Two nontrivial solutions for Robin problems driven by a $p-$Laplacian operator

G. D'Aguì, A. Sciammetta and E. Tornatore

Abstract By variational methods and critical point theorems, we show the existence of two nontrivial solutions for a nonlinear elliptic problem under Robin condition and when the nonlinearty satisfies the usual Ambrosetti-Rabinowitz condition.

Key words: Robin problem, p-Laplacian, critical point theory. *2010 Mathematics Subject Classification: 35A15, 35J60.*

1 Introduction

In this paper we study the existence of two nontrivial weak solutions of following nonlinear elliptic equation under Robin condition

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda f(x,u) \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ (with $N \geq 3$) is a non-empty bounded open set with a smooth boundary $\partial\Omega$, λ is a positive real parameter and $1 < p < N$. The differential operator in (1) is described by the p-Laplacian, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. We assume $f : \Omega \times \mathbf{R} \to \mathbf{R}$,

G. D'Aguì
Department of Engineering, University of Messina, 98166 Messina - (Italy) e-mail: gdagui@unime.it,

A. Sciammetta
Department of Mathematics and Computer Sciences, University of Palermo, 90123 Palermo - (Italy) e-mail: angela.sciammetta@unipa.it

E. Tornatore
Department of Mathematics and Computer Sciences, University of Palermo, 90123 Palermo - (Italy) e-mail: elisa.tornatore@unipa.it

$\beta \in L^\infty(\partial\Omega)$, $\beta(x) \geq 0$ a.e. on $\partial\Omega$. In the boundary condition, $\frac{\partial u}{\partial \nu}$ denotes the generalized normal derivative defined by $\frac{\partial u}{\partial \nu} = |\nabla u|^{p-2}\nabla u \cdot \nu(x)$, $\nu(x)$ being the outward unit normal at $x \in \partial\Omega$.

A special case of our main result (see Theorem 6) can be given in the following form.

Theorem 1. *Let* $g\colon \mathbf{R} \to \mathbf{R}$ *be a nonnegative and continuous function such that there exist positive constants* a_1, a_2 *and* $s \in]p, p^*[$ *such that*

$$|g(t)| \leq a_1 + a_2|t|^{s-1} \quad \textit{for all } t \in \mathbf{R},$$

and

$$\lim_{\tau \to 0^+} \frac{g(\tau)}{\tau} = +\infty.$$

Moreover, assume that there exist $\nu > p$ *and* $R > 0$ *such that*

$$0 < \nu \int_0^\tau g(t)dt \leq \tau g(\tau) \quad \textit{for all } \tau \in \mathbf{R} \textit{ with } |\tau| \geq R.$$

Then, there exists $\overline{\lambda} > 0$ *such that for each* $\lambda \in]0, \overline{\lambda}[$*, the problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda g(u) \text{ in } \Omega, \\ \\ \frac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{2}$$

has at least two nonnegative weak solutions.

The main novelty of our paper is that we apply a recent critical-points result to elliptic problems with $p-$Laplacian in the equation and with Robin conditions on the boundary. There exist several existence results to problem (1), anyway our approach is new and gives the existence of two nontrivial weak solutions. The assumptions on the nonlinear term are easy to verify and so our results could be applied to several problems of type (1).

Elliptic problems with Robin conditions have been studied by several authors by applying different tools like fixed point theorems, sub and super-solution methods, and critical point theory. We refer, without any claim to completeness, to the papers [2, 7, 12, 13, 14, 15] and the references therein.

Moreover, we observe that the derivation and application of critical point results of that used here have been initiated by the works of Ricceri [16, 17] which were the starting point of several generalizations in that direction for smooth and non-smooth functionals, we refer only to some works of Marano-Motreanu [9, 10], and Bonanno [3, 4] that inspired us in writing this paper.

The paper is organized as follows. In Section 2, we state the main definitions and tools that we are going to need to prove our main results. Especially, we recall the abstract critical point theorem of Bonanno-D'Aguì [5], which is an appropriate combination of the local minimum theorem obtained by Bonanno with the classical and seminal Ambrosetti–Rabinowitz theorem (see [1]), moreover we give a lemma about the relation of our perturbation concerning the Ambrosetti–Rabinowitz condition and the Palais-Smale condition (Lemma 1). Then, in Section 3, we are going to prove our main result which gives an answer about the existence of solutions to problem (1). To be more precise, we obtain the existence of two non-trivial solutions of (1), see Theorem 3, and the proof is based on the abstract critical points result stated in Section 2. Finally, in Section 4, we consider special problem in the autonomous case, and give an example in order to show the applicability of our results.

2 Preliminaries and basic notations

Let $(X, \|\cdot\|)$ be a Banach space; its dual space is X^* and the corresponding duality pairing is denoted by $\langle \cdot, \cdot \rangle$. Let $I : X \to \mathbf{R}$ be a Gâteaux differentiable functional; we say that I satisfies the Palais-Smale condition, (in short $(PS)-$condition), if every sequence $\{u_n\}_{n\in\mathbf{N}} \subseteq X$ such that $\{I(u_n)\}_{n\in\mathbf{N}} \subset \mathbf{R}$ is bounded, and $I'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence in X.

Let $A : X \to X^*$ be a functional. We say that A has S_+-property iff every sequence $\{u_n\}_{n\in\mathbf{N}} \subset X$ such that $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0$ implies that $u_n \to u$ in X.

We consider the usual Sobolev space $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\| = \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx\right)^{1/p}$$

and denote by $(W^{1,p}(\Omega))^*$ its dual space.

Since $1 < p < N$, $p^* = \frac{pN}{n-p}$ and it is known that, for every $u \in W^{1,p}(\Omega)$ there exists a constant $T \in \mathbf{R}_+$ such that

$$\|u\|_{L^{p^*}(\Omega)} \leq T\|u\|, \tag{3}$$

the constat T has been determined by Talenti (see [18]) and

$$T \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}},$$

where Γ is the Euler function.
Fix $s \in [1, p^*[$, by Sobolev embedding theorem and Hölder's inequality, for every $u \in W^{1,p}(\Omega)$ we have that

$$\|u\|_{L^s(\Omega)} \leq T|\Omega|^{\frac{p^*-s}{p^*s}}\|u\|, \tag{4}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω in $\mathbf{R}$. On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^p(\partial\Omega)$ $1 \leq p \leq \infty$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u_{|\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

Therefore we understand $\gamma_0(u)$ as representing the "boundary values" of an arbitrary Sobolev function u. The trace map γ_0 is compact into $L^\eta(\partial\Omega)$ for all $\eta \in [1, \frac{(N-1)p}{N-p})$. Also, we have

$$\mathrm{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega),\ (p' = \frac{p}{p-1}), \quad \ker\gamma_0 = W^{1,p}(\Omega).$$

In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions u on $\partial\Omega$ are defined in the sense of traces. In studying problem (1) we rely on the negative p-Laplacian $-\Delta_p : W^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$. It is well-known that the operator $-\Delta_p$ is continuous, bounded, pseudomonotone and has the S_+-property (see [6], [11]).

Throughout the sequel, we assume that the nonlinearity $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function i.e. $f(\cdot,t)$ is measurable for every $t \in \mathbf{R}$, $f(x,\cdot)$ is continuous for almost every $x \in \Omega$ and satisfies the subcritical growth condition and the usual Ambrosetti-Rabinowitz condition (in short (AR)-condition).

(H) There exist two non negative constants a_1, a_2, a constant $s \in]p, p^*[$ such that

$$|f(x,t)| \leq a_1 + a_2|t|^{s-1} \quad \text{for all } (x,t) \in \Omega \times \mathbf{R}.$$

Put $F(x,t) = \int_0^t f(x,\xi)d\xi$ for all $(x,t) \in \Omega \times \mathbf{R}$.

(AR) There exist two constants $\mu > p$ and $M > 0$ such that, $0 < \mu F(x,t) \leq t f(x,t)$, for all $x \in \Omega$ and for all $|t| \geq M$.

We consider the C^1-functionals Φ, $\Psi : W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u(x)|^p d\sigma, \tag{5}$$

and

$$\Psi(u) = \int_\Omega F(x,u(x))dx, \tag{6}$$

for all $u \in W^{1,p}(\Omega)$, whose Gâteaux derivatives at point $u \in W^{1,p}(\Omega)$ are given by

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx$$

$$+ \int_\Omega |u(x)|^{p-2}u(x)v(x)dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p-2}uvd\sigma,$$

and

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx,$$

for every $v \in W^{1,p}(\Omega)$. Put $I_\lambda = \Phi - \lambda\Psi$, we observe that critical points of I_λ are weak solutions of (1).

We recall that weak solution of problem (1) is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega |u(x)|^{p-2}u(x)v(x)dx$$
$$+\int_{\partial\Omega}\beta(x)|u(x)|^{p-2}u(x)v(x)d\sigma = \lambda\int_\Omega f(x,u(x))v(x)dx.$$

Finally, we recall the following two non-zero critical points theorem established in [5] that we use to point out our results.

Theorem 2. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbf{R}$ be two functionals of class C^1 such that $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbf{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{7}$$

and, for each

$$\lambda \in \Lambda = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

3 Main Results

In this section, we present our main results. To be precise, we establish the existence result of two non zero weak solutions of problem (1).

We have the following Lemma.

Lemma 1. *Assume that conditions (H)-(AR) hold. Then I_λ satisfies the $(PS)-$condition.*

Proof. Let $\{u_n\}_{n\in\mathbf{N}} \subseteq W^{1,p}(\Omega)$ be a sequence such that $\{I_\lambda(u_n)\}_{n\in\mathbf{N}} \subset \mathbf{R}$ is bounded, and $I'_\lambda(u_n) \to 0$ in $(W^{1,p}(\Omega))^*$ as $n \to +\infty$. Simple calculations show that

$$\begin{aligned}\mu I_\lambda(u_n)-||I'_\lambda(u_n)||_{(W^{1,p}(\Omega))^*}||u_n|| &\geq \mu I_\lambda(u_n)-I'_\lambda(u_n)(u_n) \\ &= \mu\Phi(u_n)-\lambda\mu\Psi(u_n)-\Phi'(u_n)(u_n)+\lambda\Psi'(u_n)(u_n) \\ &= \left(\frac{\mu}{p}-1\right)||u_n||^p+\left(\frac{\mu}{p}-1\right)\int_{\partial\Omega}\beta(x)|u_n(x)|^p d\sigma \\ &- \lambda\int_\Omega (\mu F(x,u_n(x))-f(x,u_n(x))u_n(x))\,dx \\ &\geq \left(\frac{\mu}{p}-1\right)||u_n||^p+C,\end{aligned} \tag{8}$$

where C is a constant. If $\{u_n\}_{n\in\mathbf{N}}$ is not bounded, from (8) we obtain a contradiction. Therefore $\{u_n\}_{n\in\mathbf{N}}$ is bounded in $W^{1,p}(\Omega)$. Then, using a subsequence if necessary we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_n \to u$ in $L^l(\Omega)$ where $l\in[1,p^*[$ and $u_n\to u$ in $L^\eta(\partial\Omega)$ for $\eta\in\left[1,\frac{(N-1)p}{N-p}\right[$.
Using (H) and the Hölder inequality, we obtain that

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0, \tag{9}$$

$$\lim_{n\to\infty}\int_{\partial\Omega}\beta(x)|u_n|^{p-2}u_n(u_n-u)d\sigma=0, \tag{10}$$

and

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p-2}u_n(u_n-u)dx=0. \tag{11}$$

Taking into account that such that $I'_\lambda(u_n)\to 0$ in X^* as $n\to+\infty$, we have that

$$\langle I'_\lambda(u_n),u_n-u\rangle=\langle-\Delta_p u_n,u_n-u\rangle+\int_\Omega |u_n|^{p-2}u_n(u_n-u)dx$$

$$+\int_{\partial\Omega}\beta(x)|u_n|^{p-2}u_n(u_n-u)d\sigma-\int_\Omega f(x,u_n)(u_n-u)dx\to 0.$$

From (9), (10) and (11) one has

$$\limsup_{n\to\infty}\langle-\Delta_p u_n,u_n-u\rangle\leq 0.$$

By the S_+-property of $-\Delta_p$ in $W^{1,p}(\Omega)$ we have that $u_n\to u$ in $W^{1,p}(\Omega)$. Hence I_λ fulfills $(PS)-$condition. $\square$

Put

$$k=\frac{|\Omega|+\beta_\infty|\partial\Omega|}{|\Omega|^{\frac{p}{p^*}}}T^p, \tag{12}$$

where $|\partial\Omega|=\int_{\partial\Omega}d\sigma=\sigma(\partial\Omega)$ and $\beta_\infty=ess\sup_\Omega\beta(x)$.

Theorem 3. *Assume that conditions (H) and (AR) hold. Moreover assume that there are two positive constants c and d, with $d<c$, such that*

$$a_1 c^{1-p} + \frac{a_2}{s} c^{s-p} < \frac{1}{k|\Omega|} \frac{\int_\Omega F(x,d)dx}{d^p}, \tag{13}$$

where a_1, a_2, s and k are given by (H) and (12) respectively.

Then, for each $\lambda \in \Lambda_1 := \left] \frac{k|\Omega|^{\frac{p}{p^*}}}{pT^p} \frac{d^p}{\int_\Omega F(x,d)dx}, \frac{1}{pT^p|\Omega|^{\frac{p}{N}}} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}} \right[$, problem (1) has at least two non-zero weak solutions.

Proof. Put Φ and Ψ as in (5) and (6). It is well known that Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.

Explicitly, we observe that from (13), one has $\Lambda_1 \neq \emptyset$.

Consider the constant function $\overline{u}(x) = d \in W^{1,p}(\Omega)$, taking into account (12) we have

$$\Phi(\overline{u}) = \frac{d^p}{p}\left(\int_\Omega dx + \int_{\partial\Omega} \beta(x) d\sigma\right) \leq \frac{d^p}{p}\left(|\Omega| + \beta_\infty |\partial\Omega|\right) = \frac{k|\Omega|^{\frac{p}{p^*}}}{pT^p} d^p. \tag{14}$$

On the other hand one has

$$\Psi(\overline{u}) = \int_\Omega F(x,d)\, dx,$$

hence, we obtain

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})} > \frac{pT^p}{k|\Omega|^{\frac{p}{p^*}}} \frac{\int_\Omega F(x,d)\, dx}{d^p}. \tag{15}$$

Now, set $r = \frac{1}{p}\frac{|\Omega|^{\frac{p}{p^*}}}{T^p} c^p$. For all $u \in W^{1,p}(\Omega)$ such that $u \in \Phi^{-1}(]-\infty, r])$, taking (5) into account, one has that $\|u\| \leq (pr)^{\frac{1}{p}}$ we have

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{ u \in W^{1,p}(\Omega) : \|u\| \leq (pr)^{\frac{1}{p}} \right\}. \tag{16}$$

From (H) follows

$$|F(x,t)| \leq a_1|t| + a_2\frac{|t|^s}{s} \text{ for every } (x,t) \in \Omega \times \mathbf{R}. \tag{17}$$

From (4), (16) and (17) one has

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} \leq \frac{\sup_{\|u\| \leq (pr)^{\frac{1}{p}}} \Psi(u)}{r} \tag{18}$$

$$
\begin{aligned}
&\leq \frac{\sup\limits_{\|u\|\leq (pr)^{\frac{1}{p}}} \left(a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{s} \|u\|^s_{L^s(\Omega)} \right)}{r} \\
&\leq \frac{\sup\limits_{\|u\|\leq (pr)^{\frac{1}{p}}} \left(a_1 T |\Omega|^{\frac{p^*-1}{p^*}} \|u\| + \frac{a_2}{s} T^s |\Omega|^{\frac{p^*-s}{p^*}} \|u\|^s \right)}{r} \\
&\leq \frac{a_1 T |\Omega|^{\frac{p^*-1}{p^*}} (pr)^{\frac{1}{p}} + \frac{a_2}{s} T^s |\Omega|^{\frac{p^*-s}{p^*}} (pr)^{\frac{s}{p}}}{r} \\
&= pT^p |\Omega|^{\frac{p^*-p}{p^*}} \left[a_1 \left(\frac{T^p pr}{|\Omega|^{\frac{p}{p^*}}} \right)^{\frac{1-p}{p}} + \frac{a_2}{s} \left(\frac{T^p pr}{|\Omega|^{\frac{p}{p^*}}} \right)^{\frac{s-p}{p}} \right] \\
&= pT^p |\Omega|^{\frac{p}{N}} \left[a_1 c^{1-p} + \frac{a_2}{s} c^{s-p} \right].
\end{aligned}
$$

Therefore, from (13), (15), (18) we obtain condition (7) of Theorem 2. Moreover, since $0 < d < c$ and again by virtue of (13), we infer that

$$kd^p < c^p. \tag{19}$$

Indeed, arguing by contradiction, if we assume that $kd^p \geq c^p$ and using (17) we have

$$a_1 c^{1-p} + \frac{a_2}{s} c^{s-p} \geq \frac{1}{k} \frac{a_1 d + \frac{a_2}{s} d^s}{d^p} \geq \frac{1}{k|\Omega|} \frac{\int_\Omega F(x,d)\,dx}{d^p},$$

which contradicts (13). Then from (14), (19) we obtain that

$$\Phi(\overline{u}) < r.$$

By virtue of Lemma 1, for all fix $\lambda \in \Lambda_1$ the functional I_λ satisfies the $(PS)-$condition. Using $(AR)-$condition, it is easy to prove that the functional I_λ is unbounded from below. Moreover, $\inf\limits_{u \in W^{1,p}(\Omega)} \Phi(u) = \Phi(0) = \Psi(0) = 0$, therefore, all assumptions of Theorem 2 are satisfied. So, for all $\lambda \in \Lambda_1 \subset \Lambda$ problem (1) admits at least two non-zero weak solutions. □

Finally, we point out the following result that we will use to obtain nonnegative solutions for our problem (1).

Lemma 2. *Let $f : \Omega \times \mathbf{R} \to \mathbf{R}$, assume that $f(x,0) \geq 0$ for a.e. $x \in \Omega$. Consider the problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = \lambda f_+(x,u) \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2} u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{20}$$

where

$$f_+(x,t) = \begin{cases} f(x,0), \text{ if } \quad t < 0, \\ \\ f(x,t), \text{ if } \quad t \geq 0. \end{cases} \tag{21}$$

Then, the weak solutions of problem (20) are nonnegative weak solution of problem (1).

Proof. If $\bar{u} \in W^{1,p}(\Omega)$ is a weak solution of (20), choosing $v = \bar{u}^- = \max\{-u, 0\} \in W^{1,p}(\Omega)$ as test function (see, for instance, [8, Lemma 7.6]), one has

$$\int_{\{\bar{u}<0\}} |\nabla \bar{u}(x)|^p dx + \int_{\{\bar{u}<0\}} |\bar{u}(x)|^p dx + \int_{\partial\Omega} \beta(x)|\bar{u}(x)|^p d\sigma$$

$$= \lambda \int_{\{\bar{u}<0\}} f_+(x, \bar{u}(x))\bar{u}(x) dx \leq 0,$$

that is $\bar{u} \geq 0$ for a.e. $x \in \Omega$. Then $\bar{u}$ is a nonnegative weak solution of problem (1) Hence, our claim is proved. □

Now, we present our result on the existence of at least two nonnegative solutions.

Theorem 4. *Let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a continuous functions, $f(x,0) \geq 0$ a. e. $x \in \Omega$. Assume that (H) and $(AR)-$condition hold. Moreover, there are two positive constants c and d, with $d < c$, such that*

$$a_1 c^{1-p} + \frac{a_2}{s} c^{s-p} < \frac{1}{k|\Omega|} \frac{\int_\Omega F(x,d) dx}{d^p}. \tag{22}$$

Then, for each $\lambda \in \Lambda_1 := \left] \frac{k|\Omega|^{\frac{p}{p^*}}}{pT^p} \frac{d^p}{\int_\Omega F(x,d)dx}, \frac{1}{pT^p|\Omega|^{\frac{p}{N}}} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}} \right[$ *problem (1) has at least two nonnegative solutions.*

Proof. Since all conditions of Theorem 3 are satisfied, then for each $\lambda \in \Lambda_1$ the problem (1) admits at least two non zero weak solutions in $W^{1,p}(\Omega)$ and, taking into account Lemma 2, they are also nonnegative. □

4 Some consequences

We point out a special case of Theorem 3 when the nonlinearity f does not depend on x.

Theorem 5. *Let $f : \mathbf{R} \to \mathbf{R}$ be a nonnegative continuous function such that (H) and $(AR)-$condition hold. Moreover, assume that there are two positive constants c and d, with $d < c$, such that*

$$a_1 c^{1-p} + \frac{a_2}{s} c^{s-p} < \frac{1}{k} \frac{F(d)}{d^p}. \tag{23}$$

Then, for each $\lambda \in \Lambda_2 := \left] \frac{k}{pT^p|\Omega|^{\frac{p}{N}}} \frac{d^p}{F(d)}, \frac{1}{pT^p|\Omega|^{\frac{p}{N}}} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}} \right[$ *problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda f(u) \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{24}$$

has at least two nonnegative weak solutions.

Proof. Our aim is to apply Theorem 4. We observe that from condition (23) we obtain condition (13) of Theorem 3 and moreover $f(x,0) \geq 0$ a.e. $x \in \Omega$. Then, for each $\lambda \in \Lambda_2 := \left] \frac{k}{pT^p|\Omega|^{\frac{p}{N}}} \frac{d^p}{F(d)}, \frac{1}{pT^p|\Omega|^{\frac{p}{N}}} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}} \right[$ problem (24) has at least two nonnegative weak solutions. □

Finally, we want consider the case when the nonlinear term of problem (24) has a $(p-1)$-linearity at zero.

Theorem 6. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a nonnegative continuous function such that the* (H) *and* $(AR)-$*condition hold and*

$$\limsup_{t \to 0^+} \frac{F(t)}{t^p} = +\infty, \tag{25}$$

and put $\lambda^* = \frac{1}{pT^p|\Omega|^{\frac{p}{N}}} \sup_{c>0} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}}$.

Then, for each $\lambda \in]0, \lambda^*[$, *problem (24) admits at least two nonnegative weak solutions.*

Proof. Put $\lambda \in]0, \lambda^*[$, there is $c > 0$ such that $\lambda < \frac{1}{pT^p|\Omega|^{\frac{p}{N}}} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}}$. From (25) there is $0 < d < c$ such that $\frac{pT^p|\Omega|^{\frac{p}{N}}}{k} \frac{F(d)}{d^p} > \frac{1}{\lambda}$. Hence, Theorem 5 guarantees the conclusion. □

Example 1. Let $p = 3, N = 4$ and $\Omega = B(0, 3^{\frac{1}{8}})$ and consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(t) = t^4 + 1$.

Putting $a_1 = 1$, $a_2 = 5$ and $s = 5$, we observe that conditions (H) holds. On the other hand

$$F(t) = \int_0^t (\xi^4 + 1) d\xi = \frac{t^5}{5} + t,$$

$$\limsup_{t \to 0^+} \frac{F(t)}{t^p} = \lim_{t \to 0^+} \frac{t^5 + 5t}{5t^3} = +\infty,$$

and $(AR)-$condition is satisfied as a simple computation shows.

Moreover, one has that

$$T \leq \pi^{-\frac{1}{2}} 4^{-\frac{1}{3}} 2^{\frac{2}{3}} \left(\frac{\Gamma(3)\Gamma(4)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{11}{3}\right)} \right)^{\frac{1}{4}},$$

$$\sup_{c>0} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}} = \sup_{c>0} \frac{1}{\frac{1}{c^2} + c^2} = \frac{1}{2},$$

$$\lambda^* = \frac{1}{pT^p |\Omega|^{\frac{p}{N}}} \sup_{c>0} \frac{1}{a_1 c^{1-p} + \frac{a_2}{s} c^{s-p}} \geq \frac{2^2 \cdot 5^{\frac{3}{4}} \cdot \pi^{\frac{3}{4}}}{3^{\frac{11}{2}}}.$$

Using Theorem 6, for each $\lambda \in \left]0, \frac{2^2 \cdot 5^{\frac{3}{4}} \cdot \pi^{\frac{3}{4}}}{3^{\frac{11}{2}}}\right[$, the problem

$$\begin{cases} -\Delta_3 u + |u|u = \lambda(t^4+1) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta(x)|u|u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two nonnegative weak solutions.

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