MULTIPLE SOLUTIONS FOR NONLINEAR STURM-LIOUVILLE DIFFERENTIAL EQUATIONS WITH POSSIBLY NEGATIVE VARIABLE COEFFICIENTS

ELEONORA AMOROSO, GABRIELE BONANNO, GIUSEPPINA D'AGUÌ

Abstract. The aim of this paper is to study the nonlinear differential problem with Sturm-Liouville type equation and Dirichlet boundary condition. In particular, by requiring a suitable behavior on the nonlinearity, we determine an interval of parameter λ for which the problem admits three or infinity many solutions.

1. Introduction

This paper is devoted to study the multiplicity of solutions for the following boundary value problem with a Sturm-Liouville second order differential equation and Dirichlet conditions

$$(D_\lambda) \qquad \begin{cases} -u'' + \gamma(t)u' + \delta(t)u = \lambda f(t,u) & \text{in }]a,b[, \\ u(a) = u(b) = 0. \end{cases}$$

where $\lambda \in \mathbb{R}^+$, $f \in L^1([a,b] \times \mathbb{R})$ is a function that satisfies the Carathéodory hypothesis and $\gamma, \delta \in L^\infty([a,b])$ such that $\operatorname*{ess\,inf}_{[a,b]} \delta > -\left(\frac{\pi}{b-a}\right)^2$. In literature, exists a lot of results about this kind of problems, because they are used to describe many type of physical and chemical events, for instance Boyd equation about eddies in the atmosphere [7], Laplace tidal wave equation [9] and Meissner equation which arises in a model of a one-dimensional crystal [10]. The study of existence and multiplicity of solutions for this kind of problems has been addressed by many authors by using various techniques of nonlinear analysis such as variational methods, upper-lower solutions and so on, see [12, 13] and references therein. Besides, the previous papers require that γ and δ are positive functions, which is a condition that often doesn't hold in the applied problems. In this paper, we present our results considering that γ and δ can be sign changing, in order to offer a link between pure and applied mathematics. For example, we can consider the following Sturm-Liouville differential equation describing gas dynamics in a fuel cell

$$-(tu')' - t^3u = \lambda f(t,u) \qquad \text{in }]0,b]\,, \text{ with } 0 < b < 1.$$

Fuel cells, see [2], are a clean alternative method of energy production, which convert chemical energy into electrical energy and combine hydrogen and oxygen to produce waste only in the form of water and heat. In particular, if we fix a, b such that $0 < a < b < 1$, one has the following autonomous problem

$$(FC) \qquad \begin{cases} -u'' - \frac{1}{t}u' - t^2 u = \lambda g(u) & \text{in }]a,b[, \\ u(a) = u(b) = 0. \end{cases}$$

As a particular case of our results, we obtain the following.

Theorem 1.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $G(x) = \int_0^x g(\xi)d\xi$ for all $x \in \mathbb{R}$ and assume that

$$\liminf_{\xi \to +\infty} \frac{G(\xi)}{\xi^2} < \frac{a^2\pi^2 - a^2b^2(b-a)^2}{4\,b^2\pi^2} \limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^2}. \tag{1.1}$$

Then, for each $\lambda \in \left] \frac{8b}{a(b-a)^2 \limsup\limits_{\xi \to +\infty} \frac{G(\xi)}{\xi^2}}, \frac{2a\pi^2 - 2ab^2(b-a)^2}{b\pi^2(b-a)^2 \liminf\limits_{\xi \to +\infty} \frac{G(\xi)}{\xi^2}} \right[$ the problem (FC) admits a sequence of pairwise distinct classical solutions.

The paper is organized as follows. In Section 2, we give some basic properties and we mention some critical point theorems, which are our main tools. In Section 3, we present our result on the existence of a sequence of pairwise distinct classical solutions for the problem (D_λ) and it's consequence in a particular autonomous case. The last section is dedicated to our results on the existence of three solutions, both for (D_λ) and the autonomous problem (AD_λ), giving some examples.

2. Basic properties and preliminaries

Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, that is a function such that

(i) $t \to f(t, x)$ is measurable for all $x \in \mathbb{R}$;
(ii) $x \to f(t, x)$ is continous for almost every $t \in [a, b]$;
(iii) for all $\rho > 0$ the function $\sup\limits_{|x| \le \rho} |f(t, x)|$ belongs to $L^1([a, b])$.

Consider the following Dirichlet boundary value problem

$$(D_\lambda) \qquad \begin{cases} -u'' + \gamma(t)u' + \delta(t)u = \lambda f(t, u) & \text{in }]a,b[, \\ u(a) = u(b) = 0. \end{cases}$$

where λ is a positive real number and $\gamma, \delta \in L^\infty([a, b])$, such that

$$\operatorname*{ess\,inf}_{[a,b]} \delta > -\left(\frac{\pi}{b-a}\right)^2. \tag{2.1}$$

Consider the Sobolev space $W_0^{1,2}([a, b])$, denoted in the paper by X, whose usual norm is defined as follows

$$\|u\| = \|u'\|_2 = \left(\int_a^b |u'(t)|^2 dt\right)^{\frac{1}{2}},$$

where $\|\cdot\|_2$ is the usual norm of the Lebesgue space $L^2([a,b])$. Here, we mention some results as given in [4]. We can introduce another norm in the space X, given by

$$\|u\|_X = \left(\int_a^b e^{-\Gamma(t)}|u'(t)|^2 dt + \int_a^b e^{-\Gamma(t)}\delta(t)|u(t)|^2 dt\right)^{\frac{1}{2}},$$

where

$$\Gamma(t) = \int_a^t \gamma(\xi)d\xi \qquad \forall\, t \in [a,b]\,. \tag{2.2}$$

Proposition 2.1. (see [4, Proposition 2.2]) Assume (2.1). Then $\|\cdot\|_X$ is a norm on X and it is equivalent to the usual norm. In particular, one has

$$m\|u\| \leq \|u\|_X \leq M\|u\|\,, \tag{2.3}$$

for all $u \in X$, where m, M, with $M \geq m > 0$, are given by

$$\begin{gathered}
m = \begin{cases} \left(\min\limits_{[a,b]} e^{-\Gamma}\right)^{\frac{1}{2}} & \text{if} \quad \operatorname*{ess\,inf}\limits_{[a,b]} \delta \geq 0 \\ \left[\min\limits_{[a,b]} e^{-\Gamma}\left(1 + \operatorname*{ess\,inf}\limits_{[a,b]} \delta \left(\frac{b-a}{\pi}\right)^2\right)\right]^{\frac{1}{2}} & \text{if} \quad \operatorname*{ess\,inf}\limits_{[a,b]} \delta < 0 \end{cases}, \\
M = \begin{cases} \left[\max\limits_{[a,b]} e^{-\Gamma}\left(1 + \operatorname*{ess\,sup}\limits_{[a,b]} \delta \left(\frac{b-a}{\pi}\right)^2\right)\right]^{\frac{1}{2}} & \text{if} \quad \operatorname*{ess\,sup}\limits_{[a,b]} \delta \geq 0 \\ \left(\max\limits_{[a,b]} e^{-\Gamma}\right)^{\frac{1}{2}} & \text{if} \quad \operatorname*{ess\,sup}\limits_{[a,b]} \delta < 0 \end{cases}.
\end{gathered} \tag{2.4}$$

Remark 2.1. (see [4, Remark 2.3]) The following inequality holds

$$\max_{[a,b]} |u(t)| \leq \frac{(b-a)^{\frac{1}{2}}}{2m}\|u\|_X \qquad \forall u \in X, \tag{2.5}$$

where m is given in (2.4).

Now, we recall the definition of classical and generalized solution for the problem (D_λ):

- $u : [a,b] \to \mathbb{R}$ is a classical solution if
  - $u \in C^2([a,b])$,
  - $u(a) = u(b) = 0$,
  - $-u''(t) + \gamma(t)u'(t) + \delta(t)u(t) = \lambda f(t, u(t))$ for all $t \in [a,b]$.
- $u : [a,b] \to \mathbb{R}$ is a generalized solution if
  - $u \in C^1([a,b]), u' \in AC([a,b])$,
  - $u(a) = u(b) = 0$,

– $-u''(t)+\gamma(t)u'(t)+\delta(t)u(t)=\lambda f(t,u(t))$ for almost every $t\in[a,b]$.

Clearly, a generalized solution u is a classical solution when f,γ,δ are continuous functions.
Now, put

$$F(t,x)=\int_0^x f(t,\xi)d\xi \qquad \forall (t,x)\in[a,b]\times\mathbb{R}.$$

Thanks to the hypotheses on the function f, one has

(i') $t\to F(t,x)$ is measurable for all $x\in\mathbb{R}$;
(ii') $x\to F(t,x)$ belongs to $C^1(\mathbb{R})$ for almost every $t\in[a,b]$;
(iii') $|F(t,x)|\le \sup_{|\xi|\le|x|}|f(t,\xi)|$ for almost every $t\in[a,b]$, for all $x\in\mathbb{R}$.

Furthermore, consider $\Psi: X\to\mathbb{R}$ and $\Phi: X\to\mathbb{R}$ defined by

$$\Psi(u)=\int_a^b e^{-\Gamma(t)}F(t,u(t))dt\,,$$

$$\Phi(u)=\frac{1}{2}\|u\|_X^2\,,$$

for all $u\in X$. It's well known, see for instance [4], that Ψ is well defined, Ψ and Φ are Gâteaux differentiable and one has

$$\Psi'(u)(v)=\int_a^b e^{-\Gamma(t)}f(t,u(t))v(t)dt\,,$$

$$\Phi'(u)(v)=\int_a^b e^{-\Gamma(t)}u'(t)v'(t)dt+\int_a^b e^{-\Gamma(t)}\delta(t)u(t)v(t)dt\,,$$

for all $u,v\in X$. Moreover, Φ,Ψ are C^1-functions.
Finally, put

$$I_\lambda(u)=\Phi(u)-\lambda\Psi(u) \qquad \forall u\in X.$$

Clearly, I_λ is a C^1-function and, in particular, one has

$$I'_\lambda(u)(v)=\int_a^b e^{-\Gamma(t)}u'(t)v'(t)dt+\int_a^b e^{-\Gamma(t)}\delta(t)u(t)v(t)dt-\lambda\int_a^b e^{-\Gamma(t)}f(t,u(t))v(t)dt\,,$$

for all $u,v\in X$.
We finally mention the following result.

Proposition 2.2. (see [4, Proposition 2.3]) u is a generalized solution of (D_λ) $\Longleftrightarrow$ u is a critical point of I_λ.

In order to achieve our multiplicity results, our main tools are some critical point theorems, that we recall in the following.

Let X be a non-empty set and let $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r > \inf_X \Phi$, we put

$$\varphi(r) = \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{v\in(\Phi^{-1}(-\infty,r))} \Psi(v)\right) - \Psi(u)}{r-\Phi(u)}, \tag{2.6}$$

and

$$\alpha := \liminf_{r\to+\infty} \varphi(r), \qquad \beta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r). \tag{2.7}$$

Theorem 2.1. (see [6, Theorem 2.1]) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is coercive, sequentially weakly lower semicontinuous and Gâteaux differentiable functional; $\Psi : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous and Gâteaux differentiable functional. One has

- (a) For every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X.
- (b) If $\alpha < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\alpha}\right[$, the following alternative holds: either
  - (b_1) $\Phi - \lambda\Psi$ possesses a global minimum, or
  - (b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.
- (c) If $\beta < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\beta}\right[$, the following alternative holds: either
  - (c_1) there is a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or
  - (c_2) there is a sequence of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.

Now, we present other two three critical point theorems. In particular, Theorem 2.2, obtained in [5], requires the coercivity of the functional $\Phi - \lambda\Psi$, while in Theorem 2.3, obtained in [1], a suitable sign hypothesis is assumed.

Theorem 2.2. (see [3, Theorem 3.2]) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there is a positive constant r and $\overline{v} \in X$, with $2r < \Phi(\overline{v})$, such that

(e_1) $\dfrac{\sup\limits_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)}{r} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(e_2) for all $\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)}\right[$, the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r = \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)}\right[$ the functional $\Phi - \lambda\Psi$ admits at least three distinct critical points.

Theorem 2.3. (see [3, Theorem 3.3]) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that

1. $\inf\limits_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. for each $\lambda > 0$ and for every u_1, u_2 which are local minima for the functional $\Phi - \lambda\Psi$, such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has
$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \dfrac{r_2}{2}$, such that

(f_1) $\dfrac{\sup\limits_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(f_2) $\dfrac{\sup\limits_{u\in\Phi^{-1}((-\infty,r_2))} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each $\lambda \in \Lambda_{r_1,r_2} = \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}; \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}((-\infty,r_2));} \Psi(u)}\right\}\right[$ the functional $\Phi - \lambda\Psi$ admits at least three distinct critical points which lie in $\Phi^{-1}((-\infty, r_2))$.

3. Existence of infinitely many solutions

This Section is devoted to the existence of infinitely many solutions for problem (D_λ). In particular, by requiring an appropriate oscillation of the primitive of the nonlinear term, we obtain a sequence of pairwise distinct classical solutions.

Put

$$K = \frac{1}{2}\frac{m^2}{M^2}\frac{\min\limits_{t\in[a,b]} e^{-\Gamma(t)}}{\max\limits_{t\in[a,b]} e^{-\Gamma(t)}}, \tag{3.1}$$

where m, M are given in (2.4) and Γ is defined in (2.2); we can observe that $0 < K \leq \frac{1}{2}$. Furthermore, put

$$A = \liminf_{\xi\to+\infty} \frac{\int_a^b \max\limits_{|x|\leq\xi} F(t,x)dt}{\xi^2}, \tag{3.2}$$

$$B = \limsup_{\xi\to+\infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\xi)dt}{\xi^2}, \tag{3.3}$$

and

$$\lambda_1 = \frac{4M^2}{B\,(b-a)\min\limits_{t\in[a,b]} e^{-\Gamma(t)}}, \qquad \lambda_2 = \frac{2m^2}{A\,(b-a)\max\limits_{t\in[a,b]} e^{-\Gamma(t)}}. \tag{3.4}$$

Now we can give our main result.

Theorem 3.1. Assume that

(j_1) $F(t,x) \geq 0$ for almost each $t \in \left[a, a+\frac{b-a}{4}\right] \cup \left[b-\frac{b-a}{4}, b\right]$ and for all $x \in [0,+\infty[$;

(j_2) $\liminf\limits_{\xi\to+\infty} \frac{\int_a^b \max_{|x|\leq\xi} F(t,x)dt}{\xi^2} < K \limsup\limits_{\xi\to+\infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\xi)dt}{\xi^2}$.

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are defined in (3.4), the problem (D_λ) admits a sequence of pairwise distinct classical solutions.

Proof. The proof is based on part (b) of Theorem 2.1. Take $(X, \|\cdot\|_X)$, Φ, $\Psi : X \to \mathbb{R}$ as defined in Section 2. Thanks to this choice, Φ and Ψ satisfy the hypotheses requested in Theorem 2.1. Moreover, as we have seen in Section 2, the critical points in X of the functional $I_\lambda = \Phi - \lambda\Psi$ are exactly the classical solutions of the considered problem (D_λ).

Firstly, we have to verify that $\alpha < +\infty$, where α is given in (2.7). Fix $\lambda \in]\lambda_1, \lambda_2[$ and let $\{c_n\}$ be a real sequence such that, being $\lim\limits_{n\to+\infty} c_n = +\infty$,

$$A = \lim_{n\to+\infty} \frac{\int_a^b \max\limits_{|x|\leq c_n} F(t,x)dt}{c_n^2}.$$

Put $r_n = \frac{2m^2}{b-a}c_n^2$ for all $n \in \mathbb{N}$. Taking Remark 2.1 into account, for each $u \in X$ such that $\Phi(u) = \frac{1}{2}\|u\|_X^2 < r_n$, one has

$$|u(t)| \leq \frac{(b-a)^{\frac{1}{2}}}{2m}\|u\|_X \leq \frac{(b-a)^{\frac{1}{2}}}{2m}(2r_n)^{\frac{1}{2}} = \left(\frac{b-a}{2m^2}r_n\right)^{\frac{1}{2}} = c_n \qquad \forall n \in \mathbb{N}$$

for all $t \in [a,b]$. Therefore, from (2.6), one has

$$\varphi(r_n) = \inf_{\Phi(u)<r_n} \frac{\left(\sup\limits_{\Phi(v)<r_n} \Psi(v)\right) - \Psi(u)}{r_n - \frac{\|u\|_X^2}{2}} \leq \frac{\sup\limits_{\|v\|_X^2<2r_n} \Psi(v)}{r_n} = \frac{\sup\limits_{\|v\|_X^2<2r_n} \int_a^b e^{-\Gamma(t)}F(t,v(t))dt}{\frac{2m^2}{b-a}c_n^2}.$$

Hence,

$$\varphi(r_n) \leq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max\limits_{|x|\leq c_n} F(t,x)dt}{c_n^2}.$$

Then, from (2.7) one has

$$\alpha = \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} A < +\infty .$$

At this point, we verify that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Let $\{d_n\}$ be a real sequence such that, being $\lim\limits_{n\to+\infty} d_n = +\infty$,

$$B = \lim_{n\to+\infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d_n)dt}{d_n^2}. \tag{3.5}$$

Now, for each $n \in \mathbb{N}$ consider the functions

$$w_n(t) = \begin{cases} 4d_n\,\frac{t-a}{b-a} & \text{if} \quad t \in \left[a, a+\frac{b-a}{4}\right[, \\ d_n & \text{if} \quad t \in \left[a+\frac{b-a}{4}, b-\frac{b-a}{4}\right], \\ 4d_n\,\frac{b-t}{b-a} & \text{if} \quad t \in \left]b-\frac{b-a}{4}, b\right]. \end{cases}$$

Clearly, $w_n \in X$ and

$$\|w_n\|^2 = \int_a^b |w_n'(t)|^2 dt = \int_a^{a+\frac{b-a}{4}} \left(\frac{4d_n}{b-a}\right)^2 dt + \int_{b-\frac{b-a}{4}}^b \left(\frac{4d_n}{b-a}\right)^2 dt = \frac{8d_n^2}{b-a}.$$

Hence, taking Proposition 2.1 into account one has

$$\Phi(w_n) = \frac{1}{2}\|w_n\|_X^2 \leq \frac{1}{2}M^2\|w_n\|^2 = \frac{4M^2d_n^2}{b-a}.$$

Moreover, taking (j_1) into account, one has

$$\Psi(w_n) = \int_a^b e^{-\Gamma(t)}F(t,w_n(t))dt \geq \min_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d_n)dt .$$

Therefore,

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) \le \frac{4M^2 d_n^2}{b-a} - \lambda \min_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d_n)dt\,.$$

Now, we have two possibilities.
If $B < +\infty$, let $\varepsilon \in \left]0, B - \frac{4M^2}{\lambda(b-a)\min_{t\in[a,b]} e^{-\Gamma(t)}}\right[$. From (3.5) there exists ν_ε such that

$$\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d_n)dt > (B-\varepsilon)d_n^2 \qquad \forall n > \nu_\varepsilon.$$

Hence, one has

$$I_\lambda(w_n) \le \frac{4M^2 d_n^2}{b-a} - \lambda \min_{t\in[a,b]} e^{-\Gamma(t)}(B-\varepsilon)d_n^2 = d_n^2\left(\frac{4M^2}{b-a} - \lambda \min_{t\in[a,b]} e^{-\Gamma(t)}(B-\varepsilon)\right).$$

From the choice of ε, one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty\,.$$

Besides, if $B = +\infty$, we fix $N > \frac{4M^2}{\lambda(b-a)\min_{t\in[a,b]} e^{-\Gamma(t)}}$; from (3.5) there exists ν_N such that

$$\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d_n)dt > N d_n^2 \qquad \forall n > \nu_N.$$

Consequently

$$I_\lambda(w_n) \le \frac{4M^2 d_n^2}{b-a} - \lambda \min_{t\in[a,b]} e^{-\Gamma(t)} N d_n^2 = d_n^2\left(\frac{4M^2}{b-a} - \lambda \min_{t\in[a,b]} e^{-\Gamma(t)} N\right).$$

Taking the choice of N into account, also in this case one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty\,.$$

Finally, it's easy to prove that

$$]\lambda_1, \lambda_2[\; \subseteq \left]0, \frac{1}{\alpha}\right[.$$

Then, from (b) of Theorem 2.1, for each $\lambda \in]\lambda_1, \lambda_2[$ the functional $I_\lambda = \Phi - \lambda\Psi$ admits a sequence $\{u_n\}$ of critical points such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$, which are classical solutions of the problem (D_λ). □

Remark 3.1. We observe that taking (c) of Theorem 2.1 into account and arguing as in the proof of Theorem 3.1, we obtain for problem (D_λ) for each $\lambda \in]\lambda_1, \lambda_2[$ the existence of a sequence of pairwise distinct classical solutions which converges uniformly to zero, assuming that

(j_1') $F(t,x) \geq 0$ for almost each $t \in \left[a, a+\frac{b-a}{4}\right] \cup \left[b-\frac{b-a}{4}, b\right]$ and for all $x \in [0,+\infty[$;

(j_2') $\liminf\limits_{\xi\to 0^+} \frac{\int_a^b \max_{|x|\leq\xi} F(t,x)dt}{\xi^2} < K \limsup\limits_{\xi\to 0^+} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,\xi)dt}{\xi^2}$.

Now, we consider the autonomous case. To this end, let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nonnegative function and consider the autonomous problem

$$(AD_\lambda) \qquad \begin{cases} -u'' + \gamma(t)u' + \delta(t)u = \lambda g(u) & \text{in }]a,b[, \\ u(a) = u(b) = 0. \end{cases}$$

Hence, put

$$G(x) = \int_0^x g(\xi)\,d\xi \qquad \forall x \in \mathbb{R},$$

and

$$\Psi(u) = \int_a^b e^{-\Gamma(t)} G(u(t))\,dt.$$

Moreover, put

$$\bar\lambda_1 = \frac{8M^2}{(b-a)^2 \min\limits_{t\in[a,b]} e^{-\Gamma(t)} \limsup\limits_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}}, \qquad \bar\lambda_2 = \frac{2m^2}{(b-a)^2 \max\limits_{t\in[a,b]} e^{-\Gamma(t)} \liminf\limits_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}}.$$

Here we point out a consequence of Theorem 3.1.

Corollary 3.1. Assume that

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} < \frac{1}{2} K \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}.$$

Then, for each $\lambda \in]\bar\lambda_1, \bar\lambda_2[$, the problem (AD_λ) admits a sequence of pairwise distinct classical solutions.

Remark 3.2. Now, we are able to prove Theorem 1.1 as a consequence of Corollary 3.1. In fact, if we consider the problem (AD_λ) with

$$\gamma(t) = -\frac{1}{t}, \qquad \delta(t) = -t^2,$$

we obtain the problem (FC) described in the Introduction. In particular, we observe that $\gamma(t) < 0$, $\delta(t) < 0$ for all $t \in [a,b]$. Hence, by simple calculations one has

$$\Gamma(t) = \int_a^t \gamma(\xi)d\xi = \ln\left(\frac{a}{t}\right),$$

$$m = \left[\frac{\pi^2 - b^2(b-a)^2}{\pi^2}\right]^{\frac{1}{2}}, \qquad M = \left(\frac{b}{a}\right)^{\frac{1}{2}},$$

and

$$K = \frac{a^2\pi^2 - a^2b^2(b-a)^2}{2\,b^2\pi^2}\,.$$

If (1.1) holds, then we can apply Corollary 3.1 and it follows that the problem (FC), for each $\lambda \in \left]\frac{8b}{a(b-a)^2 \limsup\limits_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}}, \frac{2a\pi^2-2ab^2(b-a)^2}{b\pi^2(b-a)^2\liminf\limits_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}}\right[$, admits a sequence of pairwise distinct classical solutions.

4. Existence of three solutions

In this Section, we present our main results on the existence of three solutions. In particular, in the first case (Theorem 4.1) an algebraic condition on the nonlinear term (hypothesis (h_2)) is required together with a sublinear growth to infinity. The second result on the existence of three solutions (Theorem 4.2) is obtained by requiring an appropriate growth on the nonlinear term in two consecutive real bounded intervals.

Theorem 4.1. Assume that there exist two positive constants c, d, with $c < d$, such that

(h_1) $F(t,x) \geq 0$ for almost each $t \in \left[a, a+\frac{b-a}{4}\right] \cup \left[b-\frac{b-a}{4}, b\right]$ and for all $x \in [0,d]$;

(h_2) $$\frac{\int_a^b \max\limits_{|x|\leq c} F(t,x)dt}{c^2} < \frac{2}{3}K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2};$$

(h_3) $$\lim_{|x|\to+\infty}\frac{F(t,x)}{|x|^2} = 0 \quad \text{for all } t\in[a,b], \text{ uniformly with respect to } t.$$

Then, for each $\lambda \in \Lambda_{c,d}$, where

$$\Lambda_{c,d} = \left]\frac{2m^2}{(b-a)\max\limits_{t\in[a,b]} e^{-\Gamma(t)}}\frac{3}{2K}\frac{d^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}, \frac{2m^2}{(b-a)\max\limits_{t\in[a,b]} e^{-\Gamma(t)}}\frac{c^2}{\int_a^b \max\limits_{|x|\leq c} F(t,x)dt}\right[,$$

the problem (D_λ) admits at least three classical solutions.

Proof. Let X, Φ, Ψ as in Section 2. Our aim is to apply Theorem 2.2 to the functional $I_\lambda = \Phi - \lambda\Psi$. Clearly, hypothesis (e_2) of Theorem 2.2 follows from (h_3). So, we are going to verify (e_1).

Put $r = \frac{2m^2}{b-a}c^2$. Taking Remark 2.1 into account, for each $u \in X$ such that $\Phi(u) = \frac{1}{2}\|u\|_X^2 < r$ one has

$$|u(t)| \leq \frac{(b-a)^{\frac{1}{2}}}{2m}\|u\|_X \leq \frac{(b-a)^{\frac{1}{2}}}{2m}(2r)^{\frac{1}{2}} = \left(\frac{b-a}{2m^2}r\right)^{\frac{1}{2}} = c\,,$$

for all $t \in [a,b]$. Hence

$$\sup_{\Phi(u)<r} \Psi(u) = \sup_{\|u\|_X^2<2r} \int_a^b e^{-\Gamma(t)} F(t,u(t))dt \le \max_{t\in[a,b]} e^{-\Gamma(t)} \int_a^b \max_{|x|\le c} F(t,x)dt\,.$$

It follows that

$$\frac{\sup\limits_{\Phi(u)<r} \Psi(u)}{r} \le \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max\limits_{|x|\le c} F(t,x)dt}{c^2}\,. \tag{4.1}$$

Now, we fix $\overline{v} = w$, where

$$w(t) = \begin{cases} 4d\,\frac{t-a}{b-a} & \text{if } \ t \in \left[a, a+\frac{b-a}{4}\right[\,, \\ d & \text{if } \ t \in \left[a+\frac{b-a}{4}, b-\frac{b-a}{4}\right]\,, \\ 4d\,\frac{b-t}{b-a} & \text{if } \ t \in \left]b-\frac{b-a}{4}, b\right]\,. \end{cases}$$

Clearly, $w \in X$ and, as seen in the proof of Theorem 3.1, one has $\|w\|^2 = \frac{8d^2}{b-a}$. Furthermore, since $c < d$ and from Proposition 2.1, it follows that $\Phi(w) > 2r$. Then, taking (h_1) into account, one has

$$\Psi(w) = \int_a^b e^{-\Gamma(t)} F(t,w(t))dt \ge \min_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt\,.$$

Moreover, from Proposition 2.1 we obtain

$$\Phi(w) = \frac{1}{2}\|w\|_X^2 \le \frac{1}{2}M^2\|w\|^2 = \frac{4M^2d^2}{b-a}\,.$$

Hence, taking into account the definition of K in (3.1), one has

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \ge \frac{2}{3} \frac{\min\limits_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{\frac{4M^2d^2}{b-a}}\,,$$

that is

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \ge \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \left(\frac{2}{3} K \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2} \right)\,. \tag{4.2}$$

Consequently, from our assumption (h_2), (4.1) and (4.2), we obtain

$$\frac{\sup\limits_{\Phi(u)<r} \Psi(u)}{r} \le \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max\limits_{|x|\le c} F(t,x)dt}{c^2} \le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}\,,$$

and (e_1) of Theorem 2.2 is satisfied.
Moreover, again from (4.1) and (4.2), one has

$$\Lambda_{c,d} \subseteq \Lambda_r\,.$$

From Theorem 2.2, for each $\lambda \in \Lambda_{c,d}$ the functional $I_\lambda = \Phi - \lambda\Psi$ admits at least three distinct critical points, which are classical solutions of the problem (D_λ). $\square$

Now, we deal with a particular case of Theorem 4.1 in the autonomous problem (AD_λ).

Corollary 4.1. Assume that there exist two positive constants c, d, with $c < d$, such that

(h_2') $\dfrac{G(c)}{c^2} < \dfrac{1}{3} K \dfrac{G(d)}{d^2}$;

(h_3') $\displaystyle\lim_{\xi \to +\infty} \frac{G(\xi)}{\xi^2} = 0$.

Then, for each $\lambda \in \bar{\Lambda}_{c,d}$, where

$$\bar{\Lambda}_{c,d} = \left] \frac{2m^2}{(b-a)^2 \max\limits_{t \in [a,b]} e^{-\Gamma(t)}} \frac{3}{K} \frac{d^2}{G(d)}, \frac{2m^2}{(b-a)^2 \max\limits_{t \in [a,b]} e^{-\Gamma(t)}} \frac{c^2}{G(c)} \right[,$$

the problem (AD_λ) admits at least three classical solutions.

Example 4.1. Consider the autonomous problem (AD_λ) in $[0, 1]$, with

$$\gamma(t) = -2t, \qquad \delta(t) = -\pi,$$

and

$$g(u) = \begin{cases} \frac{e^3}{3^2} u^2 & \text{if } u \le 3, \\ e^u & \text{if } 3 < u < 8, \\ e^8 & \text{if } u \ge 8. \end{cases}$$

Clearly, these functions satisfy the required assumptions. One has

$$\Gamma(t) = \int_0^t \gamma(\xi)\, d\xi = -t^2,$$

$$G(x) = \begin{cases} \frac{e^3}{3^3} x^3 & \text{if } x \le 3, \\ e^x & \text{if } 3 < x < 8, \\ e^8(x-7) & \text{if } x \ge 8. \end{cases}$$

and

$$K = \frac{\pi - 1}{2\pi e^2}.$$

Choosing, for instance, $c = 1$ and $d = 10$, the hypotheses of Corollary 4.1 are satisfied; then, for each $\lambda \in \left] \frac{400}{e^7}, \frac{\pi-1}{\pi} \frac{54}{e^4} \right[$, the given problem admits at least three classical solutions.

Now, we present our result on the existence of three nonnegative solutions, requiring two algebraic sign conditions on the nonlinear term.

Theorem 4.2. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function such that $f(t,x) \geq 0$ for all $t \in [a,b]$ and for all $x \geq 0$. Assume that there exist three positive constants c_1, c_2, d, with $c_1 < d < \frac{c_2}{4}\frac{m}{M}$, such that

$$(k_1)\quad \frac{\int_a^b \max_{|x|\leq c_1} F(t,x)dt}{c_1^2} < \frac{2}{3}K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2};$$

$$(k_2)\quad \frac{\int_a^b \max_{|x|\leq c_2} F(t,x)dt}{c_2^2} < \frac{1}{3}K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}{d^2}.$$

Then, for each $\lambda \in \Lambda_{c_1,c_2,d}$, where

$$\Lambda_{c_1,c_2,d} = \left]C\frac{3}{2K}\frac{d^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t,d)dt}, \min\left\{C\frac{c_1^2}{\int_a^b \max_{|x|\leq c_1} F(t,x)dt}, C\frac{c_2^2}{2\int_a^b \max_{|x|\leq c_2} F(t,x)dt}\right\}\right[,$$

with

$$C = \frac{2m^2}{(b-a)\max_{t\in[a,b]} e^{-\Gamma(t)}},$$

the problem (D_λ) admits at least three nonnegative classical solutions $u_i, i = 1,2,3$ such that $|u_i(t)| < c_2$ for all $t \in [a,b]$.

Proof. Let X, Φ, Ψ as in Section 2. The proof is based on Theorem 2.3. Clearly, Φ and Ψ satisfy condition 1. of Theorem 2.3. So, we start the proof by verifying (f_1) and (f_2).

In order to prove (f_1), put $r_1 = \frac{2m^2}{b-a}c_1^2$. Taking Remark 2.1 into account, for each $u \in X$ such that $\Phi(u) = \frac{1}{2}\|u\|_X^2 < r_1$, for all $t \in [a,b]$ one has

$$|u(t)| \leq \left(\frac{b-a}{2m^2}r_1\right)^{\frac{1}{2}} = c_1 .$$

Hence,

$$\sup_{\Phi(u)<r_1} \Psi(u) \leq \max_{t\in[a,b]} e^{-\Gamma(t)} \int_a^b \max_{|x|\leq c_1} F(t,x)dt ,$$

and

$$\frac{\sup_{\Phi(u)<r_1} \Psi(u)}{r_1} \leq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max_{|x|\leq c_1} F(t,x)dt}{c_1^2}. \tag{4.3}$$

Now, we fix $\overline{v} = w$, where

$$w(t) = \begin{cases} 4d\,\frac{t-a}{b-a} & \text{if } t \in \left[a, a+\frac{b-a}{4}\right[, \\ d & \text{if } t \in \left[a+\frac{b-a}{4}, b-\frac{b-a}{4}\right], \\ 4d\,\frac{b-t}{b-a} & \text{if } t \in \left]b-\frac{b-a}{4}, b\right]. \end{cases}$$

Clearly, $w \in X$ and, as seen in the proof of Theorem 3.1, one has $\|w\|^2 = \frac{8d^2}{b-a}$. Then, taking into account the sign hypothesis of f, one has

$$\Psi(w) = \int_a^b e^{-\Gamma(t)} F(t, w(t))dt \geq \min_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d)dt\,.$$

Moreover, from Proposition (2.1) we obtain

$$\Phi(w) = \frac{1}{2}\|w\|_X^2 \leq \frac{1}{2}M^2\|w\|^2 = \frac{4M^2d^2}{b-a}\,.$$

Hence, taking into account the definition of K in (3.1), one has

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \geq \frac{2}{3}\frac{\min\limits_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d)dt}{\frac{4M^2d^2}{b-a}}\,,$$

that is

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \geq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \left(\frac{2}{3}K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d)dt}{d^2}\right)\,. \tag{4.4}$$

Consequently, from our assumption (k_1), (4.3) and (4.4), one has

$$\frac{\sup\limits_{\Phi(u)<r_1} \Psi(u)}{r_1} \leq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max\limits_{|x|\leq c_1} F(t, x)dt}{c_1^2} \leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}\,.$$

and (f_1) of Theorem 2.2 is satisfied.
Similarly, in order to prove (f_2), put $r_2 = \frac{2m^2}{b-a}c_2^2$. As above, we can say that

$$\frac{\sup\limits_{\Phi(u)<r_2} \Psi(u)}{r_2} \leq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max\limits_{|x|\leq c_2} F(t, x)dt}{c_2^2}\,,$$

and

$$\frac{1}{3}\frac{\Psi(w)}{\Phi(w)} \geq \frac{1}{3}\frac{\min\limits_{t\in[a,b]} e^{-\Gamma(t)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d)dt}{\frac{4M^2d^2}{b-a}} = \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \left(\frac{1}{3}K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(t, d)dt}{d^2}\right)\,.$$

From (k_2) it follows that

$$\frac{\sup\limits_{\Phi(u)<r_2} \Psi(u)}{r_2} \leq \frac{b-a}{2m^2} \max_{t\in[a,b]} e^{-\Gamma(t)} \frac{\int_a^b \max\limits_{|x|\leq c_2} F(t, x)dt}{c_2^2} \leq \frac{1}{3}\frac{\Psi(w)}{\Phi(w)}\,.$$

and also (f_2) of Theorem 2.2 is satisfied.
Moreover, since $c_1 < d < \frac{c_2}{4}\frac{m}{M}$, using the result of Proposition 2.1, the inequality

$2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$ holds.
Clearly, we can also see that

$$\Lambda_{c_1,c_2,d} \subseteq \Lambda_{r_1,r_2}\,.$$

Finally, we claim that $\Phi - \lambda\Psi$ satisfies assumption 2. of Theorem 2.3.
Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, from Proposition 2.2, they are classical solutions of the problem (D_λ). Since $f(t,x) \geq 0$ for all $x \geq 0$, owing to the Weak Maximum Principle (see for instance [8]), one has

$$u_1(t) \geq 0, \quad u_2(t) \geq 0 \quad \forall\, t \in [a,b]\,. \tag{4.5}$$

Then, it follows that

$$f(t,u_1) \geq 0, \quad f(t,u_2) \geq 0 \quad \Rightarrow \quad \Psi(u_1) \geq 0, \quad \Psi(u_2) \geq 0\,.$$

Moreover, from (4.5) one has $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and as above

$$f(t, su_1 + (1-s)u_2) \geq 0 \quad \Rightarrow \quad \Psi(su_1 + (1-s)u_2) \geq 0 \quad \forall s \in [0,1].$$

From Theorem 2.3, for each $\lambda \in \Lambda_{c_1,c_2,d}$, the functional $\Phi - \lambda\Psi$ admits at least three distinct critical points $u_i, i = 1,2,3$, which are classical solutions of (D_λ), and such that $\Phi(u_i) < r_2$. Taking Remark 2.1 into account, it follows that $|u_i(t)| < c_2$ for all $t \in [a,b]$, $i = 1,2,3$. □

Finally, we point out the consequences of Theorem 4.2 for the autonomous case (AD_λ).

Corollary 4.2. Assume that there exist three positive constants c_1, c_2, d, with $c_1 < d < \frac{c_2}{4}\frac{m}{M}$, such that

(k_1') $\dfrac{G(c_1)}{c_1^2} < \dfrac{1}{3}K\dfrac{G(d)}{d^2}$;

(k_2') $\dfrac{G(c_2)}{c_2^2} < \dfrac{1}{6}K\dfrac{G(d)}{d^2}$.

Then, for each $\lambda \in \bar{\Lambda}_{c_1,c_2,d}$, where

$$\bar{\Lambda}_{c_1,c_2,d} = \left]\bar{C}\frac{3}{K}\,\frac{d^2}{G(d)}\,,\, \min\left\{\bar{C}\frac{c_1^2}{G(c_1)}\,,\, \bar{C}\frac{c_2^2}{2G(c_2)dt}\right\}\right[,$$

with

$$\bar{C} = \frac{2m^2}{(b-a)^2 \max\limits_{t\in[a,b]} e^{-\Gamma(t)}}\,,$$

the problem (AD_λ) admits at least three nonnegative classical solutions $u_i, i = 1,2,3$ such that $|u_i(t)| < c_2$ for all $t \in [a,b]$.

Corollary 4.3. Assume that

$$\lim_{\xi\to 0^+}\frac{G(\xi)}{\xi^2}=\lim_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}=0\,. \tag{4.6}$$

Then there exists $\lambda^*>0$ such that for each $\lambda\in]\lambda^*,+\infty[$ the problem (AD_λ) admits at least three nonnegative classical solutions $u_i, i=1,2,3$.

Proof. Our aim is to apply Corollary 4.2. Put

$$\lambda^*=\bar{C}\frac{3}{K}\inf_{d>0}\frac{d^2}{G(d)}\,,$$

with

$$\bar{C}=\frac{2m^2}{(b-a)^2\max\limits_{t\in[a,b]}e^{-\Gamma(t)}}\,,$$

and fix $\lambda\in]\lambda^*,+\infty[$. From our assumption (4.6), there is $c_1<d$ such that

$$\frac{G(c_1)}{c_1^2}<\frac{\bar{C}}{\lambda}\,.$$

Hence, one has

$$\frac{1}{\bar{C}}\frac{G(c_1)}{c_1^2}<\frac{1}{\lambda}<\frac{1}{\bar{C}}\frac{1}{3}K\frac{G(d)}{d^2}\,,$$

and hypothesis (k_1') is satisfied. Moreover, again from (4.6), there is $c_2>4d\frac{M}{m}$ such that

$$\frac{G(c_2)}{c_2^2}<\frac{\bar{C}}{2\lambda}\,.$$

Hence, one has

$$\frac{2}{\bar{C}}\frac{G(c_2)}{c_2^2}<\frac{1}{\lambda}<\frac{1}{\bar{C}}\frac{1}{3}K\frac{G(d)}{d^2}\,,$$

and hypothesis (k_2') is satisfied. Finally, from Corollary 4.2 there exists $\lambda^*>0$ such that for each $\lambda\in]\lambda^*,+\infty[$ the problem (AD_λ) admits at least three nonnegative classical solutions $u_i, i=1,2,3$. □

Remark 4.1. We can observe that the hypothesis (4.6) can be replaced by the following condition

$$\lim_{\xi\to 0^+}\frac{g(\xi)}{\xi}=\lim_{\xi\to+\infty}\frac{g(\xi)}{\xi}=0\,.$$

Example 4.2. Consider the autonomous problem (AD_λ) in $[0,1]$, with

$$\gamma(t)=-2t\,,\qquad \delta(t)=-\pi\,,$$

and

$$g(u)=\frac{|u|}{|u|+1}-|u|e^{-|u|}\,.$$

Clearly, these functions satisfy the required assumptions and it's easy to prove that

$$\lim_{\xi\to 0^+} \frac{g(\xi)}{\xi} = \lim_{\xi\to 0^+} \left(\frac{1}{\xi+1} - e^{-\xi}\right) = 0\,,$$

$$\lim_{\xi\to +\infty} \frac{g(\xi)}{\xi} = \lim_{\xi\to +\infty} \left(\frac{1}{\xi+1} - e^{-\xi}\right) = 0\,.$$

Finally, we observe that

$$\lambda^* = 6\,\frac{2\pi e^2}{\pi-1}\inf_{d>0}\frac{d^2}{\int_0^d \xi\left(\frac{1}{\xi+1} - e^{-\xi}\right)d\xi} =$$

$$= \frac{12\pi e^2}{\pi-1}\inf_{d>0}\frac{d^2}{\frac{d}{2}\left(\frac{d}{d+1}+1\right) + e^{-d}(d+1) - 1}\,.$$

Taking Remark 4.1 into account, we can apply Corollary 4.3 and the given problem admits, for each $\lambda \in]\lambda^*, +\infty[$, at least three nonnegative classical solutions.

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Department of Engineering, University of Messina, 98166 Messina - (Italy).
Email address: eleamoroso@unime.it bonanno@unime.it gdagui@unime.it