{ai}(\overline{f}^i + v^i), \overline{f}^a\rangle$,

and using the assumptions that $\left(\overline{f}^1,...,\overline{f}^A\right)$ is a orthonormal basis of $\overline{L}$ and

$$\left(u^1,...,u^A\right), \left(v^1,...,v^A\right) \in \left(\overline{L}^{\perp}\right)^A, \tag{3}$$

we get

$$1 = \lambda_{aa}. \tag{4}$$

Multiplying (2) by $\overline{f}^j$, with $j \in \{1,...,A\} \setminus \{a\}$, we get

$$\overline{f}^a \overline{f}^j + u^a \overline{f}^j = \sum_{i=1}^{A} \lambda_{ai}\left(\overline{f}^i \overline{f}^j + v^i \overline{f}^j\right),$$

and again using the assumptions that $\left(\overline{f}^1,...,\overline{f}^A\right)$ is a orthonormal basis of $\overline{L}$ and (3), we get

$$\text{for any } j \in \{1,...,A\} \setminus \{a\}, \quad 0 = \lambda_{aj}. \tag{5}$$

Inserting (4) and (5) in (2), we get

$$\overline{f}^a + u^a = \overline{f}^a + v^a.$$

Repeating the same argument for any $a \in \{1,...,A\}$, we get the desired result.

To prove that $\varphi_{\overline{L}}$ is continuous, define $\gamma : \left(\overline{L}^{\perp}\right)^A \to V_A\left(\mathbb{R}^S\right)$ such that

$$\gamma(u^1,...,u^A) := (\overline{f}^1+u^1,...,\overline{f}^A+u^A).$$

Then $\varphi_{\overline{L}} = q \circ \gamma$:

$$\begin{array}{ccccc} \left(\overline{L}^{\perp}\right)^A & \xrightarrow{\gamma} & V_A\left(\mathbb{R}^S\right) & \xrightarrow{q} & \mathscr{G}_{A,S} \\ \left(u^1,...,u^A\right) & \mapsto & (\overline{f}^1+u^1,...,\overline{f}^A+u^A) & \mapsto & \text{span}\left((\overline{f}^1+u^1,...,\overline{f}^A+u^A)\right) \end{array}$$

Since q and γ are continuous, then $\varphi_{\overline{L}}$ is continuous. □

Lemma 39. *(see page 169 of Bich and Cornet (2004)) For any* $u \in \left(\left(\overline{L}\right)^{\perp}\right)^{A}$, $\varphi_{\overline{L}}(0) \cap (\varphi_{\overline{L}}(u))^{\perp} = \{0\}$ *and* $\varphi_{\overline{L}}(u) \cap (\varphi_{\overline{L}}(0))^{\perp} = \{0\}$.

Moreover, we define

$$U_L^* := \varphi_L\left(\left(L^{\perp}\right)^{A}\right) := \left\{\operatorname{span}\left((\overline{f}^1 + u^1, ..., \overline{f}^A + u^A)\right) : \ (u^1, ..., u^A) \in \left(L^{\perp}\right)^{A}\right\} \subseteq \mathscr{G}_{A,S}$$

and, using the fact that $\varphi_{\overline{L}}$ is one-to-one , $\psi_{\overline{L}} : U_{\overline{L}}^* \to \left(\overline{L}^{\perp}\right)^{A}$ such that

$$\psi_{\overline{L}}(L) = (u^1, ..., u^A) \text{ with } \operatorname{span}\left(\overline{f}^1 + u^1, ..., \overline{f}^A + u^A\right) = L.$$

Proposition 40. *$\psi_{\overline{L}}$ is the inverse function of $\varphi_{\overline{L}}$ and it is continuous.*

Proof. Firstly, we want to show that $\psi_{\overline{L}} \circ \varphi_{\overline{L}} = id_{\left(\overline{L}^{\perp}\right)^{A}}$ and $\varphi_{\overline{L}} \circ \psi_{\overline{L}} = id_{U_{\overline{L}}^*}$.

(1) $(\psi_{\overline{L}} \circ \varphi_{\overline{L}})\left(u^1, ..., u^A\right) = \psi_{\overline{L}}\left(\operatorname{span}\left(\overline{f}^1 + u^1, ..., \overline{f}^A + u^A\right)\right) = (u^1, ..., u^A)$.

(2) $(\varphi_{\overline{L}} \circ \psi_{\overline{L}})(L') = \varphi_{\overline{L}}(u^1, ..., u^A)$ such that $\operatorname{span}\left(\overline{f}^1 + u^1, ..., \overline{f}^A + u^A\right) = L'$.
Then $\varphi_{\overline{L}}(u^1, ..., u^A) = \operatorname{span}(\overline{f}^1 + u^1, ..., \overline{f}^A + u^A) = L'$, hence $(\varphi_{\overline{L}} \circ \psi_{\overline{L}})(L') = L'$.

Now, we define $g = \psi_{\overline{L}} \circ q$:

$$\begin{array}{ccccc} V_A\left(\mathbb{R}^S\right) & \xrightarrow{q} & \mathscr{G}_{A,S} & \xrightarrow{\psi_{\overline{L}}} & \left(\overline{L}^{\perp}\right)^{A} \\ \left(x^1, ..., x^A\right) & \mapsto & \operatorname{span}\left(x^1, ..., x^A\right) & \mapsto & (u^1, ..., u^A) \text{ such that } \operatorname{span}\left((x^1, ..., x^A)\right) = \\ & & & & = \operatorname{span}\left((\overline{f}^1 + u^1, ..., \overline{f}^A + u^A)\right) \end{array}$$

If we show that g is continuous, then from Lemma 14 , we do have that $\varphi_{\overline{L}}$ is continuous as desired.

We want to show that g above defined is a continuous function of $\left(x^1, ..., x^A\right)$. For simplicity, define $F := \left(\overline{f}^1, ..., \overline{f}^A\right)$, $X := \left(x^1, ..., x^A\right)$ and $U := (u^1, ..., u^A)$. From the definition of g, we do have $\operatorname{span}(X) = \operatorname{span}(F + U)$. Then, from Proposition 19 , there exists an invertible matrix C such that $X = (F + U)C$ and then $UC = X - FC$. Hence

$$g(X) = U = XC^{-1} - F,$$

and therefore g is a continuous function of X. □

Let $L \in \mathscr{G}_{A,S}$ be given. Since $\mathbb{R}^S = L \oplus L^{\perp}$, for any $v \in \mathbb{R}^S$, there exists a unique $(v_L, v_{L^{\perp}}) \in L \times L^{\perp}$ such that $v = v_L + v_{L^{\perp}}$ and the following definitions are well given.

The projections on L and $L^{\perp}$ are defined as follows, respectively,

$$\pi_L := \pi : \mathbb{R}^S = L \oplus L^{\perp} \to L, \text{ such that } \pi(v) = v_L,$$
$$\pi_{L^{\perp}} := \pi_{\perp} : \mathbb{R}^S = L \oplus L^{\perp} \to L^{\perp}, \text{ such that } \pi_{\perp}(v) = v_{L^{\perp}}.$$

Furthermore,

$$U_L := \left\{ L' \in \mathscr{G}_{A,S} : L' \cap L^{\perp} = \{0\} \right\}.$$

Proposition 41. *$L' \cap L^{\perp} = \{0\}$ if and only if $L' \oplus L^{\perp} = \mathbb{R}^S$ and therefore*

$$U_L = \left\{ L' \in \mathscr{G}_{A,S} : L' \oplus L^{\perp} = \mathbb{R}^S \right\}.$$

Proof. As it is well known, if W_1 and W_2 are vector subspaces of a vector space V, then

$$V = W_1 \oplus W_2 \;\Leftrightarrow\; V = W_1 + W_2 \text{ and } W_1 \cap W_2 = \{0\} \quad .$$

(1) If $L' \oplus L^{\perp} = \mathbb{R}^S$, then $L' \cap L^{\perp} = \{0\}$, as recalled above.
(2) If $L' \cap L^{\perp} = \{0\}$, then we have to prove that $L' \oplus L^{\perp} = \mathbb{R}^S$.
It suffices to show that $L' + L^{\perp} = \mathbb{R}^S$.
From Proposition 16, $L^{\perp} \in \mathscr{G}_{S-A,S}$. Let $\left(v^1, ..., v^A\right)$ be a basis of L' and $\left(v^{A+1}, ..., v^{A+(S-A)=S}\right)$ a basis of $L^{\perp}$. We want to show that

$$\sum_{i=1}^{A} \alpha_i v^i + \sum_{j=1}^{S-A} \beta_{A+j} v^{A+j} = 0 \;\Rightarrow\; \alpha^1 = ... = \alpha^A = \beta^{A+1} = ... = \beta^S = 0.$$

Indeed, $\sum_{i=1}^{A} \alpha_i v^i = -\sum_{j=1}^{S-A} \beta_{A+j} v^{A+j} \in \{0\} = L' \cap L^{\perp}$. Then,

$$\sum_{i=1}^{A} \alpha_i v^i = 0 \text{ and } \sum_{j=1}^{S-A} \beta_{A+j} v^{A+j} = 0$$

and the desired result follows from the assumption that $\left(v^1, ..., v^A\right)$ is a basis of L' and $\left(v^{A+1}, ..., v^{A+(S-A)=S}\right)$ a basis of $L^{\perp}$.

□

Proposition 42. *Let $L' \in \mathscr{G}_{A,S}$ be such that $L' \cap L^{\perp} = \{0\}$. Then, the restriction of π_L to $L' \subseteq \mathbb{R}^S$, i.e., $\pi_{L|L'}$ denoted by $\pi_{|\,L'} : L' \to L$, such that $\pi_{|L'}(v) = v_L$ is an isomorphism.*

Proof. We have to prove that $\pi_{|L'}$ is linear, one-to-one and onto.

(1) $\pi_{|\,L'}$ is linear because it is the restriction of a linear function to a vector subspace of the domain.
(2) $\pi_{|\,L'}$ is one-to-one. We want to show for any $v', v'' \in L'$, $\pi_L(v') = \pi_L(v'')$ implies $v' = v''$.
Observe that if $v' \in L'$, then $\pi_{|L'}(v') := \pi_{L\,|L'}(v') = \pi_L(v')$. Moreover,

there exists a unique $\left(v^1, v^1_{\perp}\right) \in L \times L^{\perp}$ such that $v' = v^1 + v^1_{\perp}$

and

there exists a unique $\left(v^2, v^2_{\perp}\right) \in L \times L^{\perp}$ such that $v'' = v^2 + v^2_{\perp}$.

Then, $\pi_L(v') = \pi_L(v'')$ implies $v^1 = v^2 := v^*$ and then

$$L \ni v^* = v' - v^1_{\perp} = v'' - v^2_{\perp}.$$

Since $L' \cap L^{\perp} = \{0\}$, then from Lemma 41, we have $\mathbb{R}^S = L' \oplus L^{\perp}$.Then $v^* \in L \subseteq \mathbb{R}^S$ can be written in a unique way as the sum of vectors in L and $L^{\perp}$. Then, $v' = v''$ (and $v^1_{\perp} = v^2_{\perp}$), as desired.

(3) $\pi_{|\ L'}$ is onto. Take $v \in L$; we want to find $v' \in L'$ such that $\pi_L(v') = v$. Since $v \in L \subseteq \mathbb{R}^S = L' \oplus L^{\perp}$, then there exist a unique $(v', v_{\perp}) \in L' \oplus L^{\perp}$ such that $v = v' + v^{\perp}$.
Then, $\overset{\in L'}{v'} = \overset{\in L}{v} + \left(\overset{\in L^{\perp}}{-v^{\perp}} \right)$, and then, by definition of π_L, we do have $\pi_L(v') = v$, as desired.

□

Corollary 43. *For any basis $(h_1, .., h_A)$ of L, there exists a basis $(f_1, ..., f_A)$ of L' such that $\pi_{|L'}(f_1) = h_1, ..., \pi_{|L'}(f_A) = h_A$.*

Proof. Since $\pi_{|L'}$ is an isomorphism, then it is onto and

$$\forall a \in \{1, ..., A\}, \quad \exists f_a \in L' \text{ such that } \pi_{|L'}(f_a) = h_a. \tag{6}$$

Since $\dim L' = A$, to show that $(f_1, ..., f_A)$ is a basis of L', it is enough to show that they are linearly independent, i.e., $\sum_{a=1}^A \beta_a f_a = 0 \quad \Rightarrow \quad \beta_1 = ... = \beta_A = 0$. Indeed, $0 = \sum_{a=1}^A \beta_a f_a \overset{\pi \text{ linear}}{\Rightarrow} 0 = \sum_{a=1}^A \beta_a \pi(f_a) \overset{(6)}{=} \sum_{a=1}^A \beta_a h_a$ and the desired result follows from the fact that $(h_1, .., h_A)$ is a basis of L. □

Proposition 44. *For any $L \in \mathscr{G}_{A,S}$, one has that $U_L^* = U_L$.*

Proof. 1. $U_L^* \subseteq U_L$.

Taken $L' \in U_L^*$, then there exists $u = (u^1, ..., u^A) \in (L^{\perp})^A$ such that $\varphi_L(u) = L'$. Then, from Lemma 39 we do have $\varphi_L(u) \cap (\varphi_L(0))^{\perp} = \{0\}$, i.e., $L' \cap L^{\perp} = \{0\}$, i.e., $L' \in U_L$.

2. $U_L \subseteq U_L^*$.

Taken $L' \in U_L$, from Lemma 43, there exists a basis $(f_1, ..., f_A)$ of L' such that $\pi_{|L'}(f^1) = h_1, ..., \pi_{|L'}(f_A) = h_A$, i.e., by definition of $\pi_{|L'} := \pi_{L\ |L'}$,

$$\pi_L(f_1) = h_1, ..., \pi_L(f_A) = h_A$$

Then define $v_1 = \pi_{L^{\perp}}(f_1), ..., \ v_A = \pi_{L^{\perp}}(f_A)$. Then,

$$f_1 = \pi_L(f_1) + \pi_{L^{\perp}}(f_1) = h_1 + v_1, ..., \ f_A = \pi_L(f_A) + \pi_{L^{\perp}}(f_A) = h_A + v_A.$$

Hence

$$L' = \text{span}(f_1, ..., f_A) = \text{span}(h_1 + v_1, ..., h_A + v_A), \text{ where } (v_1, ..., v_A) \in \left(L^{\perp}\right)^A$$

and therefore $L' \in U_L^* = \varphi_L\left(\left(L^{\perp}\right)^A\right)$, as desired. □

Proposition 45. *For any $L \in \mathscr{G}_{A,S}$, U_L is open.*

Proof. First of all, given any basis $(v_1, ..., v_{S-A})$ of $L^{\perp}$, we prove that

$$q^{-1}(U_L) = \left\{(y_1, ..., y_A) \in V_A\left(\mathbb{R}^S\right) : (y_1, ..., y_A, v_1, ..., v_{S-A}) \in V_S\left(\mathbb{R}^S\right)\right\}.$$

We have

$$q^{-1}(U_L) = \left\{(y_1,...,y_A) \in V_A\left(\mathbb{R}^S\right) : \text{span}(y_1,...,y_A) \cap L^\perp = \{0\}\right\} =$$

$$= \left\{(y_1,...,y_A) \in V_A\left(\mathbb{R}^S\right) : \text{span}(y_1,...,y_A) \oplus L^\perp = \mathbb{R}^S\right\} =$$

$$= \left\{(y_1,...,y_A) \in V_A\left(\mathbb{R}^S\right) : (y_1,...,y_A,v_1,...,v_{S-A}) \text{ is a basis of } \mathbb{R}^S\right\} =$$

$$= \left\{(y_1,...,y_A) \in V_A\left(\mathbb{R}^S\right) : (y_1,...,y_A,v_1,...,v_{S-A}) \in V_S(\mathbb{R}^S)\right\}.$$

To prove that U_L is open in $\mathscr{G}_{A,S}$, we prove that $q^{-1}(U_L)$ is open in $\mathscr{G}_{A,S}$. Since

$$q^{-1}(U_L) = \left\{(y_1,...,y_A) \in V_A\left(\mathbb{R}^S\right) : (y_1,...,y_A,v_1,...,v_{S-A}) \in V_S\left(\mathbb{R}^S\right)\right\}$$

$$= \left\{(y_1,...,y_A) \in V_A\left(\mathbb{R}^S\right) : \det[y_1,...,y_A,v_1,...,v_{S-A}] \neq 0\right\},$$

$q^{-1}(U_L)$ is open in $V_A\left(\mathbb{R}^S\right)$ and then U_L is open in $\mathscr{G}_{A,S}$. □

Proposition 46. *$\mathscr{G}_{A,S}$ is a $\mathscr{C}^\infty$ manifold of dimension equal to the dimension of $\left(\overline{L}^\perp\right)^A$, i.e., $(S-A)A$ and $(\psi_{\overline{L}}, U^*_{\overline{L}}, \left(\overline{L}^\perp\right)^A)_{\overline{L}\in\mathscr{G}_{A,S}}$ defines an C^∞ atlas for $\mathscr{G}_{A,S}$.*

Proof. The thesis follows from following facts.

(1) $(U^*_{\overline{L}})_{\overline{L}\in\mathscr{G}_{A,S}}$ is an open cover of $\mathscr{G}_{A,S}$ (Proposition 45);
(2) $\psi_{\overline{L}}$ and $(\psi_{\overline{L}})^{-1}$ are continuous (Proposition 38 and Proposition 40);
(3) For any (U_E,ψ_E) and (U_F,ψ_F) two local charts at $E,F \in \mathscr{G}_{A,S}$, respectively, such that $U_E \cap U_F \neq \emptyset$, the function $\psi_F \circ (\psi_E)^{-1}$ is a C^∞ diffeomorphism (see Bich and Cornet (2004) page 169).

□

We can show that $\mathscr{G}_{A,S}$ has further topological properties.

Proposition 47. *$\mathscr{G}_{A,S}$ is a compact topological space.*

Proof. We are going to show that $V_A^0\left(\mathbb{R}^S\right)$ is compact. Then since $q_0 : V_A^0\left(\mathbb{R}^S\right) \to \mathscr{G}_{A,S}$ is onto and continuous from Remark 12 , the desired result follows.

We endow $V_A^0\left(\mathbb{R}^S\right)$ with the Euclidean metric[5] of $\mathbb{R}^{SA}$, i.e., for any $M \in V_A^0\left(\mathbb{R}^S\right)$

$$\|M\|^2 := \sum_{a=1}^{A} \langle C^a(M), C^a(M)\rangle_S = A,$$

$C^a(M)$ denote the a-th column of M. It then follows that $V_A^0\left(\mathbb{R}^S\right)$ is bounded.

Now, take a sequence $\{M_n\}_{n\in\mathbb{N}} \subseteq V_A^0\left(\mathbb{R}^S\right)$, where for any $n \in \mathbb{N}$

$$M_n := \left[\begin{array}{ccccc} C^1(M_n) & ... & C^a(M_n) & ... & C^A(M_n) \end{array}\right]$$

and $M_n \xrightarrow{n} \widetilde{M}$. We want to show that $\widetilde{M} \in V_A^0\left(\mathbb{R}^S\right)$. Observe that for any $a \in \{1,...,A\}$, we do have that $C^a(M_n) \xrightarrow{n} C^a\left(\widetilde{M}\right)$. Moreover, since for any $n \in \mathbb{N}$, $M_n \in V_A^0\left(\mathbb{R}^S\right)$, then

[5]Observe that in a finite dimensional vector space any metric induces the same topology.

for any $i, j \in \{1, .., A\}$,

$$\left\langle C^i(M_n), C^j(M_n)\right\rangle_S = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Since the inner product is a continuous function, then

$$\left\langle C^i\left(\widetilde{M}\right), C^j\left(\widetilde{M}\right)\right\rangle_S = \lim_{n\to+\infty}\left\langle C^i(M_n), C^j(M_n)\right\rangle_S = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

i.e., $\widetilde{M} \in V_A^0\left(\mathbb{R}^S\right)$, as desired.

We are left with showing that $\mathscr{G}_{A,S}$ is metrizable. □

Proposition 48. *$\mathscr{G}_{A,S}$ is second countable, i.e., it has a countable basis.*

Proof. Since $\{U_L : L \in \mathscr{G}_{A,S}\}$ is an open cover of $\mathscr{G}_{A,S}$ and $\mathscr{G}_{A,S}$ is compact, that open cover admits a finite subcover say $\{U_i : i \in \{1,...,m\}\}$, with $m \in \mathbb{N}$, and $\left(\psi_i, U_i, \left(L_i^{\perp}\right)^A\right)_{i\in\{1,...,m\}}$ is still an atlas for $\mathscr{G}_{A,S}$. Since $\left(L_i^{\perp}\right)^A$ is isomorphic to $\mathbb{R}^{(S-A)A}$, for any $i \in \{1,...,m\}$, U_i is diffeomorphic to $\mathbb{R}^{(S-A)A}$, say via α_i. Let $\mathscr{C}$ be a countable basis of $\mathbb{R}^{(S-A)A}$.

Since second countability is preserved under homeomorphisms, then, for any $i \in \{1,...,m\}$, U_i has a countable basis $\mathscr{B}_i = \{\alpha_i(C) : C \in \mathscr{C}\}$. Define now $\mathscr{B} = \cup_{i=1}^m \mathscr{B}_i$. We are left with showing that $\mathscr{B}$ is a countable basis for $\mathscr{G}_{A,S}$.

$\mathscr{B}$ is countable because it is the finite union of countably many sets. To show that $\mathscr{B}$ is a basis for $\mathscr{G}_{A,S}$ we have to check the following conditions.

1. $\mathscr{B} \subseteq \mathscr{T}$, and
2. $\forall L \in \mathscr{G}_{A,S}$, $\forall S \in \mathscr{T}$ such that $L \in S$, $\exists B \in \mathscr{B}$ such that $L \in B \subseteq S$.

1. Since $\mathscr{B}_i = \{\alpha_i(C) : C \in \mathscr{C}\}$, C is open in $\mathbb{R}^{(S-A)A}$ and α_i is a diffeomorphism from $\mathbb{R}^{(S-A)A}$ to U_i, then $\alpha_i(C)$ is open in U_i and therefore in $\mathscr{G}_{A,S}$, as desired.

2. Take $L \in \mathscr{G}_{A,S}$ and $S \in \mathscr{T}$ such that $L \in S$. We want to show that there exists $B_L \in \mathscr{B}$ such that $L \in B_L \subseteq S$. Since $\mathscr{G}_{A,S} \subseteq \cup_{i=1}^m U_i$, then there exists $j \in \{1,...,m\}$ such that $L \in U_j$ and there exists $\widehat{B}_L \in \mathscr{B}_j$ such that $L \in \widehat{B}_L \subseteq U_j$. Observe $\widehat{B}_L \cap S \subseteq \widehat{B}_L \subseteq U_j$ is an open set containing L. Then, since $\mathscr{B}_j$ is a basis of U_j, then there exists $B_L \in \mathscr{B}_j \subseteq \mathscr{B}$ such that $L \in B_L \subseteq \widehat{B}_L \cap S \subseteq S$, as desired. □

Further properties of $\mathscr{G}_{A,S}$ immediately follow from the above results and the following Proposition (for more details see Munkres (1975)).

Proposition 49. *Let X be a topological space which is second countable, i.e., it admits a countable basis, (indeed, first countable suffices), Hausdorff and compact. Then the following statements hold true.*

(1) *Every convergent sequence converges to a unique limit.*
(2) *X has a so called* nested countable local basis *at any $x \in X$, i.e., a countable local basis $\mathscr{B} = \{B_n\}_{n\in\mathbb{N}}$ such that $\forall n$, $B_{n+1} \subseteq B_n$.*
(3) *Let a nested local basis $\{B_n\}_{n\in\mathbb{N}}$ at x be given. Assume that $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is such that $\forall n \in \mathbb{N}$, $x_n \in B_n$. Then $x_n \to x$.*
(4) *Let S be a subset of X. $x \in \mathrm{Cl}(S)$ if and only if there exists a sequence of points of A converging to x.*
(5) *S is closed if and only if it sequentially closed.*

(6) *Compact sets are closed.*
(7) *Let a topological spaces* $(Y, \mathscr{T}')$ *and a function* $f : X \to Y$ *be given. Then,* f *is continuous if and only if it sequentially continuous.*
(8) *If* X *is second countable, then* $S \subseteq X$ *is sequentially compact implies it is compact.*
(9) *If* X *is Hausdorff, then* $S \subseteq X$ *is compact implies it is sequentially compact.*
(10) X *is normal.*
(11) X *is regular.*
(12) X *is metrizable.*

Remark 50. *The main consequences of above Proposition are the following ones.*
i. What we defined to be an open set in $\mathscr{T}$*, the topology we endowed* $\mathscr{G}_{A,S}$ *with, is indeed open with respect to a metric; i.e.,* $\mathscr{G}_{A,S}$ *is a metric space whose induced topology is exactly* $\mathscr{T}$ *- defined in Definition 4 and Proposition 5 above;*
ii. the fact that $\mathscr{G}_{A,S}$ *is a metric space allows to apply standard theory on set-valued functions; indeed, standard theory requires domain and codomain of set-valued functions to be metric spaces.*

Remark 51. *We conclude the section with a simple crucial consequence of all above analysis. We have seen that there exists* $m \in \mathbb{N}$ *such that*

$$\left\{ (\psi_{L_i}, U_{L_i}, \left(L_i^\perp\right)^A) \right\}_{i \in \{1,\ldots,m\}}$$

is an atlas for $\mathscr{G}_{A,S}$*. Therefore, for any* $L \in \mathscr{G}_{A,S}$ *there exists* $i \in \{1,..,m\}$ *and an associated chart* $(\psi_{L_i}, U_{L_i}, \left(L_i^\perp\right)^A)$ *such that* $L \in U_{L_i}$ *and*

$$L = \operatorname{span}\left(\mathscr{F}_{L_i} + \psi_{L_i}(L)\right),$$

where $\mathscr{F}_{L_i}$ *is a basis of* L_i*. Observe that*

$$\psi_i^* : U_{L_i} \subseteq \mathscr{G}_{A,S} \to \left(L_i^\perp\right)^A) \subseteq \left(\mathbb{R}^S\right)^A, \quad \psi_i^*(L) = \mathscr{F}_{L_i} + \psi_{L_i}(L),$$

where $\mathscr{F}_{L_i} + \psi_{L_i}(L)$ *is an* $S \times A$ *full rank matrix and it is a homeomorphism.*

3. Existence result for GQVI involving Grassmannian manifolds

In the present section, we first recall the definition of variational inequalities and then we present an existence result on generalized quasi-variational inequalities involving Grassmannian manifolds.

Definition 52. *Let* $C \subseteq \mathbb{R}^n$ *be a nonempty, closed and convex set and let* $S : C \rightrightarrows \mathbb{R}^n$ *and* $\Phi : C \rightrightarrows \mathbb{R}^n$ *be set-valued maps. A* Generalized Quasi-Variational Inequality *associated with* C, S, Φ*, denoted by* GQVI*, is the following problem*

$$\text{Find } \bar{x} \in S(\bar{x}) \text{ such that there exists } \varphi \in \Phi(\bar{x}) \text{ with } \langle \varphi, x - \bar{x} \rangle \geq 0, \ \forall x \in S(\bar{x}). \tag{7}$$

In particular, when $S(x) = C$ *for any* $x \in C$*, (7) is a* Generalized Variational Inequality, GVI*; when* Φ *is single-valued, (7) reduces to the* Quasi-Variational Inequality, QVI. *When both* $\Phi(x)$ *is singleton and* $S(x) = C$*, for any* $x \in C$*, we have the classical Stampacchia* Variational Inequality, VI.

Now, we introduce a variational inequality theorem involving the Grassmannian manifold using the result below, presented and proved by Bich and Cornet (2004).

Theorem 53. *Let C be a nonempty, convex, compact subset of $\mathbb{R}^n$; for any $a \in \{1,...,A\}$, let $\Psi_a : \mathscr{G}_{A,S} \times C \to \mathbb{R}^S$ be a continuous function and let $\Phi : \mathscr{G}_{A,S} \times C \rightrightarrows C$ be a convex valued and compact valued set-valued map which is either lower semicontinuous or upper semicontinuous. Then, there exists $\left(\overline{E},\overline{x}\right) \in \mathscr{G}_{A,S} \times C$ such that*

for any $a \in \{1,...,A\}$,

$$\Psi_a\left(\overline{E},\overline{x}\right) \in \overline{E}; \tag{8}$$

$$\text{either } \Phi\left(\overline{E},\overline{x}\right) = \varnothing \text{ or } \overline{x} \in \Phi\left(\overline{E},\overline{x}\right). \tag{9}$$

Theorem 54. *Let C be a nonempty, convex and compact subset of $\mathbb{R}^n$. Let $F : \mathscr{G}_{A,S} \times C \rightrightarrows \mathbb{R}^n$ be a nonempty valued, convex valued, compact valued and upper semicontinuous set-valued function; let $K : \mathscr{G}_{A,S} \times C \rightrightarrows C \subseteq \mathbb{R}^n$ be a nonempty valued, convex valued, compact valued, closed and lower semicontinuous set-valued function and for any $a \in \{1,...,A\}$, let $\Psi_a : \mathscr{G}_{A,S} \times C \to \mathbb{R}^S$ be a continuous function. Then there exists $\left(\overline{E},\overline{x}\right) \in K\left(\overline{E},\overline{x}\right)$ and there exists $\overline{u} \in F\left(\overline{E},\overline{x}\right)$ such that,*

$$\langle \overline{u}, z-\overline{x}\rangle_n \geq 0, \;\; \forall z \in K\left(\overline{E},\overline{x}\right) \tag{10}$$

and for any $a \in \{1,...,A\}$,

$$\Psi_a\left(\overline{E},\overline{x}\right) \in \overline{E}. \tag{11}$$

Proof. First of all observe that $\mathscr{G}_{A,S}$ is a metric space. Since C is compact by assumption, $\mathscr{G}_{A,S}$ is compact from Proposition 47 and F is upper semicontinuous and compact valued on C by assumption, we have that

$$F\left(\mathscr{G}_{A,S} \times C\right) \text{ is compact.} \tag{12}$$

Define $H := \text{conv}F\left(\mathscr{G}_{A,S} \times C\right)$ which is compact and convex. Moreover, define $T : \mathscr{G}_{A,S} \times C \times H \rightrightarrows C$ such that $T(E,x,u) = \arg\min_{z \in K(x,E)} \langle u, z-x\rangle_n$.

Observe that, by definition of K, $z \in C$ and therefore T is well defined. Observe also that $g_{u,x} : K \to \mathbb{R}$ such that $g_{u,x}(z) = \langle u, z-x\rangle_n$ is a continuous function.

From assumptions on K, we can apply the Maximum Theorem and then T is

$$\text{nonempty and compact valued, upper semicontiunuous, closed and also convex valued.} \tag{13}$$

The last property is a consequence of the following standard argument. Take $y_1, y_2 \in T(E,x,u)$ and $\lambda \in [0,1]$. Then,

$$\forall z \in K(x,E),\; \langle u, y_1 - x\rangle \leq \langle u, z-x\rangle, \tag{14}$$

and

$$\forall z \in K(x,E),\; \langle u, y_2 - x\rangle \leq \langle u, z-x\rangle. \tag{15}$$

Multiplying (14) by $(1-\lambda)$ and (15) by λ and summing up, we get

$$\forall z \in K(x,E),\; \langle u, [(1-\lambda)y_1 + \lambda y_2] - x\rangle \leq \langle u, z-x\rangle,$$

as desired.

Now, define $P : \mathscr{G}_{A,S} \times C \times H \rightrightarrows C \times H$ such that $P(E,x,u) := T(E,x,u) \times F(E,x)$.

Observe that the above definition is well given because by definition of T, $\mathrm{Im}T \subseteq C$, and $F(\mathscr{G}_{A,S} \times C) \subseteq H$.

Observe that $C \times H$ is nonempty, convex and compact and by definition, P is nonempty, convex, closed valued and compact valued. Moreover P is upper semicontinuous (from Proposition 4 page 25 of Hildebrand 1974.) Then, we can apply Theorem 53 and we have that there exists $(\overline{E}, \overline{x}, \overline{u}) \in \mathscr{G}_{A,S} \times C \times H$ such that for any $a \in \{1, ..., A\}$, $\Psi_a(\overline{E}, \overline{x}) \in \overline{E}$ and, since P is nonempty valued, $(\overline{x}, \overline{u}) \in P(\overline{x}, \overline{u}, \overline{E})$.

From the last statement, $\overline{x} \in T(\overline{x}, \overline{u}, \overline{E})$, i.e., $\overline{x} \in \arg\min_{z \in K(\overline{x}, \overline{E})} \langle \overline{u}, z - \overline{x} \rangle$, that is equivalent to $\langle \overline{u}, z - \overline{x} \rangle_n \geq \langle \overline{u}, \overline{x} - \overline{x} \rangle_n = 0$, $\forall z \in K(\overline{x}, \overline{E})$, and moreover, $\overline{u} \in F(\overline{x}, \overline{E})$. Combining all the above, we get the desired results. □

Remark 55. *A converse of Theorem 54 does hold true, i.e., if conditions (10) and (11) do hold true then condition (8) and (condition (9) with P in the place of Φ) hold true as well. Indeed, to get the above result we have to show that $(\overline{x}, \overline{u}) \in P(\overline{x}, \overline{u}, \overline{E})$, i.e., $\overline{x} \in T(\overline{x}, \overline{u}, \overline{E}) = \arg\min_{z \in K(\overline{x}, \overline{E})} \langle \overline{u}, z - \overline{x} \rangle_n$ and $\overline{u} \in F(\overline{x}, \overline{E})$, which follows immediately from assumption (10).*

4. Motivating example

In this Section, we present a framework the above analysis can be applied to. The model we analyze was first studied in the seminal paper by Duffie and Shafer (1985).

The chosen model builds on the standard two-period, general equilibrium model of pure exchange with uncertainty. In the commodity markets, $C \geq 2$ different physical commodities are traded, denoted by $c \in \mathscr{C} = \{1, 2, \ldots, C\}$. In the final period, only one among $S \geq 1$ possible states of the world, denoted by $s \in \{1, 2, \ldots, S\}$, will occur. The initial period is denoted by $s = 0$ and we define the set of all states $\mathscr{S}^0 = \{0, 1, \ldots, S\}$ and the set of uncertain states $\mathscr{S} = \{1, 2, \ldots, S\}$. In the initial period, asset markets open and $A \geq 1$ assets are traded, denoted by $a \in \mathscr{A} = \{1, 2, \ldots, A\}$. We assume $A \leq S$. Finally, there are $H \geq 2$ households, denoted by $h \in \mathscr{H} = \{1, 2, \ldots, H\}$.

The time structure of the model is as follows: in the initial period, households exchange commodities and assets, and the consumption takes place. In the final period, the uncertainty is resolved, households honor their financial obligations, exchange commodities, and then consume commodities.

We denote $x_h^{sc} \in \mathbb{R}$ and $e_h^{sc} \in \mathbb{R}_+$ as the consumption and the endowment of commodity c in state s by household h, respectively[6]. We define $x_h^s = (x_h^{sc})_{c \in \mathscr{C}} \in \mathbb{R}^C$, $x_h = (x_h^s)_{s \in \mathscr{S}^0} \in \mathbb{R}^G$ $x = (x_h)_{h \in \mathscr{H}} \in \mathbb{R}^{GH}$ and similarly $e_h^s \in \mathbb{R}_+^C$, $e_h \in \mathbb{R}_+^G$, $e \in \mathbb{R}_+^{GH}$, where $G = C(S+1)$.

Household h's preferences are represented by a utility function $u_h : \mathbb{R}^G \to \mathbb{R}$. We denote by $\mathscr{U}$ the set of vectors $u = (u_h)_{h \in \mathscr{H}}$.

Moreover, we denote by $p^{sc} \in \mathbb{R}_+$ the price of commodity c in state s, by $q^a \in \mathbb{R}$ the price of asset a and by $b_h^a \in \mathbb{R}$ the quantity of asset a held by household h. We

[6]Given $v, w \in \mathbb{R}^N$, we denote by $v \gg w$, $v \geq w$ and $v > w$ the standard binary relations between vectors. Also the definitions of the sets $\mathbb{R}_+^N$ and $\mathbb{R}_{++}^N$ are the common ones.

define $p^s = (p^{sc})_{c\in\mathscr{C}} \in \mathbb{R}^C_+$, $p^{\mathbf{1}} = (p^s)_{s\in\mathscr{S}} \in \mathbb{R}^{CS}_+$, $p = (p^s)_{s\in\mathscr{S}^0} \in \mathbb{R}^G_+$, $q = (q^a)_{a\in\mathscr{A}} \in \mathbb{R}^A$, $b_h = (b^a_h)_{a\in\mathscr{A}} \in \mathbb{R}^A$, $b = (b_h)_{h\in\mathscr{H}} \in \mathbb{R}^{AH}$.

We denote by $y^{sac} \in \mathbb{R}$ the units of commodity c delivered by one unit of asset a in state s and we define $y^{sa} = (y^{sac})_{c\in\mathscr{C}} \in \mathbb{R}^C$, $y^s = (y^{sa})_{a\in\mathscr{A}} \in \mathbb{R}^{AC}$, $y = (y^s)_{s\in\mathscr{S}} \in \mathbb{R}^{SAC}$. Note in particular that, in state s, asset a promises to deliver a vector y^{sa} of commodities. The above described assets are usually called real assets in the literature.

Define the return matrix function as follows

$$\mathscr{R} : \mathbb{R}^{CS}_+ \times \mathbb{R}^{SAC} \to \mathbb{M}(S,A),$$

$$\mathscr{R}(p^{\mathbf{1}},y) := \begin{bmatrix} \langle p^1, y^{11}\rangle_C & \cdots & \langle p^1, y^{1a}\rangle_C & \cdots & \langle p^1, y^{1A}\rangle_C \\ \cdots & & & & \\ \langle p^s, y^{s1}\rangle_C & \cdots & \langle p^s, y^{sa}\rangle_C & \cdots & \langle p^s, y^{sA}\rangle_C \\ \cdots & & & & \\ \langle p^S, y^{S1}\rangle_C & \cdots & \langle p^S, y^{Sa}\rangle_C & \cdots & \langle p^S, y^{SA}\rangle_C \end{bmatrix},$$

where $\mathbb{M}(S,A)$ denotes the set of real $S\times A$ matrices, $R^s(p^{\mathbf{1}},y) := (\langle p^s, y^{sa}\rangle_C)_{a\in\mathscr{A}}$ is the $s-$th row of matrix $\mathscr{R}(p^{\mathbf{1}},y)$ and ${}^aR\left(p^{\mathbf{1}},y\right) := (\langle p^s, y^{sa}\rangle_C)_{s\in\mathscr{S}}$ is the $a-th$ column of $\mathscr{R}\left(p^{\mathbf{1}},y\right)$.

We define the set of economies as $E := \mathbb{R}^{GH}_+ \times \mathscr{U} \times \mathbb{R}^{SAC}$, with generic element $\varepsilon := (e,u,y)$.

Definition 56. *A vector* $\left(\widetilde{x},\widetilde{b},\widetilde{p},\widetilde{q}\right) \in \mathbb{R}^{GH} \times \mathbb{R}^{AH} \times \mathbb{R}^G_+ \times \mathbb{R}^A$ *is an equilibrium for the economy* $\varepsilon \in E$ *if*

(1) *for any* $h \in \mathscr{H}$, $\left(\widetilde{x}_h,\widetilde{b}_h\right)$ *solves the following problem. Given* $\varepsilon \in E$ *and* $(\widetilde{p},\widetilde{q}) \in \mathbb{R}^G_+ \times \mathbb{R}^A$,

$$\max_{(x_h,b_h)\in\mathbb{R}^G_+\times\mathbb{R}^A} u_h(x_h)$$

$$s.t. \quad \langle \widetilde{p}^0, x^0_h - e^0_h\rangle_C + \langle \widetilde{q}, b_h\rangle_A \le 0 \tag{16}$$

$$\langle \widetilde{p}^s, x^s_h - e^s_h\rangle_C - \langle R^s(\widetilde{p}^{\mathbf{1}},y), b_h\rangle_A \le 0,\ \forall s \in \mathscr{S};$$

(2) $\left(\widetilde{x},\widetilde{b}\right)$ *satisfies market clearing conditions*

a. *for any* $s \in \mathscr{S}^0$ *and* $c \in \mathscr{C}$,

$$\sum_{h\in\mathscr{H}} \widetilde{x}^{sc}_h \le \sum_{h\in\mathscr{H}} e^{sc}_h \qquad if \quad \widetilde{p}^{sc} = 0,$$

$$\sum_{h\in\mathscr{H}} \widetilde{x}^{sc}_h = \sum_{h\in\mathscr{H}} e^{sc}_h \qquad if \quad \widetilde{p}^{sc} > 0;$$

b. *for any* $a \in \mathscr{A}$,

$$\sum_{h\in\mathscr{H}} \widetilde{b}^a_h = 0. \tag{17}$$

In what follows, we first explain why the problem of existence of an equilibrium cannot be analyzed using standard techniques employed in similar models, as Kakutani fixed point theorem or homotopy arguments. Then, we show how the concept of equilibrium is strictly related to the concept of Grassmannian manifolds. Finally, we present a variational inequality problem whose solutions, under suitable assumptions, are indeed equilibria we are trying to prove the existence of. We warn the reader that our goal below is to convey the general idea of the link between an interesting specific problem and the Variational Inequality result in terms of Grassmannian we presented in Theorem 54. Indeed, the matter is more complicated then what described below: a rigorous, complete analysis of the proof of existence is presented in a companion paper by the authors (see Donato and Villanacci (2020)).

The first step in our strategy is to introduce the definition of fixed - image equilibrium which can be shown to be equivalent to the definition of equilibrium under a simple condition. Before presenting the definition of fixed - image equilibrium, we describe and motivate the two differences between that definition and the definition of "true"equilibrium, i.e., Definition 56.

In the definition of equilibrium, as a consequence of standard monotonicity assumptions on the utility functions, for any household it is the case that each budget inequality in (16) does hold true as an equality. Therefore, it must be $\left(\langle p^s, x_h^s - e_h^s\rangle\right)_{s\in S} \in \operatorname{span}\mathscr{R}\left(\widetilde{p}^{\mathbf{1}}, y\right)$. Observe that $\operatorname{span}\mathscr{R}\left(\widetilde{p}^{\mathbf{1}}, y\right)$ is a linear subspace of $\mathbb{R}^S$ of dimension equal to $\operatorname{rank}R\left(p, y\right)$ and represents the space of potentially available values of excess demands in period 1. When the endogenous variable p changes, that dimension may change. That drop of rank leads to a discontinuity of the demand function which constitutes a major problem in the application of standard proof techniques. The above observations motivate the need to "fix the linear subspace of potentially available values of excess demand" and leads to the concept of a fixed - image equilibrium. In that concept, the value of the excess demands in period 1 has to be contained in a linear subspace of $\mathbb{R}^S$ which is independent of prices and of fixed dimension A, i.e., $\left(\langle p^s, x_h^s - e_h^s\rangle\right)_{s\in S} \in L$ where L is an element in $\mathscr{G}_{A,S}$, the Grassmannian manifold of A dimensional subspaces of $\mathbb{R}^S$.

Using the properties of the Grassmannian manifold we have seen that for any $L \in \mathscr{G}_{A,S}$ there exists $i \in \{1, ...m\}$ and an $S \times A$ matrix $\psi_i^*\left(L\right)$ such that

$$L = \operatorname{Im}\psi_i^*\left(L\right) \tag{18}$$

- see Remark 51. That observation allows to rewrite $\left(\langle p^s, x_h^s - e_h^s\rangle\right)_{s\in S} \in L$ as "there exists $b_h \in \mathbb{R}^A$ such that $\left(\langle p^s, x_h^s - e_h^s\rangle\right)_{s\in S} = \psi_i^*\left(L\right)\cdot b_h$", as we do in the definition of fixed - image equilibrium (see Definition 57).

That definition contains also the requirement that the image of the return matrix $\mathscr{R}\left(\widetilde{p}^{\mathbf{1}}, y\right)$ has to be contained in the linear subspace L, which is a crucial condition to preserve the link between the newly introduced notion of equilibrium and the original one.

Definition 57. *A vector* $\left(\widetilde{x}, \widetilde{b}, \widetilde{p}, \widetilde{L}\right) \in \mathbb{R}^{GH} \times \mathbb{R}^{AH} \times \mathbb{R}_{+}^{G} \times \mathscr{G}_{A,S}$ *is a fixed - image equilibrium for the economy* $\varepsilon \in E$ *if*

(1) *for any* $h \in \mathscr{H}$, $(\widetilde{x}_h, \widetilde{b}_h)$ *solves the following problem. Given* $\varepsilon \in E$ *and* $(\widetilde{p}, \widetilde{L}) \in \mathbb{R}^G_+ \times \mathscr{G}_{A,S}$

$$\max_{(x_h,b_h)\in\mathbb{R}^G_+\times\mathbb{R}^A} u_h(x_h)$$

$$s.t. \quad \langle \widetilde{p}^0, x^0_h - e^0_h \rangle_C + \langle 1_S \psi^*_i(\widetilde{L}), b_h \rangle_A \le 0 \tag{19}$$

$$\left(\langle p^s, x^s_h - e^s_h \rangle\right)_{s\in S} - \psi^*_i(\widetilde{L}) b_h \le 0_S,$$

(2) $\widetilde{x}$ *satisfies the market clearing conditions. For any* $s \in \mathscr{S}^0$ *and* $c \in \mathscr{C}$,

$$\sum_{h\in\mathscr{H}} \widetilde{x}^{sc}_h \le \sum_{h\in\mathscr{H}} e^{sc}_h \qquad \text{if} \qquad \widetilde{p}^{sc} = 0,$$

$$\sum_{h\in\mathscr{H}} \widetilde{x}^{sc}_h = \sum_{h\in\mathscr{H}} e^{sc}_h \qquad \text{if} \qquad \widetilde{p}^{sc} > 0;$$

(3)

$$\mathrm{Im}\mathscr{R}\left(\widetilde{p}^{\mathbf{1}}, y\right) \subseteq \widetilde{L}. \tag{20}$$

The definition of the fixed-image equilibrium presented above can be characterized in terms of a suitable generalized quasi-variational inequality, a simplified form of which is presented below.

Find $\left((\widetilde{x}_h, \widetilde{b}_h)_{h\in\mathscr{H}}, \widetilde{p}, \widetilde{L}\right) \in B\left(\widetilde{p}, \widetilde{L}\right) \times \Delta \times \mathscr{G}_{A,S}$ and $g = (g_h)_{h\in\mathscr{H}} \in \prod_{h\in\mathscr{H}} T_h(\widetilde{x}_h)$ such that

$$\langle ((-g_h)_{h\in\mathscr{H}}, 0_A, (\sum_{h\in\mathscr{H}} (\widetilde{x}_h - e^s_h))_{s\in\mathscr{S}^0}), (x, b, p, L) - (\widetilde{x}, \widetilde{b}, \widetilde{p}, \widetilde{L}) \rangle \le 0 \tag{21}$$

for any $((x_h, b_h)_{h\in\mathscr{H}}, p, L) \in B(\widetilde{p}, \widetilde{L}) \times \Delta \times \mathscr{G}_{A,S}$,
and $\mathrm{Im}\mathscr{R}(\widetilde{p}^{\mathbf{1}}, y) \subseteq \widetilde{L}$

where Δ is the simplex in $\mathbb{R}^G$, the set-valued map $T_h : \mathbb{R}^G \rightrightarrows \mathbb{R}^G$ is defined as

$$T_h(x_h) := conv\left(N^>_h(x_h) \cap S(0,1)\right),$$

with $S(0,1) = \{x \in \mathbb{R}^G : \|x\|_G = 1\}$, the unit sphere of $\mathbb{R}^G$, and $N^>_h(x_h)$, the normal cone to the strict upper level set $U^>_{u_h(x_h)} := \{x \in \mathbb{R}^n : u(x) > u_h(x_h)\}$,

$$B_h : \Delta \times \mathscr{G}_{A,S} \rightrightarrows \mathbb{R}^G_+ \times \mathbb{R}^A,$$

$$B_h((p, L) :=$$

$$\{\ (x_h, b_h) \in \mathbb{R}^G_+ \times \mathbb{R}^A : \text{there exists } (\psi_{L_i}, U_{L_i}, \left(L_i^\perp\right)^A) \in \Lambda \text{ such that } L \in U_{L_i} \text{ and}$$

$$\langle p^0, x^0_h - e^0_h \rangle_C + \langle 1_S \psi^*_i(L), b_h \rangle_A \le 0,$$

$$\left(\langle p^s, x^s_h - e^s_h \rangle\right)_{s\in S} - \psi^*_i(L)\, b_h \le 0_S\ \}$$

and $B(p,L) = \prod_{h\in\mathscr{H}} B_h(p,L)$.

Observe as the variational inequality in (21) has a simple structure, made up by two parts. A first one, involving (x,b), relates to the households' maximization problems, and the second one, involving (p,L) to market clearing conditions.

In according to Definition 52 and to Theorem 54, the problem (21) represents a generalized quasi-variational inequality where

$$C := conv(B(\Delta \times \mathscr{G}_{A,S})) \times \Delta,$$
$$K(x,b,p,L) := B(p,L) \times \Delta$$
$$F(x,b,p,L) := -(\prod_{h\in\mathscr{H}} T_h(x_h)) \times \{0_A\} \times \left\{ \sum_{h\in\mathscr{H}} (x_h - e_h) \right\}.$$

Remark 58. *As already pointed out above, the proof of the existence of equilibrium for the model presented in the section requires a further modification of the concept of equilibrium, suitable assumptions and further remarks. We analyze in detail that problem in a companion paper (see Donato and Villanacci 2020).*

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