Sharp inequalities for the Ornstein-Uhlenbeck operator

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The Ornstein-Uhlenbeck operator \( \mathcal{L} = \Delta - x \cdot \nabla \) is the natural counterpart of the Laplace operator when the ambient Euclidean space is replaced by the probability space \((\mathbb{R}^n, \gamma_n)\), where \( \gamma_n \) denotes the Gauss measure with the density

\[
d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \, dx.
\]

For any \( f \in L^1(\mathbb{R}^n, \gamma_n) \) satisfying \( \int_{\mathbb{R}^n} f \, d\gamma_n = 0 \) a unique solution to

\[
\mathcal{L}u = -f \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \text{med}(u) = 0
\]

exists (in a suitable weak sense) and the optimal transfer of integrability from \( f \) to \( u \) is available. More precisely, for a given rearrangement invariant space \( X \) (including Lebesgue, Lorentz and Orlicz), we characterise the optimal (smallest) rearrangement invariant space \( Y \) such that

\[
\|u\|_{Y(\mathbb{R}^n, \gamma_n)} \leq C \|f\|_{X(\mathbb{R}^n, \gamma_n)}
\]

for some \( C > 0 \) and every \( f \in X(\mathbb{R}^n, \gamma_n) \). Unlike in the Euclidean case, the gain of integrability is not always guaranteed.