EXISTENCE OF TWO POSITIVE SOLUTIONS FOR ANISOTROPIC NONLINEAR ELLIPTIC EQUATIONS

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Abstract. This paper deals with the existence of nontrivial solutions for a class of nonlinear elliptic equations driven by an anisotropic Laplacian operator. In particular, the existence of two nontrivial solutions is obtained, adapting a two critical point result to a suitable functional framework that involves the anisotropic Sobolev spaces.

1. Introduction

Let Ω be a nonempty bounded open set of the real Euclidean space $\mathbb{R}^N$, $N \geq 2$, with a boundary of class C^1, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a function, $\vec{p} = (p_1, p_2, \dots, p_N)$, $\vec{p} \in \mathbb{R}^N$. Put

$$p^- = \min\{p_1, p_2 \dots, p_N\} \quad \text{and} \quad p^+ = \max\{p_1, p_2 \dots, p_N\}, \tag{1.1}$$

respectively the minimum and the maximum value of the anisotropic configuration.
Let us consider the following problem

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(x, u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$D_\lambda^{\vec{p}}$}$$

where $\Delta_{\vec{p}} u = \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)$ is the anisotropic $p-$Laplacian operator and $\lambda \in]0, +\infty[$.

If $p_i = 2$, for all $i = 1, \dots, N$, we get $\displaystyle\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} = \Delta u$, the usual Laplacian operator; if $\vec{p}$ is constant (that is $p_i = p$ for all $i = 1, \dots, N$) we get

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$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = \tilde{\Delta}_p u$, which is the pseudo $p-$Laplacian operator (see, for instance, [4, 14]).
Recently, anisotropic boundary value problems have been investigated by many authors and, for an overview on these subjects, we refer to [16, 17, 18, 22, 23, 25, 27, 28, 29, 32, 34, 35, 36, 38] and references therein.
In particular, anisotropic differential problems find their applications in many field of applied sciences. For instance, the study of an epidemic disease in heterogeneous habitat is expressed by an anisotropic nonlinear system. Indeed, anisotropic operators model phenomena in which partial differential derivatives vary with direction. For more details about these arguments, we refer to [2, 5, 6, 46] and references therein.

In order to study problem $(D_\lambda^{\vec{p}})$, the functional framework is based on the theory of anisotropic Sobolev space, which was developed in [40, 41, 44] and references therein.
In [7], the authors prove a sufficient condition for the global $L^\infty-$ boundedness of solutions for some class of anisotropic differential problems and under suitable conditions on the exponents p_i. The global boundedness of the solutions is a combination of the original idea by Stampacchia with Sobolev-type inequality (see [38, 42, 44, 45]).
In [28], the authors study the following quasilinear elliptic problem

$$\begin{cases} -\Delta_{\vec{m}} u = \lambda u^{p-1} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$, $\vec{m} = (m_1, m_2, \ldots, m_N)$, $m_i > 1$ for all $i = 1, 2, \ldots N$, and $p > 1$. They give two results on the existence of at least one solution for the given problem, by applying respectively constrained minimization methods and mountain pass theorem. Moreover, nonexistence results are obtained in the critical case in domains with a specific geometric property.
In [34, 35, 36], the authors study nonhomogeneous anisotropic problem in the case when the indexes of the operator are continuous functions, i.e. they

consider the following anisotropic $\vec{p}(x)-$Laplacian operator

$$\Delta_{\vec{p}(x)} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right). \tag{1.2}$$

We observe that if $p_i(x) = p(x)$ for all $i = 1, \ldots, N$, the operator (1.2) becomes the pseudo$-p(x)-$Laplacian operator (see, for instance, [13]).
The study of nonlinear elliptic problems involving operators of the type $p(x)-$Laplacian is based on the theory of generalized Lebegue-Sobolev spaces (see for instance [21], [26], [31] and [37] for an overview on this subject). In particular, boundary value problems with variable exponent have been studied, for example, by [3, 11, 19, 24, 26, 33, 47].

Here and in the sequel, we suppose that $p^- > N$. The aim of this paper is to establish the existence of at least two nontrivial solutions for the problem $(D_\lambda^{\vec{p}})$, by using variational methods. Precisely, under suitable assumptions on the nonlinearity f, we obtain the existence of two non-zero solutions to problem $(D_\lambda^{\vec{p}})$ (see Theorem 3.1). Our main tool is a two non-zero critical points theorem (Theorem 2.1) established in [10]. Such critical point result is an appropriate combination of the local minimum theorem obtained in [8], with the classical and seminal Ambrosetti-Rabinowitz theorem (see [1]). As a way of example, here a very special case of our main result is presented (see Remark 4.2).

Theorem 1.1. *There is $\eta^* > 0$ such that, for each $\eta \in]0, \eta^*[$, the problem*

$$\begin{cases} -\Delta_{\vec{p}} u = \eta u^{(p^- - 2)} + u^{p^+} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least two positive weak solutions.

This paper is organized as follows. In Section 2, some definitions and results on anisotropic framework are collected. Precisely, an exact constant of the compact embedding of $W_0^{1,\vec{p}}(\Omega)$ in $C^0(\bar{\Omega})$ is pointed out (see Proposition 2.1) owing to the classical Talenti result (see [43]) and its consequence is emphasized (see Proposition 2.2). Moreover, the Palais-Smale condition of the energy functional associated to the problem $(D_\lambda^{\vec{p}})$ is proved (see Lemma 2.1) and the abstract critical points theorem (Theorem 2.1) is recalled. Finally, conditions on f in order to obtain nonnegative solutions (see Lemma 2.2) as well as positive solutions (see Lemma 2.3) are highlighted. The latter result is based on a strong maximum principle established in [20].

In Section 3, our main result, Theorem 3.1, is established. It ensures the existence of two non-zero solutions by requiring a suitable behaviour of the nonlinearity f (see condition (3.2)) together with the Ambrosetti-Rabinowitz condition. Its consequence (Theorem 3.2) in order to obtain two positive solutions is established. Finally, in Section 4, the autonomous case is studied. In particular, it is shown that the $(p^- - 1)-$sublinearity of f at 0 and the Ambrosetti-Rabinowitz condition ensure the existence of two positive solutions (see Theorem 4.2). A study of combined effects of concave and convex nonlinearities (see Theorem 4.3) is addressed and a concrete example is emphasized (see Example 4.1).

2. Preliminaries and basic notations

In this section, we recall some preliminaries, basic definitions and properties.

Let $(X, \|\cdot\|)$ be a Banach space, its dual space is X^* and the corresponding duality pairing is denoted by $\langle \cdot, \cdot \rangle$. Let $I : X \to \mathbb{R}$ be a Gâteaux differentiable functional. We say that I satisfies the Palais-Smale condition, (in short $(PS)-$condition), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

(P_1) $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
(P_2) $\{I'(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in X^*,

admits a convergent subsequence in X.

Denote by $W_0^{1,\vec{p}}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} .$$

It is well known (see for instance [44]) that $W_0^{1,\vec{p}}(\Omega)$, with the respective norm, is a Banach space.

We recall that, since $p^- > N$, the space $W_0^{1,p^-}(\Omega)$ is continuously embedded in $C^0(\bar{\Omega})$, such embedding is compact and one has

$$\|u\|_{C^0(\bar{\Omega})} \leq m_{p^-} \|u\|_{W_0^{1,p^-}(\Omega)} \tag{2.1}$$

for every $u \in W_0^{1,p^-}(\Omega)$, where

$$m_{p^-} = \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{N}{2}\right) \right]^{\frac{1}{N}} \left(\frac{p^- - 1}{p^- - N} \right)^{1-\frac{1}{p^-}} |\Omega|^{\frac{1}{N}-\frac{1}{p^-}} , \tag{2.2}$$

Γ denotes the Gamma function and $|\Omega|$ is the Lebesgue measure of Ω. In particular, if Ω is an $N-$dimensional ball, (2.2) is the best constant such that (2.1) is verified (see [43, Formula (6b)]).

Now, we prove the following proposition that we will use later.

Proposition 2.1. *One has*

$$\|u\|_{C^0(\bar{\Omega})} \leq T_0 \|u\|_{W_0^{1,\vec{p}}(\Omega)} , \tag{2.3}$$

for each $u \in W_0^{1,\vec{p}}(\Omega)$, *where*

$$T_0 = 2^{\frac{(N-1)(p^- -1)}{p^-}} m_{p^-} \max_{1\leq i\leq N} \left\{ |\Omega|^{\frac{p_i - p^-}{p_i p^-}} \right\} . \tag{2.4}$$

Moreover, the embedding of $W_0^{1,\vec{p}}(\Omega)$ *in* $C^0(\bar{\Omega})$ *is compact.*

Proof. It is well known that $W_0^{1,\vec{p}}(\Omega)$ is continuously embedded in $C^0(\bar{\Omega})$ (see for instance [41, Lemma 2]) and such embedding is compact (see [5, Lemma 2.2]). Here, we prove again such a property in order to have in addition a precise embedding constant, that is (2.4). To this end, fix $u \in W_0^{1,\vec{p}}(\Omega)$. In particular, one has $\dfrac{\partial u}{\partial x_i} \in L^{p_i}(\Omega)$ and then $\left|\dfrac{\partial u}{\partial x_i}\right|^{p^-} \in L^{\frac{p_i}{p^-}}(\Omega)$. If $p_i > p^-$, owing to Hölder inequality we obtain that

$$\left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p^-} dx \right)^{\frac{1}{p^-}} \leq |\Omega|^{\frac{p_i - p^-}{p_i p^-}} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} .$$

Clearly, if $p_i = p^-$ immediately one has

$$\left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p^-} dx \right)^{\frac{1}{p^-}} = \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} = |\Omega|^{\frac{p_i - p^-}{p_i p^-}} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} .$$

Hence, one has

$$\left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p^-} dx \right)^{\frac{1}{p^-}} \leq |\Omega|^{\frac{p_i - p^-}{p_i p^-}} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} , \tag{2.5}$$

for all $i = 1, \ldots, N$ and $u \in W_0^{1,p^-}(\Omega)$, for which (2.1) can be applied, ensuring the embedding of $W_0^{1,\vec{p}}(\Omega)$ in $C^0(\bar{\Omega})$.

Now, we recall the following elementary inequalities that we are using below.

$$\Big(\sum_{i=1}^{N}|a_i|^2\Big)^{1/2} \le \sum_{i=1}^{N}|a_i|; \quad \Big(\sum_{i=1}^{N}|a_i|\Big)^{1/p} \le \sum_{i=1}^{N}|a_i|^{1/p};$$

$$\Big(\sum_{i=1}^{N}|a_i|\Big)^{p} \le 2^{(N-1)(p-1)}\sum_{i=1}^{N}|a_i|^{p},$$

where $a_i \in \mathbb{R}$, $i = 1, ..., N$ and $p \ge 1$.
So, one has

$$\|u\|^{p^-}_{W_0^{1,p^-}(\Omega)} = \|\nabla u\|^{p^-}_{L^{p^-}(\Omega)} = \int_\Omega \left[\left(\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{1/2}\right]^{p^-} dx \le$$

$$\le \int_\Omega \left(\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|\right)^{p^-} dx \le \int_\Omega 2^{(N-1)(p^- -1)}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p^-} dx.$$

Hence, from (2.5) one has

$$\|u\|_{W_0^{1,p^-}(\Omega)} \le \left(2^{(N-1)(p^- -1)}\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p^-} dx\right)^{1/p^-} =$$

$$= 2^{\frac{(N-1)(p^- -1)}{p^-}}\left(\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p^-} dx\right)^{1/p^-} \le$$

$$\le 2^{\frac{(N-1)(p^- -1)}{p^-}}\sum_{i=1}^{N}\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p^-} dx\right)^{1/p^-} \le$$

$$2^{\frac{(N-1)(p^- -1)}{p^-}}\sum_{i=1}^{N}|\Omega|^{\frac{p_i-p^-}{p_ip^-}}\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} \le$$

$$\le 2^{\frac{(N-1)(p^- -1)}{p^-}}\max_{1\le i\le N}\{|\Omega|^{\frac{p_i-p^-}{p_ip^-}}\}\sum_{i=1}^{N}\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} =$$

$$= 2^{\frac{(N-1)(p^- -1)}{p^-}}\max_{1\le i\le N}\{|\Omega|^{\frac{p_i-p^-}{p_ip^-}}\}\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)},$$

that is,

$$\|u\|_{W_0^{1,p^-}(\Omega)} \leq 2^{\frac{(N-1)(p^--1)}{p^-}} \max_{1\leq i\leq N}\{|\Omega|^{\frac{p_i-p^-}{p_ip^-}}\}\,\|u\|_{W_0^{1,\vec{p}}(\Omega)}\,.$$

Hence, from (2.1), taking also into account that the embedding of $W_0^{1,p^-}(\Omega)$ in $C^0(\bar{\Omega})$ is compact, the conclusion is obtained.

□

A consequence of previous proposition is the following property.

Proposition 2.2. *Fix $r > 0$. Then for each $u \in W_0^{1,\vec{p}}(\Omega)$ such that*

$$\sum_{i=1}^{N} \frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} < r,$$

one has

$$\|u\|_{C^0(\bar{\Omega})} < T\max\{r^{1/p^-}; r^{1/p^+}\},$$

where $T = T_0\sum_{i=1}^{N} {p_i}^{1/p_i}$ and T_0 is given in (2.4).

Proof. From our assumption one has $\frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} < r$ for all $i = 1, ..., N$, that is $\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} < (p_i r)^{1/p_i}$ for all $i = 1, ..., N$. Therefore, one has

$$\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} < \sum_{i=1}^{N}(p_i r)^{1/p_i}.$$

So, taking Proposition 2.1 into account, one has

$$\|u\|_{C^0(\bar{\Omega})} \leq T_0\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} < T_0\sum_{i=1}^{N}(p_i r)^{1/p_i} \leq$$

$$\leq T_0\sum_{i=1}^{N}(p_i)^{1/p_i}\max\{r^{1/p^-}; r^{1/p^+}\},$$

that is the conclusion.

□

Throughout the sequel, we suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, i.e.:

(1) $x \mapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}$;

(2) $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in \Omega$;
(3) for every $s > 0$ there is a function $l_s \in L^1(\Omega)$ such that

$$\sup_{|\xi|\le s} |f(x,\xi)| \le l_s(x)$$

for a.e. $x \in \Omega$.

Put

$$F(x,t) = \int_0^t f(x,\xi)d\xi \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

We assume also that the nonlinearity f satisfies the following Ambrosetti-Rabinowitz condition:

(AR) *There exist constants* $\mu > p^+$ *and* $M > 0$ *such that,* $0 < \mu F(x,t) \le t f(x,t)$ *for all* $x \in \Omega$ *and for all* $|t| \ge M$.

We recall that $u : \Omega \to \mathbb{R}$ is a weak solution of problem $(D_\lambda^{\vec{p}})$ if $u \in W_0^{1,\vec{p}}(\Omega)$ satisfies the following condition

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx = \lambda \int_\Omega f(x,u(x))v(x)dx\,,$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$.

Finally, we define the functionals $\Phi, \Psi : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ by setting, for every $u \in W_0^{1,\vec{p}}(\Omega)$,

$$\Phi(u) := \sum_{i=1}^N \frac{1}{p_i}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx, \qquad \Psi(u) := \int_\Omega F(x,u(x))dx\,. \tag{2.6}$$

Clearly, Φ and Ψ are Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in W_0^{1,\vec{p}}(\Omega)$ are given by

$$\Phi'(u)(v) = \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx\,,$$

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx\,,$$

for every $v \in W_0^{1,\vec{p}}(\Omega)$.

We observe that the critical points in $W_0^{1,\vec{p}}(\Omega)$ of the functional $I_\lambda = \Phi - \lambda\Psi$ are precisely the weak solutions of problem $(D_\lambda^{\vec{p}})$.

Now we prove the following lemma.

Lemma 2.1. *Assume that the* $(AR)-$*condition holds. Then* I_λ *satisfies the* $(PS)-$*condition and it is unbounded from below.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vec{p}}(\Omega)$ be a sequence which satisfies conditions (P_1) and (P_2). Our aim is to prove that it admits a subsequence which is strongly convergent in $W_0^{1,\vec{p}}(\Omega)$. To this end, first, we verify that it is bounded in $W_0^{1,\vec{p}}(\Omega)$. Indeed, from $(AR)-$condition one has

$$\int_\Omega [\mu F(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx =$$
$$= \int_{\{x\in\Omega:|u_n(x)|\geq M\}} [\mu F(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx +$$
$$+ \int_{\{x\in\Omega:|u_n(x)|<M\}} [\mu F(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx \leq$$
$$\int_{\{x\in\Omega:|u_n(x)|<M\}} \max_{|\xi|\leq M} [\mu F(x,\xi) - f(x,\xi)\xi]\,dx \leq$$

$\leq \max_{|\xi|\leq M} [\mu F(x,\xi) - f(x,\xi)\xi]\,|\Omega|$, that is,

$$\int_\Omega [\mu F(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx \leq C_1 \qquad \forall n\in\mathbb{N} \tag{2.7}$$

for some constant C_1.
Moreover, we have

$$I_\lambda(u_n) = \left[\sum_{i=1}^N \frac{1}{p_i}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} - \lambda\int_\Omega F(x,u_n(x))dx\right] \geq$$
$$\geq \left[\frac{1}{p^+}\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} - \lambda\int_\Omega F(x,u_n(x))dx\right],$$

that is,

$$\mu I_\lambda(u_n) + \lambda\int_\Omega \mu F(x,u_n(x))dx \geq \frac{\mu}{p^+}\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i}$$

and

$$\langle I'_\lambda(u_n); u_n\rangle = \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_i}dx - \lambda\int_\Omega f(x,u_n(x))u_n(x)dx =$$
$$= \sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} - \lambda\int_\Omega f(x,u_n(x))u_n(x)dx,$$

that is,

$$\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} = \langle I'_\lambda(u_n); u_n \rangle + \lambda \int_\Omega f(x, u_n(x)) u_n(x) dx.$$

Hence, it follows that

$$\left(\frac{\mu}{p^+} - 1 \right) \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} \leq$$

$$\leq \mu I_\lambda(u_n) - \langle I'_\lambda(u_n); u_n \rangle + \lambda \int_\Omega \mu F(x, u_n(x)) dx - \lambda \int_\Omega f(x, u_n(x)) u_n(x) dx.$$

Therefore, taking (P_1) and (2.7) into account, one has

$$\left(\frac{\mu}{p^+} - 1 \right) \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} \leq \mu C_2 - \langle I'_\lambda(u_n); u_n \rangle + \lambda C_1 = -\langle I'_\lambda(u_n); u_n \rangle + C_3,$$

for which, since $-\langle I'_\lambda(u_n); u_n \rangle \leq |\langle I'_\lambda(u_n); u_n \rangle| \leq \|I'_\lambda(u_n)\|_* \|u_n\|_{W_0^{1,\vec{p}}(\Omega)}$, one has

$$\left(\frac{\mu}{p^+} - 1 \right) \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} \leq \|I'_\lambda(u_n)\|_* \|u_n\|_{W_0^{1,\vec{p}}(\Omega)} + C_3 \qquad \forall n \in \mathbb{N}. \quad (2.8)$$

As usual, we denoted

$$\|I'_\lambda(w)\|_* = \sup \left\{ |\langle I'_\lambda(w), v \rangle| : v \in W_0^{1,\vec{p}}(\Omega), \quad \|v\|_{W_0^{1,\vec{p}}(\Omega)} = 1 \right\}.$$

Now, arguing by a contradiction, assume that $\{\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}\}$ is not bounded. Since, possibly by renaming the appropriate subsequence, one has

$$\lim_{n \to +\infty} \|u_n\|_{W_0^{1,\vec{p}}(\Omega)} = +\infty,$$

from (2.8) it follows that

$$\frac{\left(\frac{\mu}{p^+} - 1 \right) \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i}}{\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}} \leq \|I'_\lambda(u_n)\|_* + \frac{C_3}{\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}}.$$

Hence, taking (P_2) into account, one has

$$\lim_{n\to+\infty} \frac{\left(\frac{\mu}{p^+}-1\right)\sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i}}{\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}} = 0. \tag{2.9}$$

Therefore, one has $\left(\frac{\mu}{p^+}-1\right)\sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} < \|u_n\|_{W_0^{1,\vec{p}}(\Omega)}$ $\forall n \in \mathbb{N}$ (by renaming the sequence), from which, by renaming again the sequence in a such way that $\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} > 1$, one has

$$\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} \leq \sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} < \frac{p^+}{\mu-p^+}\|u_n\|_{W_0^{1,\vec{p}}(\Omega)};$$

$$\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)} < \left(\frac{p^+}{\mu-p^+}\right)^{1/p_i}\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{1/p_i} \leq$$

$$\leq \max\left\{\left(\frac{p^+}{\mu-p^+}\right)^{1/p^-};\left(\frac{p^+}{\mu-p^+}\right)^{1/p^+}\right\}\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{1/p^-};$$

$$\sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)} < N\max\left\{\left(\frac{p^+}{\mu-p^+}\right)^{1/p^-};\left(\frac{p^+}{\mu-p^+}\right)^{1/p^+}\right\}\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{1/p^-} =$$

$$C_4\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{1/p^-};$$

$$\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} < C_4\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{1/p^-};$$

$$\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{p^-} < C_4^{p^-}\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}.$$

So that, one has

$$\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{p^--1} < C_4^{p^-} \quad \forall n \in \mathbb{N},$$

for which $\{\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}\}$ is bounded and this is absurd.

In conclusion, the sequence $\{u_n\}_{n\in\mathbb{N}}$ which satisfies (P_1) and (P_2) is bounded in $W_0^{1,\vec{p}}(\Omega)$, for which our first claim is verified.

Next, we verify that $\{u_n\}_{n\in\mathbb{N}}$ admits a subsequence which is strongly convergent in $W_0^{1,\vec{p}}(\Omega)$. Since $W_0^{1,\vec{p}}(\Omega)$ is reflexive (see [5, Remark 2.1]), $\{u_n\}_{n\in\mathbb{N}}$ admits a subsequence which converges weakly to some $u \in W_0^{1,\vec{p}}(\Omega)$. Clearly, since the embedding of $W_0^{1,\vec{p}}(\Omega)$ in $C^0(\bar{\Omega})$ is compact (see Proposition 2.1), the renamed sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $C^0(\bar\Omega)$. First, we prove that one has

$$\langle \Phi'(u_n); u_n - u\rangle \to 0. \tag{2.10}$$

Indeed, we have

$\langle \Phi'(u_n); u_n - u\rangle = \langle I'_\lambda(u_n); u_n - u\rangle + \lambda \int_\Omega f(x,u_n)(u_n-u)dx$. So, since one has

$\int_\Omega f(x,u_n)(u_n-u)dx \le \int_\Omega C(u_n-u)dx$,

being $f(x,u_n(x)) \le \max_{|\xi|\le k} f(x,\xi)$ since $\|u_n\|_{C^0} \le \|u_n-u\|_{C^0} + \|u\|_{C^0} \le k$, and

$\langle I'_\lambda(u_n); u_n - u\rangle \le \|I'_\lambda(u_n)\|_* \|u_n-u\|_{W_0^{1,\vec p}(\Omega)} \le M\|I'_\lambda(u_n)\|_*$

the condition (2.10) is proved.

Moreover, one has

$$\langle \Phi'(u); u_n - u\rangle \to 0, \tag{2.11}$$

since $\Phi'(u)$ is a linear operator in $W_0^{1,\vec p}(\Omega)$ and $u_n \xrightarrow{w} u$ in $W_0^{1,\vec p}(\Omega)$. Hence, from (2.10) and (2.11) one has

$$\langle \Phi'(u_n) - \Phi'(u); u_n - u\rangle \to 0. \tag{2.12}$$

Now, put

$$\mathcal{A}_i(w)(v) = \int_\Omega \left|\frac{\partial w}{\partial x_i}\right|^{p_i-2} \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}dx; \quad \mathcal{B}_i(w)(v) = \frac{1}{\|w\|^{p_i-2}_{L^{p_i}(\Omega)}}\mathcal{A}_i(w)(v)$$

for all $i = 1, ..., N$ and for all $w, v \in W_0^{1,\vec p}(\Omega)$. From (2.12), one has

$$\langle \mathcal{B}_i(u_n) - \mathcal{B}_i(u); u_n - u\rangle \to 0, \quad \forall i = 1, ..., N. \tag{2.13}$$

Observe that one has

$$\begin{aligned}
&\langle \mathcal{B}_i(u_n) - \mathcal{B}_i(u); u_n - u\rangle = \\
&= \langle \mathcal{B}_i(u_n); u_n\rangle - \langle \mathcal{B}_i(u_n); u\rangle - \langle \mathcal{B}_i(u); u_n\rangle + \langle \mathcal{B}_i(u); u\rangle = \\
&= \left\|\frac{\partial u_n}{\partial x_i}\right\|^2_{L^{p_i}(\Omega)} + \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^{p_i}(\Omega)} - \langle \mathcal{B}_i(u_n); u\rangle - \langle \mathcal{B}_i(u); u_n\rangle = \\
&= \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right)^2 + 2\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} - \\
&- \langle \mathcal{B}_i(u_n); u\rangle - \langle \mathcal{B}_i(u); u_n\rangle =
\end{aligned}$$

$$= \left(\left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} - \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right)^2 +$$

$$+ \left(\left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} - \langle \mathcal{B}_i(u_n); u \rangle \right) +$$

$$+ \left(\left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} - \langle \mathcal{B}_i(u); u_n \rangle \right).$$

Moreover, one has

$$|\langle \mathcal{B}_i(u_n); u \rangle| \leq \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$$

and $|\langle \mathcal{B}_i(u); u_n \rangle| \leq \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$.

Indeed, by Hölder inequality, the first inequality follows from steps below

$$\int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_n}{\partial x_i} \frac{\partial u}{\partial x_i} dx \leq$$

$$\leq \left[\int_\Omega \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 1} \right)^{\frac{p_i}{p_i - 1}} dx \right]^{\frac{p_i - 1}{p_i}} \left[\int_\Omega \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i} \right) dx \right]^{\frac{1}{p_i}} =$$

$$= \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i - 1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$$

and the second is established in the same way. So, it follows

$$\langle \mathcal{B}_i(u_n) - \mathcal{B}_i(u); u_n - u \rangle \geq \left(\left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} - \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right)^2,$$

for which, taking (2.13) into accont, one has

$$\lim_{n \to +\infty} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}(\Omega)} = \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}, \quad \forall i = 1, ..., N. \tag{2.14}$$

From [15, Proposition III.30], taking into account that $L^{p_i}(\Omega)$ is uniformly convex, one has

$$\lim_{n \to +\infty} \left\| \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} = 0, \quad \forall i = 1, ..., N. \tag{2.15}$$

Hence, one has

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}=0, \tag{2.16}$$

that is,

$$\lim_{n\to+\infty}\|u_n-u\|_{W_0^{1,\vec{p}}(\Omega)}=0, \tag{2.17}$$

and our claim is proved.

Finally, standard computations show that from $(AR)-$condition we obtain that the functional I_λ is unbounded from below. More precisely, from $(AR)-$condition there are two constants $A>0$ and $B\geq 0$ such that

$$F(x,t)\geq At^\mu - B, \quad \forall (x,t)\in\Omega\times\mathbb{R}$$

(see for instance [39]). Fix $\bar{u}\in W_0^{1,\vec{p}}(\Omega)$, nonnegative and $\bar{u}\not\equiv 0$, and $h>1$, then one has

$$I_\lambda(h\bar{u})=\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial h\bar{u}}{\partial x_i}\right\|^{p_i}_{L^{p_i}(\Omega)}-\lambda\int_\Omega F(x,h\bar{u}(x))dx\leq$$

$$\leq h^{p^+}\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial \bar{u}}{\partial x_i}\right\|^{p_i}_{L^{p_i}(\Omega)}-\lambda\int_\Omega (A(h|\bar{u}|)^\mu - B)dx=$$

$=D_1h^{p^+}-h^\mu\lambda A\int_\Omega|\bar{u}|^\mu dx-\lambda B|\Omega| = D_1h^{p^+}-D_2h^\mu - D_3$, with $D_1>0$, $D_2>0$, $D_3\geq 0$.

Hence,

$$\lim_{h\to+\infty} I_\lambda(h\bar{u})=-\infty$$

and the proof is complete.

□

Remark 2.1. We recall that Φ' is called an $(S)_+-$map if for every sequence $\{u_n\}\subseteq W_0^{1,\vec{p}}(\Omega)$ such that $u_n \stackrel{w}{\to} u$ and $\limsup\limits_{n\to+\infty}\langle\Phi'(u_n),u_n-u\rangle\leq 0$ one has $u_n\to u$ in $W_0^{1,\vec{p}}(\Omega)$. So, arguing as in the proof of Lemma 2.1, it is easy to verify that Φ' is an $(S)_+-$map. We also recall that, under different assumptions on $\vec{p}$, such a property for Φ' has been proved in [12, Lemma 2].

Our main tool is a two non-zero critical points theorem established in [10]. It is a consequence of a local minimum theorem obtained in [8] (see also [9]) and the classical Ambrosetti-Rabinowitz theorem established in [1]. We recall it here for convenience of the reader.

Theorem 2.1. ([10, Theorem 2.1]) *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{2.18}$$

and, for each

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

Finally, we point out the following results in order to the sign of solutions that we will use in the next sections. To this end, put

$$f^+(x,t) = \begin{cases} f(x,0), & \text{if} \quad t < 0, \\ f(x,t), & \text{if} \quad t \geq 0, \end{cases} \tag{2.19}$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and denote by $(D^{\vec{p}}_{\lambda,f^+})$ the problem having f^+ as nonlinear term.
The first result allows to obtain nonnegative solutions. Precisely, we have the following.

Lemma 2.2. *Assume that*

$$f(x,0) \geq 0 \quad for \;\; a.e. \;\; x \in \Omega.$$

Then, any weak solution of the problem $(D^{\vec{p}}_{\lambda,f^+})$ is nonnegative and it is also a weak solution of $(D^{\vec{p}}_{\lambda})$.

Proof. Let $\bar{u}$ be a weak solution of problem

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f^+(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$D^{\vec{p}}_{\lambda,f^+}$}$$

So, one has

$$\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\lambda\int_{\Omega}f^{+}(x,\bar{u}(x))v(x)dx\,,$$

for all $v\in W_0^{1,\vec{p}}(\Omega)$. We claim that $\bar{u}$ is a nonnegative function. To this end, put $\bar{u}^{-}=\min\{\bar{u},0\}$. Clearly, one has $\bar{u}^{-}\in W_0^{1,\vec{p}}(\Omega)$ (see, for instance, [30, Lemma 7.6]). Moreover, setting $A=\{x\in\Omega:\bar{u}(x)<0\}$, taking into account that $\bar{u}$ is a weak solution and choosing $v=\bar{u}^{-}$, one has

$$\begin{aligned}0\leq\sum_{i=1}^{N}\int_{A}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i}dx&=\sum_{i=1}^{N}\int_{A}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\frac{\partial \bar{u}^{-}}{\partial x_i}dx=\\&=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\frac{\partial \bar{u}^{-}}{\partial x_i}dx=\\&=\lambda\int_{\Omega}f^{+}(x,\bar{u}(x))\bar{u}^{-}(x)dx=\lambda\int_{A}f^{+}(x,\bar{u}(x))\bar{u}^{-}(x)dx=\\&=\lambda\int_{A}f(x,0)\bar{u}^{-}(x)dx\leq 0\,,\end{aligned}$$

that is,

$$\sum_{i=1}^{N}\int_{A}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i}dx=0.$$

Hence, one has

$$\int_{A}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i}dx=0\text{ for all }i=1,...,N,$$

for which

$$\int_{\Omega}\left|\frac{\partial \bar{u}^{-}}{\partial x_i}\right|^{p_i}dx=\int_{A}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i}dx+\int_{\Omega\setminus A}\left|\frac{\partial \bar{u}^{-}}{\partial x_i}\right|^{p_i}dx=0\text{ for all }i=1,...,N$$

and hence

$$\left(\int_{\Omega}\left|\frac{\partial \bar{u}^{-}}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}=0\text{ for all }i=1,...,N.$$

It follows

$$\|\bar{u}^{-}\|_{W_0^{1,\vec{p}}(\Omega)}=\sum_{i=1}^{N}\left(\int_{\Omega}\left|\frac{\partial \bar{u}^{-}}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}=0,$$

so that $\bar{u}^{-}(x)=0$ in Ω and so $\bar{u}(x)\geq 0$ in Ω. Hence, our claim is proved.

Moreover, $\bar{u}$ is a weak solution for $(D_{\lambda}^{\vec{p}})$. Indeed, one has

$$\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\lambda\int_{\Omega}f^{+}(x,\bar{u}(x))v(x)dx=$$

$$=\lambda\int_{\Omega}f(x,\bar{u}(x))v(x)dx\,,$$

for all $v\in W_0^{1,\vec{p}}(\Omega)$, and the conclusion is achieved.

□

The next result allows to obtain positive solutions. It is based on the strong maximum principle established in [20] and here, since $p^{-}>N\geq 2$, the degenerate case is applied.

Lemma 2.3. *Assume that*

$$f(x,t)\geq 0\quad for\ \ a.e.\ \ x\in\Omega,\quad for\ \ all\ \ t\geq 0.$$

Then, any non-zero weak solution of the problem $(D_{\lambda,f^+}^{\vec{p}})$ *is positive and it is also a weak solution of* $(D_{\lambda}^{\vec{p}})$.

Proof. Let $\bar{u}$ be a non-zero weak solution of problem $(D_{\lambda,f^+}^{\vec{p}})$. Owing to Proposition 2.1 and Lemma 2.2 it is bounded and nonnegative in Ω and it is also a weak solution of $(D_{\lambda}^{\vec{p}})$. Therefore, taking also our assumption into account, one has

$$\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\lambda\int_{\Omega}f(x,\bar{u}(x))v(x)dx\geq 0\,,$$

for all $v\in W_0^{1,\vec{p}}(\Omega)$, with $v\geq 0$ in Ω. So, in particular, one has

$$\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\frac{\partial v}{\partial x_i}dx\geq 0$$

for all $v\in C_0^{\infty}(\Omega)$, with $v\geq 0$ in Ω. Hence, all assumptions of [20, Corollary 4.4] are verified and the function $\bar{u}$ is positive in Ω.

□

Remark 2.2. We observe that the conclusion of Lemma 2.1 also holds for the functional I_{λ} associated to the function f^{+}, defined as before and with $f(x,0)\geq 0$ in Ω, by requiring:

(AR^+) *There exist constants* $\mu > p^+$ *and* $M > 0$ *such that,* $0 < \mu F(x,t) \leq tf(x,t)$ *for all* $x \in \Omega$ *and for all* $t \geq M$*.*

Indeed, given a $(PS)-$sequence $\{u_n\}$, one has

$$\int_\Omega \left[\mu F^+(x,u_n(x)) - f^+(x,u_n(x))u_n(x)\right] dx =$$
$$= \int_{\{x\in\Omega: u_n(x)\geq M\}} \left[\mu F(x,u_n(x)) - f(x,u_n(x))u_n(x)\right] dx +$$
$$+ \int_{\{x\in\Omega: 0\leq u_n(x)< M\}} \left[\mu F(x,u_n(x)) - f(x,u_n(x))u_n(x)\right] dx +$$
$$+ \int_{\{x\in\Omega: u_n(x)< 0\}} \left[\mu f(x,0)u_n(x) - f(x,0)u_n(x)\right] dx \leq$$
$$\int_{\{x\in\Omega: 0\leq u_n(x)< M\}} \max_{0\leq\xi\leq M} \left[\mu F(x,\xi) - f(x,\xi)\xi\right] dx +$$
$$+ \int_{\{x\in\Omega: u_n(x)< 0\}} \left[(\mu-1)f(x,0)u_n(x)\right] dx \leq$$

$\leq \max_{0\leq\xi\leq M} \left[\mu F(x,\xi) - f(x,\xi)\xi\right] |\Omega|$, that is.

$$\int_\Omega \left[\mu F^+(x,u_n(x)) - f^+(x,u_n(x))u_n(x)\right] dx \leq C_1 \qquad \forall n \in \mathbb{N}$$

for some constant C_1. At this point the same proof of Lemma 2.1 ensures the conclusion.

3. Main Results

In this section, we present our main results. First, we point out an existence result of at least two non-zero weak solutions, that is Theorem 3.1, which is based on the two critical points theorem, Theorem 2.1. Next, we point out a consequence, Theorem 3.2, that ensures the existence of two positive weak solutions.

Put

$$R := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega);$$

simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0,R) \subseteq \Omega$ and we denote by

$$\omega_R := |B(x_0,R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} R^N,$$

the measure of the N-dimensional ball of radius R.

Finally, we set

$$\mathcal{K}=\frac{1}{\left[\displaystyle\sum_{i=1}^{N}\frac{1}{p_i}\left(\frac{2}{R}\right)^{p_i}\right]\omega_R\left(\dfrac{2^N-1}{2^N}\right)\max\left\{T^{p^-};T^{p^+}\right\}},$$

which depends on T as given in Propositition 2.2, computed in turn by the Talenti constant (see (2.2) and (2.4)).

Theorem 3.1. *Assume that the $(AR)-$condition holds and there are two positive constants c and d, with $\max\left\{d^{p^-};d^{p^+}\right\}<\min\left\{c^{p^-};c^{p^+}\right\}$, such that*

$$F(x,t)\geq 0,\quad \textit{for all}\quad (x,t)\in\Omega\times[0,d],\tag{3.1}$$

and

$$\frac{\displaystyle\int_\Omega\max_{|\xi|\leq c}F(x,\xi)dx}{\min\left\{c^{p^-};c^{p^+}\right\}}<\mathcal{K}\;\frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)}F\left(x,d\right)dx}{\max\left\{d^{p^-};d^{p^+}\right\}}.\tag{3.2}$$

Then, for each

$$\lambda\in\widetilde{\Lambda}:=$$

$$\left]\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{\max\left\{d^{p^-};d^{p^+}\right\}}{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)}F\left(x,d\right)dx},\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{\min\left\{c^{p^-};c^{p^+}\right\}}{\displaystyle\int_\Omega\max_{|\xi|\leq c}F(x,\xi)dx}\right[,$$

problem $(D_\lambda^{\vec{p}})$ has at least two non-zero weak solutions.

Proof. Put Φ and Ψ as in (2.6). It is well known that Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.1 and, moreover, one has $\inf\limits_{u\in W_0^{1,\vec{p}}(\Omega)}\Phi(u)=\Phi(0)=\Psi(0)=0$. Our aim is to verify condition (2.18). To this end, put $r=\min\{\left(\frac{c}{T}\right)^{p^-};\left(\frac{c}{T}\right)^{p^+}\}$ where T is defined in Proposition 2.2, and fix

$$\tilde{u}(x)=\begin{cases}0 & \text{if } x\in\Omega\setminus B(x_0,R),\\ \dfrac{2d}{R}\left(R-|x-x_0|\right) & \text{if } x\in B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right),\\ d & \text{if } x\in B\left(x_0,\frac{R}{2}\right).\end{cases}$$

Clearly, $\tilde{u}\in W_0^{1,\vec{p}}(\Omega)$. Moreover, for all $u\in W_0^{1,\vec{p}}(\Omega)$ such that $u\in\Phi^{-1}\left(]-\infty,r]\right)$, from Proposition 2.2 one has

$$|u(x)|<T\max\{r^{1/p^-};r^{1/p^+}\}=c\quad\text{for all}\quad x\in\Omega.$$

So,

$$\Psi(u)=\int_\Omega F(x,u(x))dx\leq\int_\Omega\max_{|\xi|\leq c}F(x,\xi)dx,$$

for all $u\in X$ such that $u\in\Phi^{-1}(]-\infty,r])$. Hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)\leq\int_\Omega\max_{|\xi|\leq c}F(x,\xi)dx.$$

Therefore, one has

$$\begin{aligned}\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &\leq \frac{\int_\Omega\max\limits_{|\xi|\leq c}F(x,\xi)dx}{\min\left\{\left(\frac{c}{T}\right)^{p^-};\left(\frac{c}{T}\right)^{p^+}\right\}}\\ &\leq \max\left\{T^{p^-};T^{p^+}\right\}\frac{\int_\Omega\max\limits_{|\xi|\leq c}F(x,\xi)dx}{\min\left\{c^{p^-};c^{p^+}\right\}}. \qquad (3.3)\end{aligned}$$

On the other hand, we have

$$\Phi(\tilde u)=\sum_{i=1}^N\frac{1}{p_i}\int_\Omega\left|\frac{\partial\tilde u}{\partial x_i}\right|^{p_i}dx\leq\left[\sum_{i=1}^N\frac{1}{p_i}\left(\frac{2d}{R}\right)^{p_i}\right]\omega_R\left(\frac{2^N-1}{2^N}\right)\leq$$

$$\leq\left[\sum_{i=1}^N\frac{1}{p_i}\left(\frac{2}{R}\right)^{p_i}\right]\omega_R\left(\frac{2^N-1}{2^N}\right)\max\left\{d^{p^-};d^{p^+}\right\}$$

and, taking (3.1) into account, one has

$$\Psi(\tilde u)\geq\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)\,dx.$$

Hence, we obtain

$$\frac{\Psi(\tilde u)}{\Phi(\tilde u)}\geq\frac{1}{\left[\sum_{i=1}^N\frac{1}{p_i}\left(\frac{2}{R}\right)^{p_i}\right]\omega_R\left(\frac{2^N-1}{2^N}\right)}\frac{\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)\,dx}{\max\left\{d^{p^-};d^{p^+}\right\}}=$$

$$= \max\left\{T^{p^-}; T^{p^+}\right\} \mathcal{K} \frac{\displaystyle\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}{\max\left\{d^{p^-}; d^{p^+}\right\}},$$

that is,

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \max\left\{T^{p^-}; T^{p^+}\right\} \mathcal{K} \frac{\displaystyle\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}{\max\left\{d^{p^-}; d^{p^+}\right\}}. \tag{3.4}$$

Hence, from (3.3), (3.4) and assumption (3.2) we obtain

$$\frac{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

So, in order to satisfy condition (2.18) of Theorem 2.1, it is enough to verify that $\Phi(\tilde{u}) < r$.
To this end, since $\max\left\{d^{p^-}; d^{p^+}\right\} < \min\left\{c^{p^-}; c^{p^+}\right\}$ (for which, in particular $d < c$), and again by virtue of (3.2), we infer that

$$\frac{1}{\mathcal{K}} \max\left\{d^{p^-}; d^{p^+}\right\} < \min\left\{c^{p^-}; c^{p^+}\right\}. \tag{3.5}$$

Indeed, arguing by contradiction, if we assume that $\frac{1}{\mathcal{K}} \max\left\{d^{p^-}; d^{p^+}\right\} \geq \min\left\{c^{p^-}; c^{p^+}\right\}$, we have

$$\frac{\displaystyle\int_\Omega \max_{|\xi| \leq c} F(x,\xi) dx}{\min\left\{c^{p^-}; c^{p^+}\right\}} \geq \mathcal{K} \frac{\displaystyle\int_\Omega \max_{|\xi| \leq c} F(x,\xi) dx}{\max\left\{d^{p^-}; d^{p^+}\right\}} \geq \mathcal{K} \frac{\displaystyle\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}{\max\left\{d^{p^-}; d^{p^+}\right\}}$$

which contradicts (3.2).
Hence, taking into account that

$$\begin{aligned} \Phi(\tilde{u}) \quad &\leq \quad \left[\sum_{i=1}^{N} \frac{1}{p_i} \left(\frac{2}{R}\right)^{p_i}\right] \omega_R \left(\frac{2^N - 1}{2^N}\right) \max\left\{d^{p^-}; d^{p^+}\right\} \\ &= \frac{1}{\mathcal{K}} \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \max\left\{d^{p^-}; d^{p^+}\right\} \end{aligned}$$

and

$$r = \min\{\left(\frac{c}{T}\right)^{p^-}; \left(\frac{c}{T}\right)^{p^+}\} \geq \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \min\left\{c^{p^-}; c^{p^+}\right\},$$

condition (3.5) ensures as claimed.

Finally, observing that Lemma 2.1 establishes that the functional I_λ is unbounded from below and it satisfies the $(PS)-$condition, we can apply Theorem 2.1. Therefore, I_λ admits two non-zero critical points that are two non-zero weak solutions to problem $(D_\lambda^{\vec{p}})$ for all $\lambda \in \tilde{\Lambda} \subset \Lambda$ and the conclusion is achieved.

□

A special case of our main result is the following theorem.

Theorem 3.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$.*
Assume that the $(AR^+)-$condition holds. Moreover, assume that there are two positive constants c and d, with $d < 1 \leq c$, such that

$$\frac{\displaystyle\int_\Omega F(x,c)dx}{c^{p^-}} < \mathcal{K} \frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx}{d^{p^-}}. \tag{3.6}$$

Then, for each $\lambda \in \tilde{\Lambda} :=$

$$\left]\frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{d^{p^-}}{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx}, \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{c^{p^-}}{\displaystyle\int_\Omega F(x,c)dx}\right[,$$

problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Proof. We apply Theorem 3.1 to the function f^+, as defined in (2.19), taking into account that the $(AR^+)-$condition is enough to verify the Palais-Smale condition of the associated functional (see Remark 2.2). Therefore, problem $(D_{\lambda,f^+}^{\vec{p}})$ admits two non-zero solutions for each $\lambda \in \tilde{\Lambda}$. Hence, Lemma 2.3 ensures that they are positive weak solutions of $(D_\lambda^{\vec{p}})$.

□

Remark 3.1. Taking into account condition (3.2) of Theorem 3.1, we can give other two versions of Theorem 3.2 depending on whether the positive constants c and d are such that either $d < c \leq 1$ or $1 \leq d < c$. To be precise, it is enough to substitute in Theorem 3.2 the condition (3.6) and interval $\tilde{\Lambda}$ with the followings

$$\frac{\int_\Omega F(x,c)dx}{c^{p^+}} < \mathcal{K}\frac{\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx}{d^{p^-}} \tag{3.6'}$$

and

$$\left]\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx},\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{c^{p^+}}{\int_\Omega F(x,c)dx}\right[,$$

in the first case. Similarly, in the other case, they become

$$\frac{\int_\Omega F(x,c)dx}{c^{p^-}} < \mathcal{K}\frac{\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx}{d^{p^+}} \tag{3.6''}$$

and

$$\left]\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^+}}{\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx},\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{c^{p^-}}{\int_\Omega F(x,c)dx}\right[.$$

Example 3.1. Fix $N = 3$ and $\Omega = B(0,2)$, put $p_1 = 4$, $p_2 = 5$, $p_3 = 6$ and consider the following problem

$$\begin{cases} -\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = 10^{-12}(x^2+y^2+z^2)u^8 + 10^{-12}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) =$$

$$= \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^2 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^3 \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^4 \frac{\partial u}{\partial z}\right).$$

Theorem 3.2 ensures that the problem (3.7) admits two positive weak solutions.
Indeed, by choosing $f(x,y,z,t) = (x^2+y^2+z^2)t^8+t^2$ for which $F(x,y,z,t) = (x^2+y^2+z^2)\frac{t^9}{9} + \frac{t^3}{3}$, it is easy to verify that the $(AR^+)-$condition holds. Moreover, taking into account that in this case one has $m_{p^-} = \sqrt[4]{\frac{3^3}{2\pi}}$, $T_0 = \sqrt[3]{\frac{2^5 \cdot 3^2}{\sqrt{\pi}}}$, $T = (\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})\sqrt[3]{\frac{2^5 \cdot 3^2}{\sqrt{\pi}}}$, $\max\left\{T^{p^-};T^{p^+}\right\} = T^6 = (\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6\frac{\left(2^5 \cdot 3^2\right)^2}{\pi}$, $\mathcal{K} = \frac{5}{2^{10}\cdot 3^2 \cdot 7 \cdot 37(\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6}$, by choosing $c = 1$ and $d = 10^{-14}$, we have

$$\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx} =$$

$$= \frac{7\cdot 37}{5}\frac{1}{\frac{2^2}{5}d^5 + \frac{2^2}{d}} \leq \frac{7\cdot 37}{4}d = \frac{7\cdot 37}{4}10^{-14} < 10^{-12}$$

and

$$\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{c^{p^-}}{\displaystyle\int_\Omega F(x,c)dx} = \frac{5}{(\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6 2^{15}3^4} > 10^{-12},$$

for which we obtain

$$\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F\left(x,d\right)dx} < 10^{-12} <$$

$$< \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{c^{p^-}}{\displaystyle\int_\Omega F(x,c)dx}.$$

Hence, all assumptions of Theorem 3.2 are satisfied and our claim is proved.

Remark 3.2. To the best of our knowledge, we do not know regularity results of solutions for a general problem as $(D_\lambda^{\vec{p}})$, which in particular involves nonlinearities. In other completely different contexts, regularity results are obtained for anisotropic problems under suitable assumptions and we refer, for instance, to [28, Theorem 5], where a classical C^2-solution is obtained, and to [16], where the local boundedness of solutions (as local minimizers of the energy functional) is established.

4. SOME CONSEQUENCES

In this section, we point out some consequences of our main results in autonomous case. To be precise, let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function and consider the following anisotropic Dirichlet problem

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$AD_\lambda^{\vec{p}}$}$$

The usual Ambrosetti-Rabinowitz condition, given in Section 2, becomes:

(AR_1^+) *there exist constants $\mu > p^+$ and $M > 0$ such that, $0 < \mu F(t) \leq t f(t)$ for all $t \geq M$.*

Moreover, put

$$\mathcal{K}^* = \frac{\omega_R}{2^N |\Omega|}\mathcal{K},$$

where $\mathcal{K}$ is given in Section 3.

A special case of Theorem 3.2 is the following result.

Theorem 4.1. *Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function such that the $(AR_1^+)-$condition holds. Moreover, assume that there are two positive constants c and d, with $d < 1 \leq c$, such that*

$$\frac{F(c)}{c^{p^-}} < \mathcal{K}^* \frac{F(d)}{d^{p^-}}. \tag{4.1}$$

Then, for each

$$\lambda \in \tilde{\Lambda}_1 := \left] \frac{1}{\max\{T^{p^-}; T^{p^+}\}} \frac{1}{|\Omega|} \frac{1}{\mathcal{K}^*} \frac{d^{p^-}}{F(d)}, \frac{1}{\max\{T^{p^-}; T^{p^+}\}} \frac{1}{|\Omega|} \frac{c^{p^-}}{F(c)} \right[,$$

the problem $(AD_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Proof. We can assume f defined in $\mathbb{R}$ by setting $f(t) = f(0)$ for all $t < 0$. Hence, the conclusion follows from Theorem 3.2.

□

Now, by setting

$$\lambda^* = \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \sup_{c \geq 1} \frac{c^{p^-}}{F(c)},$$

we point out the following consequence of Theorem 4.1.

Theorem 4.2. *Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function such that the $(AR_1^+)-$condition holds. Assume that*

$$\limsup_{t \to 0^+} \frac{F(t)}{t^{p^-}} = +\infty. \tag{4.2}$$

Then, for each $\lambda \in \,]0, \lambda^*[$, the problem $(AD_\lambda^{\vec{p}})$ admits at least two positive weak solutions.

Proof. Fix a positive number $\lambda < \lambda^*$. So, there is $c \geq 1$ such that $\lambda < \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \frac{c^{p^-}}{F(c)}$, that is,

$$\max\left\{T^{p^-}; T^{p^+}\right\} |\Omega| \frac{F(c)}{c^{p^-}} < \frac{1}{\lambda}.$$

On the other hand, from (4.2) one has $\limsup_{t \to 0^+} \max\left\{T^{p^-}; T^{p^+}\right\} |\Omega| \mathcal{K}^* \frac{F(t)}{t^{p^-}} = +\infty$, for which there is a positive sequence $\{d_n\}$ such that $d_n \to 0^+$ and

$$\max\left\{T^{p^-}; T^{p^+}\right\} |\Omega| \mathcal{K}^* \frac{F(d_n)}{d_n^{p^-}} > \frac{1}{\lambda}$$

for all $n > \nu$. Fix $\bar{n} > \nu$ such that $d_{\bar{n}} < 1$ and put $d = d_{\bar{n}}$. One has $d < 1 \leq c$ and

$$\max\left\{T^{p^-}; T^{p^+}\right\} |\Omega| \frac{F(c)}{c^{p^-}} < \frac{1}{\lambda} < \max\left\{T^{p^-}; T^{p^+}\right\} |\Omega| \mathcal{K}^* \frac{F(d)}{d^{p^-}}.$$

Hence, all assumption of Theorem 4.1 are satisfied and, taking also into account that $\lambda \in \tilde{\Lambda}_1$, the conclusion is achieved.

□

Remark 4.1. Actually, taking into account the generality of (3.2) in Theorem 3.1 (see Remark 3.1), the parameter λ^* can be chosen in a more precise way, that is,

$$\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{|\Omega|}\max\left\{\sup_{c\geq 1}\frac{c^{p^-}}{F(c)};\sup_{0<c<1}\frac{c^{p^+}}{F(c)}\right\}.$$

As a consequence of Theorem 4.2 we point out the following result which deals with a problem having combined effects of concave and convex nonlinearities.

Theorem 4.3. *Fix s,q such that $0\leq s<p^- -1$ and $p^+-1<q$. Put*

$$\eta^*=\min\left\{\frac{1-\frac{p^+}{q+1}}{\frac{p^+}{s+1}-1},\left[\frac{(s+1)(q+1)}{\max\left\{T^{p^-};T^{p^+}\right\}|\Omega|}\frac{\left(\frac{p^+}{s+1}-1\right)^{\frac{p^+-(s+1)}{q-s}}\left(1-\frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1-\frac{p^+}{q+1}\right)+(s+1)\left(\frac{p^+}{s+1}-1\right)}\right]^{\frac{q-s}{(q+1)-p^+}}\right\}.$$

Then, for each $\eta\in]0,\eta^*[$ the problem

$$\begin{cases} -\Delta_{\vec{p}}u=\eta u^s+u^q & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases} \tag{$AD^{\vec{p}}_{\eta}$}$$

has at least two positive weak solutions.

Proof. Our aim is to apply Theorem 4.2 to the function $f(u)=\eta u^s+u^q$. Indeed, fix μ such that $p^+<\mu<q+1$, then one has $\lim\limits_{t\to+\infty}\dfrac{\mu F(t)}{f(t)t}=\lim\limits_{t\to+\infty}\dfrac{\mu(\eta\frac{t^{s+1}}{s+1}+\frac{t^{q+1}}{q+1})}{(\eta t^s+t^q)t}=\dfrac{\mu}{q+1}<1$, for which there is $M>0$ such that $\mu(\eta\frac{t^{s+1}}{s+1}+\frac{t^{q+1}}{q+1})<(\eta t^s+t^q)t$ for all $t\geq M$ and so (AR_1^+)−condition is verified. Moreover, $\lim\limits_{t\to 0^+}\dfrac{f(t)}{t^{p^--1}}=\lim\limits_{t\to 0^+}\dfrac{\eta t^s+t^q}{t^{p^--1}}=+\infty$, for which also (4.2) is satisfied. Hence, Theorem 4.2 ensures that the problem

$$\begin{cases} -\Delta_{\vec{p}}u=\lambda\left(\eta u^s+u^q\right) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

admits two positive weak solutions for all positive $\lambda<\lambda^*$. Therefore, in order to obtain the conclusion it is enough to prove that $\lambda^*>1$. To this end, we observe that the function $\dfrac{c^{p^+}}{F(c)}$ in $]0,1[$ assumes the maximum in $\bar{c} = \left(\dfrac{\frac{p^+}{s+1}-1}{1-\frac{p^+}{q+1}}\eta\right)^{\frac{1}{q-s}}$, being $\eta < \eta^* \leq \dfrac{1-\frac{p^+}{q+1}}{\frac{p^+}{s+1}-1}$ and taking into account that $D\frac{c^{p^+}}{F(c)} = c^{(p^+ + s)}\dfrac{\left(\frac{p^+}{s+1}-1\right)\eta + \left(\frac{p^+}{q+1}-1\right)c^{q-s}}{\left[\eta\frac{c^{s+1}}{s+1}+\frac{c^{q+1}}{q+1}\right]^2}$. It follows that (see also Remark 4.1) one has

$$\begin{aligned}\lambda^* \quad &\geq \quad \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{|\Omega|}\sup_{0<c<1}\frac{c^{p^+}}{F(c)} = \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{|\Omega|}\frac{\bar{c}^{p^+}}{\eta\frac{\bar{c}^{s+1}}{s+1}+\frac{\bar{c}^{q+1}}{q+1}}\\ &= \frac{(s+1)(q+1)}{\max\left\{T^{p^-};T^{p^+}\right\}|\Omega|}\frac{\left(\frac{p^+}{s+1}-1\right)^{\frac{p^+-(s+1)}{q-s}}\left(1-\frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1-\frac{p^+}{q+1}\right)+(s+1)\left(\frac{p^+}{s+1}-1\right)}\frac{1}{\eta^{\frac{(q+1)-p^+}{q-s}}} > 1,\end{aligned}$$

being

$$\eta<\eta^*\leq\left[\frac{(s+1)(q+1)}{\max\left\{T^{p^-};T^{p^+}\right\}|\Omega|}\frac{\left(\frac{p^+}{s+1}-1\right)^{\frac{p^+-(s+1)}{q-s}}\left(1-\frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1-\frac{p^+}{q+1}\right)+(s+1)\left(\frac{p^+}{s+1}-1\right)}\right]^{\frac{q-s}{(q+1)-p^+}},$$

for which the conclusion is achieved.

□

Remark 4.2. Theorem 1.1 in Introduction is an immediate consequence of Theorem 4.3. Indeed, it is enough to choose $s = p^- - 2$ and $q = p^+$.

Finally, we present an example of problem that admits two positive weak solutions, applying Theorem 4.3.

Example 4.1. Put $p_1 = 3$, $p_2 = 4$, $N = 2$ and $\Omega = B(0,1)$. Theorem 4.3 ensures that for each $\eta \in \left]0, \frac{3}{2^8(2^{\frac{1}{2}}+3^{\frac{1}{3}})^8}\right[$, the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_1}\left(\left|\dfrac{\partial u}{\partial x_1}\right|\dfrac{\partial u}{\partial x_1}\right) - \dfrac{\partial}{\partial x_2}\left(\left|\dfrac{\partial u}{\partial x_2}\right|^2\dfrac{\partial u}{\partial x_2}\right) = \eta u + u^5 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

admits at least two positive weak solutions.

Indeed, in this case one has $m_{p^-} = \left(\frac{2}{\pi}\right)^{\frac{1}{3}}$; $T_0 = \frac{2}{\pi^{\frac{1}{4}}}$; $T = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})\frac{2}{\pi^{\frac{1}{4}}}$; $\max\left\{T^{p^-}; T^{p^+}\right\} = T^4 = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4\frac{2^4}{\pi}$; $\max\left\{T^{p^-}; T^{p^+}\right\}|\Omega| = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4 2^4$; $(s+1)(q+1) = 12$;

$$\frac{\left(\frac{p^+}{s+1} - 1\right)^{\frac{p^+-(s+1)}{q-s}}\left(1 - \frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1 - \frac{p^+}{q+1}\right) + (s+1)\left(\frac{p^+}{s+1} - 1\right)} = \frac{1}{3^{\frac{1}{2}}4}, \text{ for which}$$

$$\eta^* = \min\left\{\frac{1}{3}; \left[\frac{3^{\frac{1}{2}}}{(3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4 2^4}\right]^2\right\} = \frac{3}{(3^{\frac{1}{3}} + 4^{\frac{1}{4}})^8 2^8}.$$

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