Sharp inequalities for the Ornstein-Uhlenbeck operator

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The **Ornstein-Uhlenbeck operator** $\mathcal{L} = \Delta - x \cdot \nabla$ is the natural counterpart of the Laplace operator when the ambient Euclidean space is replaced by the probability space $(\mathbb{R}^n, \gamma_n)$, where γ_n denotes the **Gauss measure** with the density

$$\mathrm{d}\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \,\mathrm{d}x.$$

For any $f \in L^1(\mathbb{R}^n, \gamma_n)$ satisfying $\int_{\mathbb{R}^n} f \,\mathrm{d}\gamma_n = 0$ a unique solution to

$$\mathcal{L}u = -f \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \operatorname{med}(u) = 0$$

exists (in a suitable weak sense) and the **optimal transfer of integrability** from f to u is available. More precisely, for a given rearrangement invariant space X (including Lebesgue, Lorentz and Orlicz), we characterise the optimal (smallest) rearrangement invariant space Y such that

$$\|u\|_{Y(\mathbb{R}^n, \gamma_n)} \leq C \|f\|_{X(\mathbb{R}^n, \gamma_n)}$$

for some $C > 0$ and every $f \in X(\mathbb{R}^n, \gamma_n)$. Unlike in the Euclidean case, the **gain** of integrability is **not always guaranteed**.