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Research article

The anisotropic integrability logarithmic regularity criterion to the 3D micropolar fluid equations

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Abstract: The aim of this paper is to establish the regularity criterion of weak solutions to the 3D micropolar fluid equations by one directional derivative of the pressure in anisotropic Lebesgue spaces. We improve the regularity criterion for weak solutions previously given by Jia, Zhang and Dong in [21].

Keywords: micropolar fluid equations; regularity criterion; anisotropic Lebesgue spaces; a priori estimates
Mathematics Subject Classification: 35Q35, 35B65

1. Introduction

Let us consider the following Cauchy problem of the incompressible micropolar fluid equations in three-spatial dimensions :

$$\begin{cases} \partial_t u + (u\cdot\nabla)\, u - \Delta u + \nabla\pi - \nabla\times\omega = 0, \\ \partial_t \omega - \Delta\omega - \nabla(\nabla\cdot\omega) + 2\omega + (u\cdot\nabla)\omega - \nabla\times u = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ \omega(x,0) = \omega_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$, $\omega = \omega(x,t) \in \mathbb{R}^3$ and $\pi = \pi(x,t)$ denote the unknown velocity vector field, the micro-rotational velocity and the unknown scalar pressure of the fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,T)$, respectively, while u_0, ω_0 are given initial data with $\nabla \cdot u_0 = 0$ in the sense of distributions.

Theory of micropolar fluid equations was first proposed by Eringen [11] in 1966, which have important applications in fluid mechanics and material sciences and which enables to consider some physical phenomena that cannot be treated by the classical Navier-Stokes equations for the viscous incompressible fluids, for example of animal blood, liquid crystals and delute aqueous polymer solutions, etc. (see [28, 29, 31]). If $\omega = 0$, then (1.4) reduces to be the well-known Navier-Stokes equations. Besides its physical applications, the Navier-Stokes equations are also mathematically significant. Since Leray [24] and Hopf [23] constructed the so-called well-known Leray-Hopf weak solution $u(x,t)$ of the incompressible Navier-Stokes equation for arbitrary $u_0(x) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0(x) = 0$ in last century, the problem on the uniqueness and regularity of the Leray-Hopf weak solutions is one of the most challenging problem of the mathematical community. Later on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Different criteria for regularity of the weak solutions have been proposed and many interesting results were established (see e.g. [12–14, 22, 33] and references there in).

Due to the importance of both physics and mathematics, the question of smoothness and uniqueness of weak solutions to (1.4) is one of the most challenging problems in the theory of PDE's. Galdi and Rionero [19], Łukaszewicz [26] considered the existence of weak solutions of the micropolar fluid flows (1.4). While the existence of regular solutions is still open problem, there are many interesting sufficient conditions which guarantee that a given weak solution is smooth (see [4, 9, 15–18, 32] and references there in). In particular, as for the pressure criterion, Dong et al. [10] (see also Yuan [31]) showed that the weak solution becomes regular if the pressure satisfies

$$\pi \in L^q(0,T; L^{p,\infty}(\mathbb{R}^3)), \quad \text{for} \quad \frac{2}{q} + \frac{3}{p} \leq 2, \quad \frac{3}{2} < p \leq \infty,$$

or

$$\pi \in L^1(0,T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3)),$$

where $L^{p,\infty}$ and $\dot{B}^{0}_{\infty,\infty}$ denote Lorents space and homogeneous Besov space. Later on, Jia et al. [20] extended and improved Serrin's regularity criterion to the largest critical Besov spaces as

$$\pi \in L^q(0,T; B^r_{p,\infty}(\mathbb{R}^3)),$$

where $\frac{2}{q} + \frac{3}{p} = 2 + r$, $\frac{3}{2+r} < p < \infty$ and $-1 < r \leq 1$.

Besides, some interesting logarithmical pressure regularity criteria of micropolar fluid equations are studied. In particular, in [21], Jia et al. refined this question by establishing a regularity criterion in terms of the partial derivative of the pressure in the Lebesgue space. More precisely, they showed that if the partial derivative of the pressure $\partial_3 \pi$ satisfies

$$\int_0^T \frac{\|\partial_3 \pi\|_{L^q}^p}{1 + \ln(e + \|\omega\|_{L^4})}\, dt < \infty, \quad \frac{2}{p} + \frac{3}{q} = \frac{7}{4} \ \text{and} \ \frac{12}{7} < q \leq \infty, \tag{1.2}$$

then the weak solution (u, ω) becomes a regular solution on $(0,T]$. (see, for instance [1–3] and the more recent papers [5–8] and the references therein).

Here we would like to give an improvement of the anisotropic regularity criterion of (1.5). Before giving the main result, we recall the definition of weak solutions for micropolar fluid equations (1.4) (see [26, 27]).

Definition 1.1 (weak solutions)**.** *Let* $(u_0, \omega_0) \in L^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *in the sense of distribution and* $T > 0$*. A measurable function* $(u(x,t), \omega(x,t))$ *on* $\mathbb{R}^3 \times (0,T)$ *is called a weak solution of (1.4) on* $[0,T)$ *if* (u,ω) *satisfies the following properties :*

(i) $(u,\omega) \in L^\infty\Big((0,T)\,;L^2(\mathbb{R}^3)\Big) \cap L^2\Big((0,T)\,;H^1(\mathbb{R}^3)\Big)$;

(ii) $\nabla \cdot u = 0$ *in the sense of distribution;*

(iii) (u,ω) *verifies (1.4) in the sense of distribution.*

(iv) (u,ω) *satisfies the energy inequality, that is,*

$$\begin{aligned} &\|u(\cdot,t)\|_{L^2}^2 + \|\omega(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^2\,d\tau + 2\int_0^t \|\nabla \omega(\cdot,\tau)\|_{L^2}^2\,d\tau \\ \le\ & \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2\,,\ \textit{for all } t \in [0,T]. \end{aligned}$$

We endow the usual Lebesgue space $L^p(\mathbb{R}^3)$ with the norm $\|\cdot\|_{L^p}$. We denote by $\partial_i = \frac{\partial}{\partial x_i}$ the partial derivative in the x_i-direction. Recall that the anisotropic Lebesgue space consists on all the total measurable real valued functions $h = h(x_1,x_2,x_3)$ with finite norm

$$\Big\|\|h\|_{L^p_{x_i}}\Big\|_{L^q_{x_jx_k}} = \left(\int_{\mathbb{R}^2}(\int_{\mathbb{R}}|h(x)|^p\,dx_i)^{\frac{q}{p}}dx_jdx_k\right)^{\frac{1}{q}},$$

where (i,j,k) belongs to the permutation group S =span$\{1,2,3\}$. Our main result is as follows:

Theorem 1.2. *Let* $(u_0,\omega_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *in* $\mathbb{R}^3$*. Suppose that* (u,ω) *is a weak solution of (1.4) in* $(0,T)$*. If the pressure satisfies the condition*

$$\int_0^T \frac{\Big\|\|\partial_3\pi(\cdot,t)\|_{L^\gamma_{x_3}}\Big\|^q_{L^\alpha_{x_1x_2}}}{1+\ln\left(e+\|\omega(\cdot,t)\|_{L^4}\right)}\,dt < \infty, \tag{1.3}$$

where

$$\frac{1}{\gamma}+\frac{2}{q}+\frac{2}{\alpha} = \lambda \in [2,3) \ \textit{ and }\ \frac{3}{\lambda} \le \gamma \le \alpha < \frac{1}{\lambda-2},$$

then the weak solution (u,ω) *becomes a regular solution on* $(0,T]$.

This allows us to obtain the regularity criterion of weak solutions via only one directional derivative of the pressure. This extends and improve some known regularity criterion of weak solutions in term of one directional derivative, including the notable works of Jia et al. [21].

Remark 1.1. *Criterion (1.6) can be viewed as a generalization of the recent result (1.5) of Jia-Zhang-Dong in [21]. Moreover, thanks to the fact that micropolar fluid equations (1.4) with* $\omega = 0$ *reduce to the 3D Navier-Stokes equations, we notice that our criterion (1.6) becomes the recent result of Liu-Dai [25] for the Navier-Stokes equations.*

As an application of Theorem 1.5, we also obtain the following regularity criterion of weak solutions.

Corollary 1.3. *Let* $(u_0, \omega_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *in the sense of distributions. Assume that* (u, ω) *is a weak solution of (1.4) in* $(0, T)$. *If the pressure satisfies the condition*

$$\int_0^T \frac{\|\partial_3 \pi(\cdot, t)\|_{L^\alpha}^q}{1 + \ln(e + \|\omega(\cdot, t)\|_{L^4})} \, dt < \infty,$$

where

$$\frac{2}{q} + \frac{3}{\alpha} = \lambda \in [2, 3) \ \text{ and } \ \frac{3}{\lambda} \le \alpha < \frac{1}{\lambda - 2},$$

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then the weak solution (u,ω) becomes a regular solution on $(0,T]$.

Here we would like to give an improvement of the anisotropic regularity criterion of (1.5). Before giving the main result, we recall the definition of weak solutions for micropolar fluid equations (1.4) (see [26]).

Definition 1.4 (weak solutions)**.** *Let* $(u_0,\omega_0)\in L^2(\mathbb{R}^3)$ *with* $\nabla\cdot u_0=0$ *in the sense of distribution and* $T>0$. *A measurable function* $(u(x,t),\omega(x,t))$ *on* $\mathbb{R}^3\times(0,T)$ *is called a weak solution of (1.4) on* $[0,T)$ *if* (u,ω) *satisfies the following properties :*

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$$\begin{aligned}
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\leq\ &\|u_0\|^2_{L^2}+\|\omega_0\|^2_{L^2},\ \textit{ for all } t\in[0,T].
\end{aligned}$$

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$$\int_0^T \frac{\left\| \|\partial_3 \pi(\cdot, t)\|_{L^\gamma_{x_3}} \right\|^q_{L^\alpha_{x_1 x_2}}}{1 + \ln(e + \|\omega(\cdot, t)\|_{L^4})} \, dt < \infty, \tag{1.6}$$

where

$$\frac{1}{\gamma} + \frac{2}{q} + \frac{2}{\alpha} = \lambda \in [2, 3) \ \ \textit{and} \ \ \frac{3}{\lambda} \leq \gamma \leq \alpha < \frac{1}{\lambda - 2},$$

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where

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then the weak solution (u, ω) *becomes a regular solution on* $(0, T]$.

2. Preliminaries

Before to prove our main result, we first recall the following result proved in [30].

Lemma 2.1. *Let* $n \geq 2$ *be a natural number,* $\gamma_i, t_i \geq 0$, $p_i \in (1, +\infty)$, $i = 1, 2, 3, ..., n$ *and suppose that*

$$\sum_{i=1}^n \gamma_i = 1, \quad \sum_{i=1}^n \frac{1}{p_i} > 1 \ \ \textit{and} \ \ \delta = \frac{1 + t_i}{\gamma_i(n-1) - 1 + \frac{1}{p_i}} > 0.$$

Then there exists $C > 0$ *such that for every* $f \in C^\infty \cap L^2$

$$\left(\int |f(x)|^\delta \, dx \right)^{\left(\sum\limits_{i=1}^n \frac{1}{p_i} \right) - 1} \leq C \prod_{i=1}^n \left(\int |f(x)|^{t_i p_i} |\partial_i f(x)|^{p_i} \, dx \right)^{\frac{1}{p_i}}.$$

Next, we recall the following Gagliardo-Nirenberg interpolation inequality in $\mathbb{R}^1$.

Lemma 2.2. *Let* $1 \leq \kappa, \mu, \nu < \infty$ *satisfy*

$$\frac{1}{\kappa} = (\frac{1}{\nu} - 1)\theta + \frac{1-\theta}{\mu} \quad \textit{for some} \ \ \theta \in [0,1].$$

Assume that $\varphi \in H^1(\mathbb{R}^3)$*. Then there exists a constant* $C > 0$ *such that*

$$\|\varphi\|_{L^{\kappa}_{x_3}} \leq C \|\partial_3\varphi\|^{\theta}_{L^{\nu}_{x_3}} \|\varphi\|^{(1-\theta)}_{L^{\mu}_{x_3}}. \tag{2.1}$$

The crucial tool in this paper is the following result, which plays important role in proving our main result.

Lemma 2.3. *Let* $r > 1$ *and* $1 < \gamma \leq \alpha < \infty$*. Then for* $f, g, \varphi \in C_0^\infty(\mathbb{R}^3)$*, we have*

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} fg\varphi dx_1dx_2dx_3\right| \\ \leq\ & C \left\|\|\partial_3\varphi\|_{L^{\gamma}_{x_3}}\right\|^{\frac{1}{r}}_{L^{\alpha}_{x_1x_2}} \left\|\|\partial_3\varphi\|_{L^{\gamma}_{x_3}}\right\|^{\frac{\theta(r-1)}{r}}_{L^{a\theta(r-1)}_{x_1x_2}} \left\|\|\varphi\|_{L^{\beta}_{x_3}}\right\|^{\frac{(1-\theta)(r-1)}{r}}_{L^{b(1-\theta)(r-1)}_{x_1x_2}} \\ & \times \|f\|^{\frac{r-1}{r}}_{L^2} \|\partial_1 f\|^{\frac{1}{2r}}_{L^2} \|\partial_2 f\|^{\frac{1}{2r}}_{L^2} \|g\|^{\frac{r-1}{r}}_{L^2} \|\partial_1 g\|^{\frac{1}{2r}}_{L^2} \|\partial_2 g\|^{\frac{1}{2r}}_{L^2}. \end{aligned}$$

where $0 \leq \theta \leq 1$ *satisfying*

$$\frac{1}{a} + \frac{1}{b} = \frac{\alpha - 1}{\alpha}, \tag{2.2}$$

and

$$\frac{1}{\gamma(r-1)} + \frac{\theta}{\gamma} = \frac{1-\theta}{\beta(\gamma-1)}. \tag{2.3}$$

and C *is a constant independent of* f, g, φ*.*

Proof: Invoking Hölder's inequality and Fubini's theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} |fg\varphi|\, dx_1dx_2dx_3 \\ \leq\ & \int_{\mathbb{R}^2} \left\{ \max_{x_3\in\mathbb{R}} |\varphi| \left(\int_{\mathbb{R}} |f|^2\, dx_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |g|^2\, dx_3\right)^{\frac{1}{2}} \right\} dx_1dx_2 \\ \leq\ & \left\{ \int_{\mathbb{R}^2} \left(\max_{x_3\in\mathbb{R}} |\varphi|\right)^r dx_1dx_2 \right\}^{\frac{1}{r}} \times \left\{ \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |f|^2\, dx_3\right)^{\frac{r}{r-1}} dx_1dx_2 \right\}^{\frac{r-1}{2r}} \\ & \times \left\{ \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |g|^2\, dx_3\right)^{\frac{r}{r-1}} dx_1dx_2 \right\}^{\frac{r-1}{2r}}. \end{aligned}$$

Notice that

$$\max_{x_3\in\mathbb{R}} |\varphi|^r \leq r \int_{\mathbb{R}} |\partial_3\varphi| \cdot |\varphi|^{r-1}\, dx_3.$$

Moreover, by Hölder's inequality and (2.1), we obtain

$$\begin{aligned}
\left\{\int_{\mathbb{R}^2}\left(\max_{x_3\in\mathbb{R}}|\varphi|\right)^r dx_1dx_2\right\}^{\frac{1}{r}} &\le r^{\frac{1}{r}}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}}|\partial_3\varphi|\cdot|\varphi|^{r-1}\,dx_3dx_1dx_2\right\}^{\frac{1}{r}}\\
&\le r^{\frac{1}{r}}\left\|\|\partial_3\varphi\|_{L^\gamma_{x_3}}\right\|^{\frac{1}{r}}_{L^\alpha_{x_1x_2}}\left\|\left\||\varphi|^{r-1}\right\|_{L^{\frac{\gamma}{\gamma-1}}_{x_3}}\right\|^{\frac{1}{r}}_{L^{\frac{\alpha}{\alpha-1}}_{x_1x_2}}\\
&\le r^{\frac{1}{r}}\left\|\|\partial_3\varphi\|_{L^\gamma_{x_3}}\right\|^{\frac{1}{r}}_{L^\alpha_{x_1x_2}}\left\|\|\partial_3\varphi\|^{\theta(r-1)}_{L^\gamma_{x_3}}\|\varphi\|^{(1-\theta)(r-1)}_{L^\beta_{x_3}}\right\|^{\frac{1}{r}}_{L^{\frac{\alpha}{\alpha-1}}_{x_1x_2}}\\
&\le r^{\frac{1}{r}}\left\|\|\partial_3\varphi\|_{L^\gamma_{x_3}}\right\|^{\frac{1}{r}}_{L^\alpha_{x_1x_2}}\left\|\|\partial_3\varphi\|^{\theta(r-1)}_{L^\gamma_{x_3}}\right\|^{\frac{1}{r}}_{L^a_{x_1x_2}}\left\|\|\varphi\|^{(1-\theta)(r-1)}_{L^\beta_{x_3}}\right\|^{\frac{1}{r}}_{L^b_{x_1x_2}}\\
&\le r^{\frac{1}{r}}\left\|\|\partial_3\varphi\|_{L^\gamma_{x_3}}\right\|^{\frac{1}{r}}_{L^\alpha_{x_1x_2}}\left\|\|\partial_3\varphi\|_{L^\gamma_{x_3}}\right\|^{\frac{\theta(r-1)}{r}}_{L^{a\theta(r-1)}_{x_1x_2}}\left\|\|\varphi\|_{L^\beta_{x_3}}\right\|^{\frac{(1-\theta)(r-1)}{r}}_{L^{b(1-\theta)(r-1)}_{x_1x_2}},
\end{aligned}$$

where we use the interpolation theorem

$$\frac{1}{a}+\frac{1}{b}=\frac{\alpha-1}{\alpha}, \tag{2.4}$$

and

$$\frac{1}{\gamma(r-1)}+\frac{\theta}{\gamma}=\frac{1-\theta}{\beta(\gamma-1)}.$$

We can use Lemma 2.1, where we put $n=2$, $\delta=\frac{2r}{r-1}$, $p_1=p_2=\frac{2}{r}$, $\gamma_1=\gamma_1=\frac{1}{2}$, $t_1=t_2=r-1$ and estimate by applying Hölder's inequality

$$\begin{aligned}
\left(\int_{\mathbb{R}^2}|f|^{\frac{2r}{r-1}}\,dx_1dx_2\right)^{\frac{r-1}{r}} &= \left\{\left(\int_{\mathbb{R}^2}|f|^{\frac{2r}{r-1}}\,dx_1dx_2\right)^{r-1}\right\}^{\frac{1}{r}}\\
&\le \left\{\prod_{i=1}^{2}\left(\int_{\mathbb{R}^2}|f|^{\frac{2(r-1)}{r}}\,|\partial_i f|^{\frac{2}{r}}\,dx_1dx_2\right)^{\frac{r}{2}}\right\}^{\frac{1}{r}}\\
&\le \left\{\prod_{i=1}^{2}\left(\int_{\mathbb{R}^2}|f|^2\,dx_1dx_2\right)^{\frac{r-1}{2}}\left(\int_{\mathbb{R}^2}|\partial_i f|^2\,dx_1dx_2\right)^{\frac{1}{2}}\right\}^{\frac{1}{r}}\\
&= \left(\int_{\mathbb{R}^2}|f|^2\,dx_1dx_2\right)^{\frac{r-1}{r}}\left(\int_{\mathbb{R}^2}|\partial_1 f|^2\,dx_1dx_2\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^2}|\partial_2 f|^2\,dx_1dx_2\right)^{\frac{1}{r}}\\
&= \|f\|^{\frac{2(r-1)}{r}}_{L^2(\mathbb{R}^2)}\|\partial_1 f\|^{\frac{1}{r}}_{L^2(\mathbb{R}^2)}\|\partial_2 f\|^{\frac{1}{r}}_{L^2(\mathbb{R}^2)}
\end{aligned}$$

So by applying Minkowski's inequality, we obtain

$$\left\{\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|f|^2\,dx_3\right)^{\frac{r}{r-1}}dx_1dx_2\right\}^{\frac{r-1}{2r}} \le \left\{\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|f|^{\frac{2r}{r-1}}\,dx_1dx_2\right)^{\frac{r-1}{r}}dx_3\right\}^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le \left\{ \int_{\mathbb{R}} \|f\|_{L^2(\mathbb{R}^2)}^{\frac{2(r-1)}{r}} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{r}} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{r}} \, dx_3 \right\}^{\frac{1}{2}} \\
&\le \|f\|_{L^2}^{\frac{r-1}{r}} \|\partial_1 f\|_{L^2}^{\frac{1}{2r}} \|\partial_2 f\|_{L^2}^{\frac{1}{2r}} .
\end{aligned}
$$

Similarly, we have

$$
\left\{ \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |g|^2 \, dx_3 \right)^{\frac{r}{r-1}} dx_1 dx_2 \right\}^{\frac{r-1}{2r}} \le \|g\|_{L^2}^{\frac{r-1}{r}} \|\partial_1 g\|_{L^2}^{\frac{1}{2r}} \|\partial_2 g\|_{L^2}^{\frac{1}{2r}} .
$$

Thus,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |f g \varphi| \, dx_1 dx_2 dx_3 \\
\le\ & C \left\| \|\partial_3 \varphi\|_{L^\gamma_{x_3}} \right\|_{L^\alpha_{x_1 x_2}}^{\frac{1}{r}} \left\| \|\partial_3 \varphi\|_{L^\gamma_{x_3}} \right\|_{L^{\theta(r-1)a}_{x_1 x_2}}^{\frac{\theta(r-1)}{r}} \left\| \|\varphi\|_{L^\beta_{x_3}} \right\|_{L^{(1-\theta)(r-1)b}_{x_1 x_2}}^{\frac{(1-\theta)(r-1)}{r}} \\
& \times \|f\|_{L^2}^{\frac{r-1}{r}} \|\partial_1 f\|_{L^2}^{\frac{1}{2r}} \|\partial_2 f\|_{L^2}^{\frac{1}{2r}} \|g\|_{L^2}^{\frac{r-1}{r}} \|\partial_1 g\|_{L^2}^{\frac{1}{2r}} \|\partial_2 g\|_{L^2}^{\frac{1}{2r}} .
\end{aligned}
$$

□

We recall the following result according to Dong et al. [10], that will be used in the proof of Theorem 1.5.

Lemma 2.4. *Suppose* $(u_0, \omega_0) \in L^s(\mathbb{R}^3)$, $s > 3$ *with* $\nabla \cdot u_0 = 0$ *in* $\mathbb{R}^3$. *Then there exists* $T > 0$ *and a unique strong solution* (u, ω) *of the 3D micropolar fluid equations (1.4) such that*

$$
(u, \omega) \in (L^\infty \cap C) \left([0, T); L^s(\mathbb{R}^3) \right).
$$

Moreover, let $(0, T_0)$ *be the maximal interval such that* (u, ω) *solves (1.4) in* $C\left((0, T_0); L^s(\mathbb{R}^3) \right)$, $s > 3$. *Then, for any* $t \in (0, T_0)$,

$$
\|(u, \omega)(\cdot, t)\|_{L^s} \ge \frac{C}{(T_0 - t)^{\frac{s-3}{2s}}}
$$

with the constant C *independent of* T_0 *and* s.

By a strong solution we mean a weak solution (u, ω) such that

$$
(u, \omega) \in L^\infty \left((0, T) \, ; H^1(\mathbb{R}^3) \right) \cap L^2 \left((0, T) \, ; H^2(\mathbb{R}^3) \right).
$$

It is well-known that strong solution are regular (say, classical) and unique in the class of weak solutions.

3. Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5.

Proof: First, we multiply both sides of the equation $(1.4)_1$ by $u|u|^2$, and integrate over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4+\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|u|^2\right|^2dx \\ \le\ &\left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|u|^2u)dx\right|+\int_{\mathbb{R}^3}|\omega|\,|u|^2\,|\nabla u|\,dx, \end{aligned} \tag{3.1}$$

where we used the following identities due to divergence free condition:

$$\begin{aligned} \int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot|u|^2\,udx &= \frac{1}{4}\int_{\mathbb{R}^3}u\cdot\nabla|u|^4\,dx=0, \\ \int_{\mathbb{R}^3}(\Delta u)\cdot|u|^2\,udx &= -\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx-2\int_{\mathbb{R}^3}|\nabla|u||^2|u|^2\,dx \\ &= -\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|u|^2\right|^2dx, \end{aligned}$$

$$\int_{\mathbb{R}^3}\nabla\times\omega\cdot|u|^2\,udx=-\int_{\mathbb{R}^3}|u|^2\,\omega\cdot\nabla\times udx-\int_{\mathbb{R}^3}\omega\cdot\nabla|u|^2\times udx.$$

Note that

$$|\nabla\times u|\le|\nabla u|\,,\quad |\nabla|u||\le|\nabla u|\,.$$

Multiplying the second equation of (1.4) by $\omega|\omega|^2$, then integrating the resulting equation with respect to x over $\mathbb{R}^3$ and using integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\|\omega(\cdot,t)\|_{L^4}^4+\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|^2\,dx+\int_{\mathbb{R}^3}\left|\nabla|\omega|^2\right|^2dx \\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|^2\,dx+2\int_{\mathbb{R}^3}|\omega|^4\,dx \\ =\ &\int_{\mathbb{R}^3}\nabla\times u\cdot|\omega|^2\,\omega dx, \end{aligned} \tag{3.2}$$

where we have used the fact that $\nabla\mathrm{div}\omega=\nabla\times(\nabla\times\omega)+\Delta\omega$ yields

$$\begin{aligned} &-\int_{\mathbb{R}^3}\nabla\mathrm{div}\omega\cdot|\omega|^2\,\omega dx \\ =\ &-\int_{\mathbb{R}^3}(\nabla\times(\nabla\times\omega)+\Delta\omega)\cdot|\omega|^2\,\omega dx \\ =\ &\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|^2\,dx+\int_{\mathbb{R}^3}\nabla\times\omega\cdot\nabla|\omega|^2\times\omega dx+\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|\omega|^2\right|^2dx \\ \ge\ &\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla\times\omega|^2|\omega|^2+\left|\nabla|\omega|^2\right|^2)dx+\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|\omega|^2\right|^2dx \\ =\ &-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|\omega|^2\right|^2dx. \end{aligned}$$

Combining (3.1) and (3.2) together, it follows that

$$\frac{1}{4}\frac{d}{dt}\left(\|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4\right)+\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla|u|^2\right|^2dx$$

$$
\begin{aligned}
& + \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 \, dx + \int_{\mathbb{R}^3} \left| \nabla |\omega|^2 \right|^2 dx + 2 \int_{\mathbb{R}^3} |\omega|^4 \, dx \\
\leq & \left| \int_{\mathbb{R}^3} \nabla \pi \cdot (|u|^2 u) dx \right| + \int_{\mathbb{R}^3} |\omega| \, |u|^2 \, |\nabla u| \, dx + \int_{\mathbb{R}^3} |u| \, |\omega|^2 \, |\nabla \omega| \, dx \\
= & \; A_1 + A_2 + A_3. \qquad (3.3)
\end{aligned}
$$

With the use of Hölder's inequality and Young's inequality, the first two terms on the right-hand side of (3.3) is bounded by

$$
\begin{aligned}
& \int_{\mathbb{R}^3} |\omega| \, |u|^2 \, |\nabla u| \, dx + \int_{\mathbb{R}^3} |u| \, |\omega|^2 \, |\nabla \omega| \, dx \\
\leq & \; \| |\omega| |u| \|_{L^2} \, \| |u| |\nabla u| \|_{L^2} + \| |\omega| |u| \|_{L^2} \, \| |\omega| |\nabla \omega| \|_{L^2} \\
\leq & \; \frac{1}{2} \| |u| |\nabla u| \|_{L^2}^2 + \frac{1}{2} \| |\omega| |u| \|_{L^2}^2 + \frac{1}{2} \| |\omega| |\nabla \omega| \|_{L^2}^2 + \frac{1}{2} \| |\omega| |u| \|_{L^2}^2 \\
\leq & \; \frac{1}{2} \| |u| |\nabla u| \|_{L^2}^2 + \frac{1}{2} \| |\omega| |\nabla \omega| \|_{L^2}^2 + \|u\|_{L^4}^2 \, \|\omega\|_{L^4}^2 \\
\leq & \; \frac{1}{2} \| |u| |\nabla u| \|_{L^2}^2 + \frac{1}{2} \left(\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4 \right). \qquad (3.4)
\end{aligned}
$$

Let us now estimate the integral A_1. The Cauchy inequality implies that

$$
\begin{aligned}
A_1 = & \left| \int_{\mathbb{R}^3} \nabla \pi \cdot (|u|^2 u) dx \right| = \left| \int_{\mathbb{R}^3} \pi \cdot \operatorname{div}(|u|^2 u) dx \right| \\
\leq & \; 2 \int_{\mathbb{R}^3} |\pi| \, |u|^2 \, |\nabla u| \, dx \leq 2 \, \|\pi u\|_{L^2} \, \| |u| |\nabla u| \|_{L^2} \\
\leq & \; C \int_{\mathbb{R}^3} |\pi|^2 \, |u|^2 \, dx + \frac{1}{2} \| |u| |\nabla u| \|_{L^2}^2 . \qquad (3.5)
\end{aligned}
$$

Let us estimate the integral

$$
I = \int_{\mathbb{R}^3} |\pi|^2 \, |u|^2 \, dx
$$

on the right-hand side of (3.5). Before turning to estimate I, it is well-known that for the micropolar fluid equations in $\mathbb{R}^3$, we have the following relationship between π and u and Calderón-Zygmund inequality

$$
-\Delta \pi = \operatorname{div}(u \cdot \nabla u) = \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j),
$$

$$
\|\pi\|_{L^q} \leq C \, \|u\|_{L^{2q}}^2 , \quad 1 < q < \infty.
$$

We select that $a = \frac{\alpha(\gamma+\alpha\gamma-\alpha)}{\alpha-\gamma}$ and $b = \frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma-1)}$ in Lemma 2.3, then the selected a and b satisfy (2.4). Then we can estimate I as follows

$$
\begin{aligned}
I = & \int_{\mathbb{R}^3} |\pi| \, |\pi| \, |u|^2 \, dx_1 dx_2 dx_3 \\
\leq & \; C \left\| \|\partial_3 \pi\|_{L^{\gamma}_{x_3}} \right\|_{L^{\alpha}_{x_1 x_2}}^{\frac{\gamma}{\gamma+\beta(\gamma-1)}} \|\pi\|_{L^{\beta}}^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}} \|\pi\|_{L^2}^{\frac{\alpha-\gamma+\alpha\beta(\gamma-1)}{\alpha(\gamma+\beta(\gamma-1))}} \|\partial_1 \pi\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \|\partial_2 \pi\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}}
\end{aligned}
$$

$$
\begin{aligned}
& \times \left\| |u|^2 \right\|_{L^2}^{\frac{\alpha-\gamma+\alpha\beta(\gamma-1)}{\alpha(\gamma+\beta(\gamma-1))}} \left\| \partial_1 |u|^2 \right\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \left\| \partial_2 |u|^2 \right\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{2\alpha(\gamma+\beta(\gamma-1))}} \\
\leq\ & C \left\| \|\partial_3 \pi\|_{L^{\gamma}_{x_3}} \right\|_{L^{\alpha}_{x_1 x_2}}^{\frac{\gamma}{\gamma+\beta(\gamma-1)}} \|\pi\|_{L^{\beta}}^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}} \|u\|_{L^4}^{\frac{4(\alpha-\gamma+\alpha\beta(\gamma-1))}{\alpha(\gamma+\beta(\gamma-1))}} \|\nabla \pi\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \left\| \nabla |u|^2 \right\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \\
\leq\ & C \left\| \|\partial_3 \pi\|_{L^{\gamma}_{x_3}} \right\|_{L^{\alpha}_{x_1 x_2}}^{\frac{\gamma}{\gamma+\beta(\gamma-1)}} \|\pi\|_{L^{\beta}}^{\frac{\beta(\gamma-1)}{\gamma+\beta(\gamma-1)}} \|u\|_{L^4}^{\frac{4(\alpha-\gamma+\alpha\beta(\gamma-1))}{\alpha(\gamma+\beta(\gamma-1))}} \||u|\,|\nabla u|\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \left\| \nabla |u|^2 \right\|_{L^2}^{\frac{\gamma+\alpha\gamma-\alpha}{\alpha(\gamma+\beta(\gamma-1))}} \\
\leq\ & C \left\| \|\partial_3 \pi\|_{L^{\gamma}_{x_3}} \right\|_{L^{\alpha}_{x_1 x_2}}^{\frac{\alpha\gamma}{\alpha-\gamma+\alpha\beta(\gamma-1)}} \|\pi\|_{L^{\beta}}^{\frac{\alpha\beta(\gamma-1)}{\alpha-\gamma+\alpha\beta(\gamma-1)}} \|u\|_{L^4}^4 + \frac{1}{4}\left(\||u|\,|\nabla u|\|_{L^2}^2 + \left\| \nabla |u|^2 \right\|_{L^2}^2 \right),
\end{aligned}
$$

where α, β, r and θ satisfy the following identities

$$
\begin{cases}
\alpha = \theta(r-1)a, \\
\beta = (1-\theta)(r-1)b, \\
r = \frac{\alpha\gamma+\alpha\beta(\gamma-1)}{\gamma+\alpha\gamma-\alpha} \\
\theta = \frac{\alpha-\gamma}{\alpha\beta(\gamma-1)+\alpha-\gamma}.
\end{cases}
\tag{3.6}
$$

Using the fact that $2 \leq \lambda < 3$, we choose $r = \frac{(4-\lambda)\alpha\gamma}{\gamma+\alpha\gamma-\alpha}$, then it follows from $(3.6)_3$ that $\beta = \frac{(3-\lambda)\gamma}{\gamma-1}$. Now, on the one hand, observe that

$$
\begin{aligned}
\gamma \quad < \quad & \frac{1}{\lambda-2} \Leftrightarrow \lambda\gamma - 3 < 2(3-\lambda)\gamma \\
\Leftrightarrow \quad & \lambda\alpha\gamma - 3\alpha < 2(3-\lambda)\alpha\gamma \\
\Leftrightarrow \quad & \lambda\alpha\gamma - \alpha - 2\gamma < 2(3-\lambda)\alpha\gamma - 2\gamma + 2\alpha \\
\Leftrightarrow \quad & \frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} < 1.
\end{aligned}
$$

On the other hand, since

$$
\gamma \geq \frac{3}{\lambda} \Leftrightarrow \lambda\alpha\gamma \geq 3\alpha \Leftrightarrow \lambda\alpha\gamma - \alpha - 2\gamma \geq 2\alpha - 2\gamma
$$

and since $\alpha \geq \gamma$, we get

$$
\lambda\alpha\gamma - \alpha - 2\gamma \geq 0.
$$

But you know, λ must be less than 3, hence

$$
\begin{cases}
(3-\lambda)\alpha\gamma > 0 \\
\alpha - \gamma \geq 0
\end{cases}
$$

which implies that $(3-\lambda)\alpha\gamma + (\alpha-\gamma) > 0$. Gathering these estimates together, we obtain

$$
0 \leq \frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} < 1,
$$

and it is clear that

$$
\frac{\lambda\alpha\gamma - \alpha - 2\gamma}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} + \frac{2(3-\lambda)\alpha\gamma - \alpha(\lambda\gamma - 3)}{2[(3-\lambda)\alpha\gamma - \gamma + \alpha]} = 1,
$$

Now using Hölder inequality with exponents $\frac{\lambda\alpha\gamma-\alpha-2\gamma}{2[(3-\lambda)\alpha\gamma-\gamma+\alpha]}$ and $\frac{2(3-\lambda)\alpha\gamma-\alpha(\lambda\gamma-3)}{2[(3-\lambda)\alpha\gamma-\gamma+\alpha]}$, I can be further estimated as

$$\begin{aligned} I &\leq \frac{1}{4}(\left\|\nabla |u|^2\right\|_{L^2}^2 + \||u|\nabla |u|\|_{L^2}^2) + C\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{1}{3-\lambda}} \|\pi\|_{L^3}\|u\|_{L^4}^4 \\ &\leq \frac{1}{4}(\left\|\nabla |u|^2\right\|_{L^2}^2 + \||u|\nabla |u|\|_{L^2}^2) + C\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{1}{3-\lambda}} \|u\|_{L^6}^2\|u\|_{L^4}^4 \\ &\leq \frac{1}{4}(\left\|\nabla |u|^2\right\|_{L^2}^2 + \||u|\nabla |u|\|_{L^2}^2) + C\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{1}{3-\lambda}} \|\nabla u\|_{L^2}^2\|u\|_{L^4}^4, \end{aligned}$$

when $\frac{\lambda\alpha\gamma-\alpha-2\gamma}{2[(3-\lambda)\alpha\gamma-\gamma+\alpha]} = 0$ (i.e. $\alpha = \gamma = \frac{3}{\lambda}$) or

$$\begin{aligned} I &\leq \frac{1}{4}(\left\|\nabla |u|^2\right\|_{L^2}^2 + \||u|\nabla |u|\|_{L^2}^2) + C\left(\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}} + \|\pi\|_{L^\beta}^{\frac{2\gamma(3-\lambda)}{2\gamma(3-\lambda)-3(\gamma-1)}}\right)\|u\|_{L^4}^4 \\ &\leq \frac{1}{4}(\left\|\nabla |u|^2\right\|_{L^2}^2 + \||u|\nabla |u|\|_{L^2}^2) + C\left(\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}} + \|u\|_{L^{2\beta}}^{\frac{4\gamma(3-\beta)}{2\gamma(3-\lambda)-3(\gamma-1)}}\right)\|u\|_{L^4}^4, \end{aligned}$$

when $0 < \frac{\lambda\alpha\gamma-\alpha-2\gamma}{2[(3-\lambda)\alpha\gamma-\gamma+\alpha]} < 1$ (i.e. $\frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}$) and $\beta = \frac{(3-\lambda)\gamma}{\gamma-1}$.

Combining all the estimates from above, we get

$$\frac{d}{dt}\left(\|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4\right) \leq \begin{cases} C\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{1}{3-\lambda}} \|\nabla u\|_{L^2}^2\|u\|_{L^4}^4 + C\left(\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4\right), \\ \qquad \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\ \left(\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}} + \|u\|_{L^{2\beta}}^{\frac{4\gamma(3-\lambda)}{2\gamma(3-\lambda)-3(\gamma-1)}}\right)\|u\|_{L^4}^4 \\ \quad +C\left(\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4\right), \\ \qquad \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}. \end{cases} \tag{3.7}$$

Defining

$$H(t) = e + \|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4,$$

and thanks to

$$\begin{aligned} 1 + \ln(1 + \|\omega\|_{L^4}) &\leq 1 + \ln(e + \|\omega\|_{L^4}^4) \\ &\leq 1 + \ln(e + \|u\|_{L^4}^4 + \|\omega\|_{L^4}^4), \end{aligned}$$

inequality (3.7) implyes that

$$\frac{d}{dt}H(t) \leq \begin{cases} C\frac{\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{1}{3-\lambda}}}{1+\ln(1+\|\omega\|_{L^4})} \|\nabla u\|_{L^2}^2 H(t)(1+\ln H(t)), \\ \qquad \text{if } \gamma = \alpha = \frac{3}{\lambda}, \\ \left(\frac{\left\|\|\partial_3\pi\|_{L^\gamma_{x_3}}\right\|_{L^\alpha_{x_1x_2}}^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}}}{1+\ln(1+\|\omega\|_{L^4})} + \|u\|_{L^{2\beta}}^{\frac{4\gamma(3-\lambda)}{2\gamma(3-\lambda)-3(\gamma-1)}}\right) H(t)(1+\ln H(t)), \\ \qquad \text{if } \frac{3}{\lambda} < \gamma \leq \alpha < \frac{1}{\lambda-2}, \end{cases}$$

and hence

$$\frac{d}{dt}(1+\ln H(t)\leq \begin{cases} C\dfrac{\left\|\|\partial_3\pi\|_{L^{\gamma}_{x_3}}\right\|^{\frac{1}{3-\lambda}}_{L^{\alpha}_{x_1x_2}}}{1+\ln(1+\|\omega\|_{L^4})}\|\nabla u\|^2_{L^2}(1+\ln H(t)), & \text{if } \gamma=\alpha=\frac{3}{\lambda},\\ \left(\dfrac{\left\|\|\partial_3\pi\|_{L^{\gamma}_{x_3}}\right\|^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}}_{L^{\alpha}_{x_1x_2}}}{1+\ln(1+\|\omega\|_{L^4})}+\|u\|^{\frac{4\gamma(3-\lambda)}{2\gamma(3-\lambda)-3(\gamma-1)}}_{L^{2\beta}}\right)(1+\ln H(t)), & \text{if } \frac{3}{\lambda}<\gamma\leq\alpha<\frac{1}{\lambda-2}. \end{cases}$$

Thanks to (u,ω) is a weak solution of the 3D micropolar equations (1.4), that is

$$u\in L^{\infty}(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3)), \tag{3.8}$$

together with the interpolation inequality yields that

$$u\in L^s(0,T;L^r(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{s}+\frac{3}{r}=\frac{3}{2} \text{ and } 2\leq r\leq 6.$$

On the other hand, since

$$\gamma<\frac{1}{\lambda-2}\Leftrightarrow 3\gamma-\lambda\gamma>\gamma-1\Leftrightarrow\frac{(3-\lambda)\gamma}{\gamma-1}>1$$

and

$$\gamma>\frac{3}{\lambda}\Leftrightarrow 3\gamma-\lambda\gamma<3\gamma-3\Leftrightarrow\frac{(3-\lambda)\gamma}{\gamma-1}<3,$$

it is easy to see that

$$2<\frac{2(3-\lambda)\gamma}{\gamma-1}<6 \quad \text{if} \quad \frac{3}{\lambda}<\gamma\leq\alpha<\frac{1}{\lambda-2}$$

and consequently

$$\frac{2}{\frac{4\gamma(3-\lambda)}{2(3-\lambda)\gamma-3(\gamma-1)}}+\frac{3}{\frac{2\gamma(3-\lambda)}{\gamma-1}}=\frac{2(3-\lambda)\gamma-3(\gamma-1)}{2\gamma(3-\lambda)}+\frac{3(\gamma-1)}{2\gamma(3-\lambda)}=\frac{3}{2}.$$

Hence, one has

$$u\in L^{\frac{4\gamma(3-\lambda)}{2(3-\lambda)\gamma-3(\gamma-1)}}(0,T;L^{\frac{2\gamma(3-\lambda)}{\gamma-1}}(\mathbb{R}^3)), \quad \text{if} \quad \frac{3}{\lambda}<\gamma\leq\alpha<\frac{1}{\lambda-2}. \tag{3.9}$$

Applying the Gronwall inequality yields that

$$\ln(H(t))\leq C(T,u_0,\omega_0)\begin{cases} \exp\left\{C\sup\limits_{0\leq t\leq T}\left(\dfrac{\left\|\|\partial_3\pi(\cdot,t)\|_{L^{\gamma}_{x_3}}\right\|^{\frac{1}{3-\lambda}}_{L^{\alpha}_{x_1x_2}}}{1+\ln(1+\|\omega(\cdot,t)\|_{L^4})}\right)\int_0^t\|\nabla u(\cdot,\tau)\|^2_{L^2}\,d\tau\right\}, \\ \qquad \text{if } \gamma=\alpha=\frac{3}{\lambda},\\ \exp\left\{\int_0^t\left(\dfrac{\left\|\|\partial_3\pi(\cdot,\tau)\|_{L^{\gamma}_{x_3}}\right\|^{\frac{2\alpha\gamma}{\lambda\alpha\gamma-\alpha-2\gamma}}_{L^{\alpha}_{x_1x_2}}}{1+\ln(1+\|\omega(\cdot,\tau)\|_{L^4})}+\|u(\cdot,\tau)\|^{\frac{4\gamma(3-\lambda)}{2\gamma(3-k)-3(\gamma-1)}}_{L^{2\beta}}\right)d\tau\right\}, \\ \qquad \text{if } \frac{3}{\lambda}<\gamma\leq\alpha<\frac{1}{\lambda-2}. \end{cases} \tag{3.10}$$

Now, we are in a position to complete the proof of Theorem 1.5. From Lemma 2.4, it follows that there exists $T_0 > 0$ and the smooth solution $(\widetilde{u}, \widetilde{\omega})$ of (1.4) satisfying

$$(\widetilde{u}, \widetilde{\omega})(t) \in (L^\infty \cap C)([0, T_0); L^4(\mathbb{R}^3)), \quad (\widetilde{u}, \widetilde{\omega})(0) = (u_0, \omega_0).$$

Since the weak solution (u, ω) satisfies the energy inequality, we may apply Serrin's uniqueness criterion to conclude that

$$(u, \omega) \equiv (\widetilde{u}, \widetilde{\omega}) \text{ on } [0, T_0).$$

Thus, it is sufficient to show that $T_0 = T$. Suppose that $T_0 < T$. Without loss of generality, we may assume that T_0 is the maximal existence time for $(\widetilde{u}, \widetilde{\omega})(t)$. By lemma 2.4 again, we find that

$$\|u(\cdot, t)\|_{L^4} + \|\omega(\cdot, t)\|_{L^4} \geq \frac{C}{(T_0 - t)^{\frac{1}{8}}} \quad \text{for any } t \in (0, T_0). \tag{3.11}$$

On the other hand, from (3.10), we know that

$$\sup_{0 \leq t \leq T_0} \left(\|u(\cdot, t)\|_{L^4}^4 + \|\omega(\cdot, t)\|_{L^4}^4 \right) \leq C(T, u_0, \omega_0) \tag{3.12}$$

which contradicts with (3.11). Thus, $T_0 = T$. This completes the proof of Theorem 1.5. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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