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Positive solutions for nonlinear Robin problems with indefinite potential and competing nonlinearities

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Abstract

We consider a nonlinear Robin problem associated to the p-Laplacian plus an indefinite potential. In the reaction we have the competing effects of two nonlinear terms. One is parametric and strictly $(p-1)$-sublinear. The other is $(p-1)$-linear. We prove a bifurcation-type theorem describing the dependence of the set of positive solutions on the parameter $\lambda > 0$. We also show that for every admissible parameter the problem has a smallest positive solution $\bar{u}_\lambda$ and we study monotonicity and continuity properties of the map $\lambda \to \bar{u}_\lambda$.

Keywords Competing nonlinearities · Truncation · Nonlinear regularity · Nonlinear maximin principle · Strong comparison principle · Bifurcation-type result · Minimal positive solutions

Mathematics Subject Classification 35J20 · 35J60

1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^N$, with C^2-boundary $\partial\Omega$, we examine the following nonlinear parametric Robin problem:

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$$\begin{cases} -\Delta_p u(z)+\xi(z)u(z)^{p-1} = \lambda f(z,u(z))+g(z,u(z)), \quad u(z)>0, \quad \lambda>0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

By $\Delta_p u$ we denote the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < +\infty.$$

The potential function $\xi(z) \in L^\infty(\Omega)$ and in general it is sign changing. So, the left-hand side in (1.1) is not coercive. The reaction (i.e. the right-hand side of (1.1) exhibits the competing effects of two terms. One is the parametric term $\lambda f(z,x)$, with $\lambda > 0$ being the parameter, and $f(z,x)$ being a Carathéodory function which has strictly $(p-1)$-sublinear growth in $x \in \mathbb{R}$ near $+\infty$. The perturbation $g(z,x)$ is $(p-1)$-linear near $+\infty$ and, asymptotically as $x \to +\infty$, the quotient $\frac{g(z,x)}{x^{p-1}}$ stays above $\hat{\lambda}_1$ the principal eigenvalue of the Robin p-Laplacian.

In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u corresponding to the p-Laplacian defined by extension of the map

$$C^1(\bar{\Omega}) \ni u \to |Du|^{p-2}(Du,n)_{\mathbb{R}^n} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with n being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(z) \in C^{0,\alpha}(\partial\Omega)$ ($\alpha \in]0,1[$) and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$ we have the usual Neumann problem.

We are looking for positive solutions of problem (1.1) and our aim is to describe how the set of positive solutions changes as the parameter $\lambda > 0$ moves in the positive semiaxis $]0,+\infty[$. So we prove a bifurcation-type result establishing the existence of a critical parameter value $\lambda^* > 0$ such that

- for all $\lambda \in]0,\lambda^*[$ problem (1.1) has at least two positive solutions;
- for $\lambda = \lambda^*$ problem (1.1) has at least one positive solution;
- for all $\lambda > \lambda^*$ problem (1.1) has no positive solution.

Moreover, we show that for every admissible parameter $\lambda \in]0,\lambda^*]$ problem (1.1) has a smallest positive solution $\bar{u}_\lambda$ and we study the monotonicity and the continuity properties of the map $\lambda \to \bar{u}_\lambda$.

Such results for the set of positive solutions of elliptic equations were proved by Ambrosetti–Brezis–Cerami [2] (for semilinear Dirichlet problems) and by Garcia Azorero–Manfredi–Peral Alonso [5], Guo–Zhang [8], Hu–Papageorgiou [10], Marano–Papageorgiou [14] (for nonlinear Dirichlet problems).

In the aforementioned papers $\xi \equiv 0$, hence the differential operator of the equation is coercive and the competition is between a concave (strictly sublinear) term and a convex (superlinear) term. So they deal with the well-known "concave–convex" problem.

Recently Papageorgiou–Radulescu–Repovs [20] studied semilinear Robin problems with indefinite linear part and a reaction having the combined effects of concave and convex terms. They proved a bifurcation-type result as described above. There are also the recent works of Candito–Livrea–Papageorgiou [4] (Neumann problems) and Papageorgiou–Radulescu–Repovs [19] (Robin problems). Both treat concave–convex problems. In [4] the emphasis is on the existence of nodal solutions. In [19] the differential operator is nonhomogeneous, the potential function is nonnegative (thus the left-hand side is coercive) and the reaction has the form $\lambda f(z, x)$ (that is $g \equiv 0$).

Our approach uses variational tools based on the critical point theory combined with suitable truncation, perturbation and comparison techniques.

For other kind of operators with lower order terms see also [11,12]

2 Mathematical preliminaries: hypotheses

Let X be a Banach space and let X^* be its topological dual. By $< \cdot, \cdot >$ we denote the duality brackets for the dual pair (X^*, X).

Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the C-condition for short) if the following property holds:

"Every sequence $\{u_n\} \subset X$ such that

$$\{\varphi(u_n)\} \subset \mathbb{R} \quad \text{is bounded}$$

and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \ \text{ in } X^*, \ \text{ as} n \to +\infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ which compensates for the fact that X is in general infinite dimensional and so it is not locally compact. The C-condition leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. A major result in this theory is the so-called "mountain pass theorem", which we recall here.

Theorem 2.1 *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) \ : \ \|u_1 - u_0\| = \rho\} = \eta_\rho$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) \ : \ \gamma(0) = u_0, \ \gamma(1) = u_1\}$, then $c \ge \eta_\rho$ and c is a critical value of φ (that is, there exists $\hat{u} \in X$ such that $\varphi'(\hat{u}) = 0$, $\varphi(\hat{u}) = c$).

By $\| \cdot \|$ we denote the norm of $W^{1,p}(\Omega)$ given by

$$\|u\| := \left[\|u\|_{L^p}^p + \|Du\|_{L^p}^p\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\bar{\Omega})$ is an ordered space with positive (order) cone

$$C_+ = \{u \in C^1(\bar{\Omega}) \, : \, u(z) \geq 0 \; \forall z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ \, : \, u(z) > 0 \; \forall z \in \bar{\Omega}\}.$$

Also we will use another open cone in $C^1(\bar{\Omega})$, namely

$$int\, \hat{C}_+ = \left\{ u \in C^1(\bar{\Omega}) : u(z) > 0 \quad \text{for all } z \in \Omega, \; \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure we can define in the usual way the "boundary" Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq +\infty$. From the theory of Sobolev spaces, we know that there exists a continuous linear map $\gamma_0 : W^{1,p} \to L^p(\partial\Omega)$, known as the "trace map" , such that

$$\gamma_0(u) = u_{|\partial\Omega} \;\; \forall u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}) \, .$$

So, the trace map defines boundary values for every Sobolev function. We know that

$$im\, \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \; \big(1/p + 1/p' = 1\big) \; \text{ and } ker\, \gamma_0 = W_0^{1,p}(\Omega).$$

The trace map $\gamma_0(\cdot)$ is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p}[$, if $p < N$, and into $L^q(\partial\Omega)$ for all $q \in [1, +\infty[$, if $p \geq N$.

In the sequel, for notational economy, we drop the use of trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$< A(u), h >= \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \, dz \; \text{ for all } u, h \in W^{1,p}(\Omega). \qquad (2.1)$$

The next proposition shows the main properties of this map (see, for example, [7], Gasinski–Papageorgiou , Problem 2.192, p. 279).

Proposition 2.2 *If*

$$A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$$

is defined by (2.1)*, then* $A(\cdot)$ *is bounded, continuous, monotone and of type* $(S)_+$ *(that is, if*

$$u_n \xrightarrow{w} u \;\; in \; W^{1,p}(\Omega)$$

and

$$\limsup_{n\to+\infty} < A(u_n), u_n - u > \leq 0$$

then

$$u_n \to u \ \ in \ W^{1,p}(\Omega)).$$

We introduce the conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

($\boldsymbol{\xi}$) $\xi \in L^\infty(\Omega)$.
($\boldsymbol{\beta}$) $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$ and $\beta(z) \geq 0$, $\forall z \in \partial\Omega$.

Remark 2.3 When $\beta \equiv 0$ we recover the Neumann problem.

In what follows by $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma_p(u) = \|Du\|_{L^p}^p + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma, \quad \forall\, u \in W^{1,p}(\Omega).$$

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying

$$|f_0(z,x)| \leq \alpha_0(z)[1 + |x|^{r-1}] \text{ for a.a. } z \in \Omega \text{ and } \forall x \in \mathbb{R},$$

with $\alpha_0 \in L^\infty(\Omega)$ and $1 < r \leq p^*$, where $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}$.

We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega F_0(z,u)\,dz, \quad \forall u \in W^{1,p}(\Omega)\,.$$

The next result is a special case of a more general one of Papageorgiou–Radulescu [17] (see also Brezis–Nirenberg [3], Garcia Azorero–Manfredi–Peral Alonso [5], Guo–Zhang [8] for earlier results of this nature).

Proposition 2.4 *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\bar{\Omega})$-minimizer of φ_0, that is there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi(u_0 + h) \ \ \textit{for all } h \in C^1(\bar{\Omega}) \textit{ and } \|h\|_{C^1(\bar{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\eta}(\bar{\Omega})$ for some $0 < \eta < 1$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi(u_0) \leq \varphi(u_0 + h) \ \ \textit{for all } h \in W^{1,p}(\Omega),\ \|h\| \leq \rho_1.$$

The above result is essentially an outgrowth of the nonlinear regularity theory of Lieberman [13]. To make effective use of Proposition 2.4 we need the following strong comparison principle. Again, the result is a special case of a more general result due to Papageorgiou–Radulescu–Repovs [19]

Proposition 2.5 *If* $\hat{\xi} \in L^\infty(\Omega)$, $\hat{\xi}(z) \geq 0$ *for a.a.* $z \in \Omega$, $h_1, h_2 \in L^\infty(\Omega)$ *satisfy*

$$0 < \hat{c} \leq h_2(z) - h_1(z) \ \ \textit{for a.a. } z \in \Omega,$$

$u, v \in C^1(\bar{\Omega})\backslash\{0\}$ *satisfy* $u \leq v$ *and*

$$\begin{aligned} -\Delta_p u(z) + \hat{\xi}(z)|u(z)|^{p-2}u(z) = h_1(z) \ \ \textit{for a.a. } z \in \Omega \\ -\Delta_p v(z) + \hat{\xi}(z)|v(z)|^{p-2}v(z) = h_2(z) \ \ \textit{for a.a. } z \in \Omega \end{aligned}$$

then $v - u \in int\hat{C}_+$.

We will also need some facts about the spectrum of the differential operator $u \to -\Delta_p u + \xi(z)|u|^{p-2}u$. So, we consider the following nonlinear eigenvalue problem.

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

We say that $\hat{\lambda}$ is an "eigenvalue" if the problem admits a nontrivial solution $\hat{u}$ known as "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. This eigenvalue problem was studied by Papageorgiou–Radulescu [16] (Robin problems) and Mugnai–Papageorgiou [15] (Neumann problems). We know that problem (2.2) admits a smallest eigenvalue $\hat{\lambda}_1$ which has the following properties:

- $\hat{\lambda}_1$ is isolated [that is, there exists $\varepsilon > 0$ such that the open interval $]\hat{\lambda}_1, \hat{\lambda}_1 + \varepsilon[$ contains no eigenvalue of (2.2)];
- $\hat{\lambda}_1$ is simple (that is, if $\hat{u}, \hat{v}$ are eigenfunctions corresponding to the eigenvalue $\hat{\lambda}_1$, then $\hat{u} = \theta\hat{v}$ for some $\theta \in \mathbb{R}\backslash\{0\}$);
- $$\hat{\lambda}_1 = \inf\left[\frac{\gamma_p(u)}{\|u\|_{L^p}^p} : u \in W^{1,p}(\Omega), u \neq 0\right]. \tag{2.3}$$

The nonlinear regularity theory implies that all eigenfunctions of (2.2) belong to $C^1(\Omega)$. Moreover, the above properties of $\hat{\lambda}_1$ imply that all the eigenfunctions corresponding to $\hat{\lambda}_1$ have fixed sign.

Let $\hat{u}_1$ be the L^p-normalized (that is, $\|\hat{u}_1\|_{L^p} = 1$), positive eigenfunction corresponding to $\hat{\lambda}_1$. Then form the nonlinear maximum principle (see Pucci–Serrin [21]), we have $\hat{u}_1 \in D_+$.

In (2.3) the infimum is realized on the corresponding one dimensional eigenspace $\mathbb{R}\hat{u}_1$. An eigenfunction $\hat{u} \in C^1(\bar{\Omega})$ corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$ is necessarily nodal (that is, sign changing). The Ljusternik–Schnirelmann minimax scheme gives us in addition to $\hat{\lambda}_1$, a whole strictly increasing sequence $\{\hat{\lambda}_k\}$ of distinct eigenvalues (known as "variational eigenvalues") such that $\hat{\lambda}_k \to +\infty$. We do not know if this sequence of variational eigenvalues exhausts the spectrum of (2.2). This is the case if $N = 1$ or if $p = 2$.

We will also encounter a weighted version of the eigenvalue problem (2.2). So, let $m \in L^\infty(\Omega)\backslash\{0\}$ and consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \tilde{\lambda}\, m(z)|u(z)|^{p-2}u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

We can have a smallest eigenvalue $\tilde{\lambda}_1(m)$ which now has the following variational characterization

$$\tilde{\lambda}_1(m) = \inf \left[\frac{\gamma_p(u)}{\displaystyle\int_\Omega m(z)|u|^p\,dz} : u \in W^{1,p}(\Omega), u \neq 0 \right]. \tag{2.5}$$

The corresponding eigenfunctions $\tilde{u}$ have constant sign. As before, by $\tilde{u}_1$ we denote the positive, L^p-normalized eigenfunction. We have $\tilde{u}_1 \in D_+$ and the infimum in (2.5) is realized on $\mathbb{R}\tilde{u}_1$

Lemma 2.6 *If $m_1, m_2 \in L^\infty(\Omega)\backslash\{0\}$, $m_1(z) \le m_2(z)$ for a.a. $z \in \Omega$ and $m_1 \neq m_2$, then $\tilde{\lambda}_1(m_2) < \tilde{\lambda}_1(m_1)$*

Proof Using (2.5) and recalling that $\tilde{u}_1 \in D_+$, we have

$$\tilde{\lambda}_1(m_2) \le \frac{\gamma_p(\tilde{u}_1)}{\displaystyle\int_\Omega m_2(z)\tilde{u}_1^p\,dz} < \frac{\gamma_p(\tilde{u}_1)}{\displaystyle\int_\Omega m_1(z)\tilde{u}_1^p\,dz} = \tilde{\lambda}_1(m_1)\,.$$

□

Finally let us fix some basic notation that we will use in the sequel.

If $x \in \mathbb{R}$ then we set $x^\pm = \max\{\pm x, 0\}$. For $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-\,.$$

If $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then $N_g(\cdot)$ denotes the Nemytskii (superposition) operator for $g(z, x)$ defined by $N_g(u)(\cdot) = g(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$.

Given $u, v \in W^{1,p}(\Omega)$ with $u(z) \le v(z)$ for a.a. $z \in \Omega$, we define

$$[u, v] = \{y \in W^{1,p}(\Omega) : u(z) \le y(z) \le v(z) \text{ for a.a. } z \in \Omega\}.$$

By $int_{C^1(\bar{\Omega})}[u, v]$ we denote the interior in the $C^1(\bar{\Omega})$-norm topology of the set $[u, v] \cap C^1(\bar{\Omega})$.

Also, if $u \in W^{1,p}(\Omega)$, we set

$$[u[= \{y \in W^{1,p}(\Omega) : u(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X \ : \ \varphi'(u) = 0\}.$$

Now we will introduce our hypotheses on the two competing nonlinearities $f(z, x)$ and $g(z, x)$ in the reaction problem (1.1).

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

(**f1**) one has

$$f(z, 0) = 0 \quad \text{for a.a. } z \in \Omega;$$

(**f2**) for every $\rho > 0$ there exists a positive function $\alpha_\rho \in L^\infty(\Omega)$ such that

$$0 \leq f(z, x) \leq \alpha_\rho(z) \quad \text{for a. a. } z \in \Omega \text{ and all } x \in [0, \rho];$$

(**f3**) we have

$$\lim_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} = 0,$$

uniformly for a.a. $z \in \Omega$;

(**f4**) there exists $\delta_0 > 0$ and $q \in]1, p[$ such that

$$c_1 x^{q-1} \leq f(z, x) \quad \text{for a. a. } z \in \Omega, \text{ all } x \in [0, \delta_0]$$

and for every $s > 0$ there exists $\tilde{\eta}_s > 0$ such that

$$\tilde{\eta}_s \leq f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq s.$$

Remark 2.7 Since we are looking for positive solutions and all the above hypotheses concern the semiaxis $[0, +\infty[$, without any loss of generality, we may assume that

$$f(z, x) = 0 \text{ for a.a. } z \in \Omega \text{ and } \forall x < 0. \tag{2.6}$$

Hypothesis (f3) implies that for a.a. $z \in \Omega$, $f(z, \cdot)$ is strictly $(p-1)$-sublinear near $+\infty$. Hypothesis (f4) reveals the presence of a concave nonlinearity near 0^+.

In turn, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

($\boldsymbol{g1}$) one has

$$g(z,0)=0 \quad \text{for a.a. } z\in\Omega;$$

($\boldsymbol{g2}$) there exists a positive function $\alpha\in L^\infty(\Omega)$ such that

$$|g(z,x)|\leq \alpha(z)(1+x^{p-1})$$

for a.a. $z\in\Omega$ and all $x\geq 0$;

($\boldsymbol{g3}$) there exists $\hat{\eta}>\hat{\lambda}_1$ such that

$$\hat{\eta}\leq \liminf_{x\to+\infty}\frac{g(z,x)}{x^{p-1}},$$

uniformly for a.a. $z\in\Omega$;

($\boldsymbol{g4}$) there exist $c_2,c_3,c_4,\delta_1>0$ and $r\in]p,p^*[$ such that

$$-c_2x^{p-1}\leq g(z,x)$$

for a.a. $z\in\Omega$, all $x\in[0,\delta_1]$ and

$$g(z,x)\leq c_3x^{r-1}-c_4x^{p-1}$$

for a.a. $z\in\Omega$, all $x\geq 0$.

Remark 2.8 As we did for $f(z,\cdot)$, without any loss of generality, we may assume that

$$g(z,x)=0 \quad \text{for a.a. } z\in\Omega,\ \text{ all } x\leq 0 \tag{2.7}$$

Hypotheses (g2), (g3) imply that, for a.a. $z\in\Omega$, $g(z,\cdot)$ is $(p-1)$-linear near $+\infty$ and, asymptotically as $x\to+\infty$, the quotient $\dfrac{g(z,x)}{x^{p-1}}$ stays above $\hat{\lambda}_1$.

So, in the present work the competition is between a concave term and a $(p-1)$-linear perturbation. Evidently in hypothesis (g4), by appropriately modifying c_2, we can always assume that $c_4>\|\xi\|_{L^\infty}$.

Finally, for every $\rho>0$ and every $B\subseteq]0,+\infty[$ bounded, we can find $\hat{\xi}_\rho^B>0$ such that for all $\lambda\in B$ and a.a. $z\in\Omega$, the function

$$x\to\lambda f(z,x)+g(z,x)+\hat{\xi}_\rho^B x^{p-1} \tag{2.8}$$

is nondecreasing on $[0,\rho]$.

If $p=2$, then this hypothesis is a one-sided local Lipschitz condition on the reaction. If for a.a. $z\in\Omega$, $f(z,\cdot)$ and $g(z,\cdot)$ are both differentiable and for every $\rho>0$ and every $B\subseteq]0,+\infty[$ bounded, we can find $\hat{\xi}_\rho^B>0$ such that

$$\left[\lambda f_x'(z,x)+g_x'(z,x)\right]x^2\geq -\hat{\xi}_\rho^B x^p$$

for a.a. $z\in\Omega$, all $x\in[0,\rho]$, all $\lambda\in B$, then hypothesis (2.8) is satisfied.

Examples The following functions satisfy hypotheses $(f1)$–$(f4)$ and $(g1)$–$(g4)$. For the sake of simplicity we drop the z-dependence.

$$f_1(x) = x^{q-1} \quad \text{for all } x \geq 0, \quad \text{with } 1 < q < p < +\infty,$$
$$g_1(x) = \begin{cases} \hat{\eta}(2x^{r-1} - x^{p-1}) & \text{if } x \in [0,1] \\ \hat{\eta}x^{p-1} & \text{if } x > 1 \end{cases}$$

with $\hat{\eta} > \hat{\lambda}_1, r > p$;

$$f_2(x) = \begin{cases} x^{q-1} & \text{if } x \in [0,1] \\ \dfrac{x^{p-1}}{\ln(1+x)} + \dfrac{\ln 2 - 1}{\ln 2} & \text{if } x > 1 \end{cases}$$
$$g_2(x) = \begin{cases} c(2x^{r-1} - x^{p-1}) & \text{if } x \in [0,1] \\ \hat{\eta}(x^{p-1} - x^{\tau-1}) & \text{if } x > 1 \end{cases}$$

with $c > 0, \eta > \hat{\eta}_1, \tau < p < r$.

3 Positive solutions of problem (1.1)

We introduce the following two sets

$$\mathscr{L} = \{\lambda > 0 \, : \, \text{problem (1.1) admits a positive solution}\},$$
$$S(\lambda) = \{\text{positive solutions of problem (1.1)}\}.$$

Also, we define

$$\lambda^* = \sup \mathscr{L}.$$

Proposition 3.1 *If hypotheses* (ξ), (β), *(f1)–(f4), (g1)–(g4) and* (2.8) *hold, then* $\mathscr{L} \neq \emptyset$ *and, for every* $\lambda \in \mathscr{L}$, $S(\lambda) \subseteq D_+$.

Proof We leave at the end the proof that $\mathscr{L} \neq \emptyset$ and we start proving the second part of the statement.

Let $\lambda \in \mathscr{L}$. Then we can find $u \in S(\lambda)$ such that

$$\begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = \lambda f(z, u(z)) + g(z, u(z)) & \text{for a.a. } x \in \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

(see Papageorgiou–Radulescu [16]).

Form (3.1) and Proposition 7 of Papageorgiou–Radulescu [17], we deduce

$$u \in L^\infty(\Omega).$$

Let $\rho = \|u\|_{L^\infty}$, $B = \{\lambda\}$ and let $\hat{\xi}_\rho^B$ be as postulated by hypothesis (2.8).

From (3.1) and hypothesis (2.8), for a.a. $z \in \Omega$, we have

$$\begin{aligned} \Delta_p u(z) &\leq \left[\|\xi\|_{L^\infty} + \hat{\xi}_\rho^B\right] u(z)^{p-1} \\ &\Rightarrow u \in D_+ \quad \text{(see [6], p. 738)} \\ &\Rightarrow S(\lambda) \subseteq D_+ . \end{aligned}$$

Next we show that $\mathscr{L} \neq \emptyset$.

Let $F(z,x) = \int_0^x f(z,s)\,ds$, $G(z,s) = \int_0^x g(z,s)\,ds$, $\mu > \|\xi\|_{L^\infty}$ and consider the C^1-functional

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u^-\|_{L^p}^p - \int_\Omega \lambda F(z,u)\,dz - \int_\Omega G(z,u)\,dz, \quad \forall u \in W^{1,p}(\Omega).$$

On account of hypotheses (f2), (f3) and (f4), we see we can find a constant $c_5 > 0$ such that

$$F(z,x) \leq c_5 x^q + x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.2}$$

Also, from hypothesis (g4) we have

$$G(z,x) \leq \frac{c_3}{r}x^r - \frac{c_4}{p}x^p \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geq 0. \tag{3.3}$$

Recall that we can take $c_4 > \|\xi\|_{L^\infty}$. We have

$$\begin{aligned} \hat{\varphi}_\lambda(u) \geq\ & c_6\|u^-\|^p + \frac{1}{p}\left[\gamma_p(u^+) + c_4\|u^+\|_{L^p}^p\right] - \lambda\|u^+\|^p \\ & - c_7\left[\lambda\|u^+\|^q + c_4\|u^+\|^r\right] \end{aligned} \tag{3.4}$$

for some $c_6, c_7 > 0$ (see (2.6), (2.7), (3.2), (3.3) and recall that $\mu > \|\xi\|_{L^\infty}$)

Since $c_4 > \|\xi\|_{L^\infty}$, choosing $\lambda > 0$ sufficiently small, we have

$$c_6\|u^-\|^p + \frac{1}{p}\left[\gamma_p(u^+) + c_4\|u^+\|_{L^p}^p\right] - \lambda\|u^+\|^p \geq c_8\|u\|^p \tag{3.5}$$

for some $c_8 > 0$.

Merging (3.5) in (3.4), we obtain for $\lambda > 0$ sufficiently small

$$\begin{aligned} \hat{\varphi}_\lambda(u) &\geq c_8\|u\|^p - c_7\left[\lambda\|u\|^q + c_4\|u\|^r\right] \\ &= \left[c_8 - c_7\left(\lambda\|u\|^{q-p} + c_4\|u\|^{r-p}\right)\right]\|u\|^p \end{aligned} \tag{3.6}$$

for all $u \in W^{1,p}(\Omega)$.

We now set

$$\theta_\lambda(t) = \lambda t^{q-p} + t^{r-p} \quad \text{for all } t \geq 0$$

and we observe that we can find $t_0 > 0$ such that

$$\begin{aligned}\theta_\lambda(t_0) = \inf_{t\geq 0}\theta_\lambda(t) &\Rightarrow \theta'_\lambda(t_0) = 0\\ &\Rightarrow \lambda(p-q)t_0^{q-p-1} = (r-p)t_0^{q-p-1}\\ &\Rightarrow t_0 = \left[\frac{\lambda(p-q)}{r-p}\right]^{\frac{1}{r-q}}.\end{aligned}$$

It follows that

$$\theta_\lambda(t_0) \to 0^+ \quad \text{as } \lambda \to 0^+.$$

So, we can find $\lambda_0 > 0$ small enough such that

$$\begin{aligned}&c_8 > c_7\theta_\lambda(t_0) \quad \text{for all } \lambda \in]0, \lambda_0[\\ &\Rightarrow \inf\left\{\hat{\varphi}_\lambda(u) : \|u\| = \rho_\lambda = t_0(\lambda)\right\} = \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0)\end{aligned} \tag{3.7}$$

for all $\lambda \in]0, \lambda_0[$ [see (3.6)].

Hypotheses (f3) and (g3) imply that

$$\hat{\varphi}_\lambda(t\hat{u}_1) \to -\infty \quad \text{as } t \to +\infty \text{ (recall } \hat{\eta} > \hat{\lambda}_1). \tag{3.8}$$

We now claim that the functional $\hat{\varphi}_\lambda$ satisfies the C–condition.

We consider a sequence $\{u_n\} \subset W^{1,p}(\Omega)$ such that $\{\hat{\varphi}_\lambda(u_n)\} \subset \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\hat{\varphi}'_\lambda(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \text{ as } n \to +\infty.$$

So, we have

$$\begin{aligned}&\Bigg| < A(u_n), h > + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\, dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\, d\sigma\\ &\quad - \int_\Omega \mu(u_n^-)^{p-1} h\, dz - \int_\Omega [\lambda f(z, u_n) + g(z, u_n)]\, h\, dz\Bigg|\\ &\leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|}\end{aligned} \tag{3.9}$$

for all $h \in W^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$.

In (3.9) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\gamma_p(u_n^-) + \mu\|u_n^-\|_{L^p}^p \leq \varepsilon_n \quad \text{[see(2.6)and(2.7)]}\\ &\Rightarrow c_9\|u_n^-\|^p \leq \varepsilon_n \quad \text{for some constant } c_9 > 0, \text{ all } n \in \mathbb{N} \text{ (recall that } \mu > \|\xi\|_{L^\infty})\\ &\Rightarrow u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega).\end{aligned} \tag{3.10}$$

We plug (3.10) in (3.9). Then

$$\begin{aligned} &\Big| < A(u_n^+), h > + \int_\Omega \xi(z)(u_n^+)^{p-1} h\,dz + \int_{\partial\Omega} \beta(z)(u_n^+)^{p-1} h\,d\sigma \\ &\quad - \int_\Omega \big[\lambda f(z, u_n^+) + g(z, u_n^+)\big] h\,dz \Big| \\ &\le \varepsilon_n' \|h\| \end{aligned} \tag{3.11}$$

for all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n' \to 0^+$.

We show that $\{u_n^+\} \subseteq W^{1,p}(\Omega)$ is bounded.

Arguing indirectly, suppose that, at least for a subsequence, we have

$$\|u_n^+\| \to +\infty. \tag{3.12}$$

Let $y_n = \dfrac{u_n^+}{\|u_n^+\|}$ for all $n \in \mathbb{N}$. We have $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \text{ and} y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega),\ y \ge 0. \tag{3.13}$$

From (3.11), we obtain

$$\begin{aligned} &\Big| < A(y_n), h > + \int_\Omega \xi(z) y_n^{p-1} h\,dz + \int_{\partial\Omega} \beta(z) y_n^{p-1} h\,d\sigma \\ &\quad - \int_\Omega \frac{\big[\lambda N_f(u_n^+) + N_g(u_n^+)\big]}{\|u_n^+\|^{p-1}} h\,dz \Big| \\ &\le \frac{\varepsilon_n' \|h\|}{\|u_n^+\|^{p-1}} \end{aligned} \tag{3.14}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

Hypotheses (f2) and (f3) imply that

$$\left\{ \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \right\} \subseteq L^{p'}(\Omega) \quad \text{is bounded } (1/p + 1/p' = 1). \tag{3.15}$$

Similarly hypothesis (g2) and (3.12) imply that

$$\left\{ \frac{N_g(u_n^+)}{\|u_n^+\|^{p-1}} \right\} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \tag{3.16}$$

So, if in (3.14) we choose $h = y_n - y \in W^{1,p}(\Omega)$, we pass to the limit as $n \to +\infty$ and use (3.13), (3.15) and (3.16), then we obtain

$$\begin{aligned}&\lim_{n\to+\infty} < A(y_n), y_n - y >= 0\\ &\Rightarrow y_n \to y \text{ in } W^{1,p}(\Omega)(\text{see Proposition 2.2})\\ &\Rightarrow \|y\| = 1,\ y \ge 0\,.\end{aligned} \tag{3.17}$$

From (3.15), (3.16) and by passing to a subsequence if necessary, we deduce

$$\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} 0 \quad \text{in } L^{p'}(\Omega) \text{ (see hypotheses (f1) and (f3))} \tag{3.18}$$

$$\frac{N_g(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \eta_0(z)y^{p-1} \quad \text{in } L^{p'}(\Omega) \tag{3.19}$$

with $\hat{\eta} \le \eta_0(z) \le c_{10}$ for a.a. $z \in \Omega$, some constant $c_{10} > 0$ (see hypotheses (g1), (g3) and [1], proof of Proposition 16).

So, if in (3.14) we pass to the limit as $n \to +\infty$ and we use (3.17), (3.18) and (3.19) then we obtain

$$\begin{aligned}&< A(y), h > + \int_\Omega \xi(z)y^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)y^{p-1}h\,d\sigma\\ &= \int_\Omega \eta_0(z)y^{p-1}h\,dz,\ \ \forall h \in W^{1,p}(\Omega)\\ &\Rightarrow \begin{cases} -\Delta_p y(z) + \xi(z)y(z)^{p-1} = \eta_0(z)y(z)^{p-1} & \text{for a.a.} z \in \Omega\\ \dfrac{\partial y}{\partial n_p} + \beta(z)y^{p-1} = 0 & \text{on } \partial\Omega\end{cases}\end{aligned} \tag{3.20}$$

(see [16]).

From Lemma 2.6 we know that

$$\begin{aligned}&\tilde{\lambda}_1(\eta_0) \le \tilde{\lambda}_1(\hat{\eta}) < \tilde{\lambda}_1(\hat{\lambda}_1) = 1 \quad [\text{see}(3.19)]\\ &\Rightarrow y \text{ must be nodal or zero } [\text{see}(3.20)].\end{aligned}$$

This contradicts (3.17). Hence we have proved that

$$\begin{aligned}&\{u_n^+\} \subseteq W^{1,p}(\Omega) \text{ is bounded}\\ &\Rightarrow \{u_n\} \subseteq W^{1,p}(\Omega) \text{ is bounded } [\text{see}(3.10)].\end{aligned}$$

So we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.21}$$

We return to (3.9), we choose $h = u_n - u \in W^{1,p}(\Omega)$, we pass to the limit as $n \to +\infty$ and use (3.21). Then we get

$$\begin{aligned}&\lim_{n\to+\infty} < A(u_n), u_n - u >= 0\\ &\Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega)(\text{see Proposition2.2})\\ &\Rightarrow \hat{\varphi}_\lambda \text{ satisfies the C-condition.}\end{aligned}$$

This proves the claim.

Thus, (3.7), (3.8) and the claim permit the use of Theorem 2.1 on $\hat{\varphi}_\lambda$, $\lambda \in]0, \lambda_0[$. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\hat{\varphi}_\lambda} \text{ and } \hat{m}_\lambda \le \hat{\varphi}_\lambda(u_\lambda). \tag{3.22}$$

From (3.22) and (3.7), we have

$$u_\lambda \ne 0 \text{ and } \hat{\varphi}'_\lambda(u_\lambda) = 0. \tag{3.23}$$

Then

$$\begin{aligned}&< A(u_\lambda), h > + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h\, dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h\, d\sigma - \int_\Omega \mu(u_\lambda^-)^{p-1} h\, dz \\ &= \int_\Omega [\lambda f(z, u_\lambda) + g(z, u_\lambda)]\, h\, dz.\end{aligned} \tag{3.24}$$

In (3.24) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\gamma_p(u_\lambda^-) + \mu\|u_\lambda^-\|_{L^p}^p = 0 \quad \text{[see (2.6) and (2.7)]} \\ &\Rightarrow c_{11}\|u_\lambda^-\|^p \le 0 \quad \text{for some constant } c_{11} > 0 \text{(recall that } \mu > \|\xi\|_{L^\infty}) \\ &\Rightarrow u_\lambda \ge 0,\ u_\lambda \ne 0 \quad \text{[see (3.23)]} \\ &\Rightarrow u_\lambda \in S(\lambda) \subseteq D_+ \quad \forall \lambda \in]0, \lambda_0[\\ &\Rightarrow]0, \lambda_0[\subseteq \mathscr{L} \\ &\Rightarrow \mathscr{L} \ne \emptyset.\end{aligned}$$

□

Proposition 3.2 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and* (2.8) *hold, then* $\lambda^* < +\infty$.

Proof On account of hypothesis (g3), we can find $\tilde{\eta} > \lambda_1$ and a constant $M > 0$ such that

$$g(z, x) \ge \tilde{\eta}x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M \tag{3.25}$$

Also by hypothesis (g4) we have

$$g(z, x) \ge -c_2 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta_1]. \tag{3.26}$$

Finally hypotheses (f2) and (f3) imply that we can find a constant $c_{12} > 0$ such that

$$g(z, x) \ge -c_{12} x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [\delta_1, M]. \tag{3.27}$$

Then on account of hypothesis (f4) and since $q < p$, using (3.25), (3.26) and (3.27) we see that for $\bar{\lambda} > 0$ big enough we can have that

$$\bar{\lambda} f(z,x) + g(z,x) \geq \tilde{\eta} x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.28}$$

Let $\lambda > \bar{\lambda}$ and suppose that $\lambda \in \mathscr{L}$.The we can find $u_\lambda \in S(\lambda) \subseteq D_+$ (see Proposition 3.1). We have

$$\begin{aligned} &-\Delta_p u_\lambda(z) + \xi(z) u_\lambda^{p-1} = \lambda f(z, u_\lambda(z)) + g(z, u_\lambda(z)) \\ &\quad > \bar{\lambda} f(z, u_\lambda(z)) + g(z, u_\lambda(z)) \quad (\text{since } \lambda > \bar{\lambda}) \\ &\quad \geq \tilde{\eta} u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega \, [(\text{see } (3.28)]. \end{aligned} \tag{3.29}$$

We consider the Carathéodory function $k_\lambda((z,x)$ defies by

$$k_\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ (\tilde{\eta} + \mu) x^{p-1} & \text{if } 0 \leq x \leq u_\lambda(z) \\ (\tilde{\eta} + \mu) u_\lambda(z)^{p-1} & \text{if } x > u_\lambda(z). \end{cases} \tag{3.30}$$

We set

$$K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\, ds$$

and introduce the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) := \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u\|_{L^p}^p - \int_\Omega K_\lambda(z,u)\, dz, \quad \forall u \in W^{1,p}(\Omega).$$

Using (3.30) and the fact that $\mu > \|\xi\|_{L^\infty}$, we see that

$$\psi_\lambda(\cdot) \quad \text{is coercive.}$$

Also, using the Sobolev embedding theorem and the compactness of the trace map, we infer that

$$\psi_\lambda(\cdot) \quad \text{is sequentially weakly lower semicontinuous.}$$

So, by the Weierstrass–Tonelli theorem, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(\tilde{u}) = \inf\{\psi_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{3.31}$$

We choose $t \in]0,1[$ small enough such that

$$t\hat{u}_1 \in [0, u_\lambda] \quad (\text{recall that } \hat{u}_1 \in D_+).$$

Then we have

$$\begin{aligned} \psi_\lambda(t\hat{u}_1) &= \frac{t^p}{p}[\gamma_p(\tilde{u}_1) - \tilde{\eta}] \quad [\text{see } (3.30)] \\ &= \frac{t^p}{p}[\hat{\lambda}_1 - \tilde{\eta}] < 0 \\ &\Rightarrow \psi_\lambda(\tilde{u}) < 0 = \psi_\lambda(0) \quad [\text{see } (3.31)] \\ &\Rightarrow \tilde{u} \neq 0. \end{aligned}$$

From (3.31) we have

$$\begin{aligned} &\psi'_\lambda(\tilde{u}) = 0 \\ &\quad \Rightarrow < A(\tilde{u}), h > + \int_\Omega [\xi(z) + \mu]|\tilde{u}|^{p-2}\tilde{u}h\,dz + \int_{\partial\Omega} \beta(z)|\tilde{u}|^{p-2}\tilde{u}h\,d\sigma \\ &\quad = \int_\Omega k_\lambda(z,\tilde{u})h\,dz, \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{3.32}$$

In (3.32) we choose $h = \tilde{u}^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\gamma_p(\tilde{u}^-) + \mu\|\tilde{u}^-\|^p_{L^p} = 0 \quad [\text{see}(3.29)] \\ &\quad \Rightarrow \tilde{u} \geq 0,\ \tilde{u} \neq 0 \quad (\text{recall that } \mu > \|\xi\|_{L^\infty}). \end{aligned}$$

Next in (3.32) we choose $h = (\tilde{u} - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &< A(\tilde{u}), (\tilde{u}-u_\lambda)^+ > + \int_\Omega [\xi(z) + \mu]\tilde{u}^{p-1}(\tilde{u}-u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1}(\tilde{u}-u_\lambda)^+\,d\sigma \\ &= \int_\Omega [\tilde{\eta} + \mu]u_\lambda^{p-1}(\tilde{u} - u_\lambda)^+\,dz \quad [\text{see } (3.30)] \\ &\leq \int_\Omega \left[\bar{\lambda} f(z,u_\lambda) + g(z,u_\lambda) + \mu u_\lambda^{p-1}\right](\tilde{u} - u_\lambda)^+\,dz \quad [\text{see } (3.29)] \\ &\leq \int_\Omega \left[\lambda f(z,u_\lambda) + g(z,u_\lambda) + \mu u_\lambda^{p-1}\right](\tilde{u} - u_\lambda)^+\,dz \quad (\text{since } \lambda > \bar{\lambda},\ f \geq 0) \\ &=< A(u_\lambda), (\tilde{u} - u_\lambda)^+ > + \int_\Omega [\xi(z) + \mu]u_\lambda^{p-1}(\tilde{u} - u_\lambda)^+\,dz \\ &+ \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(\tilde{u} - u_\lambda)^+\,d\sigma \quad (\text{since } u_\lambda \in S(\lambda)) \\ &\Rightarrow \tilde{u} \leq u_\lambda \quad (\text{since } \mu > \|\xi\|_{L^\infty} \text{ and using hypothesis}(\beta)). \end{aligned}$$

So we have proved that

$$\tilde{u} \in [0, u_\lambda],\ \tilde{u} \neq 0. \tag{3.33}$$

From (3.30), (3.32) and (3.33), we obtain

$$< A(\tilde{u}), h > + \int_{\Omega} \xi(z)\tilde{u}^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1} h\, d\sigma = \int_{\Omega} \tilde{\eta}\tilde{u}^{p-1} h\, dz, \quad \forall h \in W^{1,p}(\Omega)$$

$$\Rightarrow \begin{cases} -\Delta_p \tilde{u}(z) + \xi(z)\tilde{u}(z)^{p-1} = \tilde{\eta}\tilde{u}(z)^{p-1} & \text{for a.a. } z \in \Omega \\ \dfrac{\partial \tilde{u}}{\partial n_p} + \beta(z)\tilde{u}^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.34}$$

Recall that $\tilde{\eta} > \hat{\lambda}_1$. Then from (3.34) and Lemma 2.6, we infer that $\tilde{u}$ must be nodal, a contradiction to (3.33). Therefore $\lambda \notin \mathscr{L}$ and so

$$\lambda^* = \sup \mathscr{L} \leq \bar{\lambda} < +\infty.$$

□

Proposition 3.3 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and (2.8) hold, $\lambda \in \mathscr{L}$ and $\tau \in]0, \lambda[$ then $\tau \in \mathscr{L}$.*

Proof Since $\lambda \in \mathscr{L}$, we can find $u_\lambda \in S(\lambda) \subseteq D_+$ (see Proposition 3.1).

Let $e_\tau : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$e_\tau(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ \tau f(z, x) + g(z, x) + \mu x^{p-1} & \text{if } 0 \leq x \leq u_\lambda(z) \\ \tau f(z, u_\lambda(z)) + g(z, u_\lambda(z)) + \mu u_\lambda(z)^{p-1} & \text{if } x > u_\lambda(z). \end{cases} \tag{3.35}$$

We set $E_\tau(z, x) = \int_0^x e_\tau(z, s)\, ds$ and consider the C^1-functional $\hat{\psi}_\tau : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\tau(u) := \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u\|_{L^p}^p - \int_{\Omega} E_\tau(z, u)\, dz, \quad \forall u \in W^{1,p}(\Omega).$$

From (3.30) and since $\mu > \|\xi\|_{L^\infty(\Omega)}$, we see that $\hat{\psi}_\tau(\cdot)$ is coercive.

Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\tau \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\tau(u_\tau) = \inf\{\hat{\psi}_\tau(u) : u \in W^{1,p}(\Omega)\}. \tag{3.36}$$

Let $u \in D_+$ and choose $t \in]0, 1[$ small enough such that

$$0 < tu(z) \leq \delta \quad \text{for a.a. } z \in \bar{\Omega}, \tag{3.37}$$

with $\delta = \min\{\delta_0, \delta_1\}$ [see (f4) and (g4)]. Then on account of hypotheses (g2) and (g4), we have

$$\hat{\psi}_\tau(tu) \leq \frac{t^p}{p}[\gamma_p(u) + c_{13}]\|u\|_{L^p}^p - \frac{\tau c_1 t^q}{q}\|u\|_{L^q}^q \tag{3.38}$$

for some constant $c_{13} > \mu > \|\xi\|_{L^\infty}$ [see (3.37)].

Since $q < p$, if in (3.38) we choose $t \in]0, 1[$ even smaller, we deduce that

$$\begin{aligned} &\hat{\psi}_\tau(tu) < 0 \\ &\quad \Rightarrow \hat{\psi}_\tau(u_\tau) < 0 = \hat{\psi}_\tau(0) \quad \text{[see (3.36)]} \\ &\quad \Rightarrow u_\tau \neq 0. \end{aligned}$$

From (3.36) we have

$$\begin{aligned} &\hat{\psi}'_\tau(u_\tau) = 0 \\ &\quad \Rightarrow < A(u_\tau), h > + \int_\Omega [\xi(z) + \mu]|u_\tau|^{p-2}u_\tau h\, dz + \int_{\partial\Omega} \beta(z)|u_\tau|^{p-2}u_\tau h\, d\sigma \\ &\quad = \int_\Omega e_\tau(z, u_\tau)h\, dz, \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{3.39}$$

As before, choosing in (3.39) first $h = -u_\tau^- \in W^{1,p}(\Omega)$ and then $h = (u_\tau - u_\lambda)^+ \in W^{1,p}(\Omega)$, we show that

$$u_\tau \in [0, u_\lambda],\ u_\tau \neq 0. \tag{3.40}$$

From (3.35), (3.39) and (3.40) we conclude that

$$u_\tau \in S(\tau) \subseteq D_+ \Rightarrow \tau \in \mathscr{L}.$$

□

An interesting byproduct of this proof is the following corollary.

Corollary 1 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and (2.8) hold, $\lambda \in \mathscr{L}$, $\tau \in]0, \lambda[$ and $u_\lambda \in S(\lambda)$, then $\tau \in \mathscr{L}$ and there exists $u_\tau \in S(\tau)$ such that*

$$u_\lambda - u_\tau \in C_+ \backslash \{0\}.$$

In fact, using Proposition 2.5, we can improve the conclusion of this corollary. The following stronger version will be used in the analysis of the minimal solution map which we conduct later.

Proposition 3.4 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and (2.8) hold, $\lambda \in \mathscr{L}$, $\tau \in]0, \lambda[$ and $u_\lambda \in S(\lambda)$, then $\tau \in \mathscr{L}$ and there exists $u_\tau \in S(\tau)$ such that*

$$u_\lambda - u_\tau \in int\hat{C}_+.$$

Proof From Corollary 1 we already know that $\tau \in \mathscr{L}$ and that we can find $u_\tau \in S(\tau)$ such that

$$u_\lambda - u_\tau \in C_+ \backslash \{0\}.$$

Let $\rho = \|u_\lambda\|_{L^\infty}$, $B = [\tau, \lambda]$ and let $\hat{\xi}_\rho^B > 0$ be as postulated by hypothesis (2.8). We have

$$\begin{aligned} &-\Delta_p u_\tau + [\xi(z) + \hat{\xi}_\rho^B] u_\tau^{p-1} \\ &\quad = \tau f(z, u_\tau) + g(z, \tau) + \hat{\xi}_\rho^B u_\tau^{p-1} \\ &\quad = \lambda f(z, u_\tau) + g(z, u_\tau) + \hat{\xi}_\rho^B u_\tau^{p-1} - (\lambda - \tau) f(z, u_\tau). \end{aligned} \tag{3.41}$$

Since $u_\tau \in D_+$, $m_\tau = \min_{\bar{\Omega}} u_\tau > 0$ and then from hypothesis (g4) we have

$$0 < \tilde{\eta}_\tau = c_1 m_\tau^{q-1} \leq f(z, u_\tau(z)) \quad \text{foe a.a } z \in \Omega.$$

Therefore

$$\begin{aligned} &\lambda f(z, u_\tau) + g(z, u_\tau) + \hat{\xi}_\rho^B u_\tau^{p-1} - (\lambda - \tau) f(z, u_\tau) \\ &\quad \leq \lambda f(z, u_\tau) + g(z, u_\tau) + \hat{\xi}_\rho^B u_\tau^{p-1} - (\lambda - \tau)\tilde{\eta}_\tau \\ &\quad \leq \lambda f(z, u_\lambda) + g(z, u_\lambda) + \hat{\xi}_\rho^B u_\lambda^{p-1} \quad [\text{since } u_\tau \leq u_\lambda, \text{ see hypothesis} (2.8)] \\ &\quad = -\Delta_p u_\lambda + [\xi(z) + \hat{\xi}_\rho^B] u_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{3.42}$$

Since $(\lambda - \tau)\tilde{\eta}_{m_\tau} > 0$, from (3.41), (3.42) and Proposition 2.5, we deduce that

$$u_\lambda - u_\tau \in int\hat{C}_+.$$

□

Next we show that for all $\lambda \in]0, \lambda^*[$ problem (1.1) has at least two positive solutions.

Proposition 3.5 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and* (2.8) *hold,* $\lambda \in]0, \lambda^*[$, *then problem* (1.1) *has at least two positive solutions*

$$u_0, \hat{u} \in D_+, \; u_0 \neq \hat{u}.$$

Proof Let $0 < \tau < \lambda < \theta < \lambda^*$. We know that $\tau, \theta \in \mathscr{L}$ (see Proposition 3.3). According to Proposition 3.4, we can find $u_\theta \in S(\theta) \subseteq D_+$ and $u_\tau \in S(\tau) \subseteq D_+$ such that $u_\theta - u_\tau \in intC_+$. Using these two solutions we introduce the Carathéodory function $l_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ define by

$$l_\lambda(z, x) = \begin{cases} \lambda f(z, u_\tau(z)) + g(z, u_\tau(z)) + \mu u_\tau(z)^{p-1} & \text{if } x < u_\tau(z) \\ \lambda f(z, x) + g(z, x) + \mu x^{p-1} & \text{if } u_\tau(z) \leq x \leq u_\theta(z) \\ \lambda f(z, u_\theta(z)) + g(z, u_\theta(z)) + \mu u_\theta(z)^{p-1} & \text{if } x > u_\theta(z). \end{cases} \tag{3.43}$$

As always $\mu > \|\xi\|_{L^\infty}$. We set $L_\lambda(z, x) = \displaystyle\int_0^x l_\lambda(z, s)\, ds$ and consider the C^1-functional $\tilde{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda(u) := \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u\|_{L^p}^p - \int_\Omega L_\lambda(z, u)\, dz, \quad \forall u \in W^{1,p}(\Omega).$$

This functional is coercive [see (3.43)] and sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\tilde{\varphi}_\lambda(u_0) = \inf\{\tilde{\psi}_\lambda(u) : u \in W^{1,p}(\Omega)\} \\ &\quad \Rightarrow \tilde{\varphi}'_\lambda(u_0) = 0 \qquad (3.44) \\ &\Rightarrow < A(u_0), h > + \int_\Omega [\xi(z) + \mu]|u_0|^{p-2}u_0 h\, dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h\, d\sigma \\ &\quad = \int_\Omega l_\lambda(z, u_0)h\, dz, \quad \forall\, h \in W^{1,p}(\Omega). \qquad (3.45) \end{aligned}$$

In (3.45) we choose $h = (u_0 - u_\theta)^+ \in W^{1,p}(\Omega)$ and $h = (u_\tau - u_0)^+ \in W^{1,p}(\Omega)$, and using (3.43) we show that

$$u_0 \in [u_\tau, u_\theta] \cap D_+,\ u_0 \in S(\lambda) \quad \text{[see (3.43)]}.$$

In fact, as in the proof of Proposition 3.4, exploiting Proposition 2.5, we obtain

$$u_0 \in int_{C^1(\Omega)}[u_\tau, u_\theta],\ u_0 \in S(\lambda) \tag{3.46}$$

We consider the following Carathéodory function

$$r_\lambda(z, x) = \begin{cases} \lambda f(z, u_\tau(z)) + g(z, u_\tau(z)) + \mu u_\tau(z)^{p-1} & \text{if } x \le u_\tau(z) \\ \lambda f(z, x) + g(z, x) + \mu x^{p-1} & \text{if } x > u_\tau(z). \end{cases} \tag{3.47}$$

We set $R_\lambda(z, x) = \displaystyle\int_0^x r_\lambda(z, s)\, ds$ and consider the C^1-functional $\tilde{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\psi}_\lambda(u) := \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u\|^p_{L^p} - \int_\Omega R_\lambda(z, u)\, dz, \quad \forall\, u \in W^{1,p}(\Omega).$$

From (3.43) and (3.47) we see that

$$\begin{aligned} &\tilde{\varphi}_\lambda|_{[u_\tau, u_\theta]} = \tilde{\psi}_\lambda\Big|_{[u_\tau, u_\theta]} \\ &\quad \Rightarrow u_0 \ \text{ is a local } C^1\text{-minimizer of } \tilde{\psi}_\lambda \text{[see (3.44) and (3.46)]} \\ &\quad \Rightarrow u_0 \ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of} \tilde{\psi}_\lambda \text{(see Proposition 2.4)}. \end{aligned} \tag{3.48}$$

Using (3.47) we can show that

$$K_{\tilde{\psi}_\lambda} \subseteq [u_\tau[\cap D_+. \tag{3.49}$$

On account of (3.47) and (3.49), we see that we may assume that $K_{\tilde{\psi}_\lambda}$ is finite. Otherwise we already have an infinity of positive solutions for problem (1.1).

The (3.48) implies that we can find $\rho \in]0, 1[$ small enough such that

$$\tilde{\psi}_\lambda(u_0) < \inf\{\tilde{\psi}_\lambda(u) : \|u - u_0\| = \rho\} = \tilde{m}_\lambda \quad \text{(see [1])} \tag{3.50}$$

Hypotheses (f3) and (g3) imply that

$$\tilde{\psi}_\lambda(t\hat{u}) \to -\infty \ \text{ as } t \to +\infty. \tag{3.51}$$

Moreover, as in the proof of Proposition 3.1, we show that

$$\tilde{\psi}_\lambda(\cdot) \ \text{ satisfies the C-condition.} \tag{3.52}$$

Then (3.50), (3.51) and (3.52) permit the use of Theorem 2.1. So we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\tilde{\psi}_\lambda} \ \text{ and } \tilde{m}_\lambda \leq \tilde{\psi}_\lambda(\hat{u}) \text{ [see (3.50)]}. \tag{3.53}$$

From (3.49), (3.50) and (3.53) we conclude that

$$\hat{u} \in S(\lambda) \subseteq D_+ \ \text{ and } \hat{u} \neq u_0.$$

□

Next we show that the critical parameter value λ^* is admissible and hence $\mathscr{L} = [0, \lambda^*]$.

Proposition 3.6 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and (2.8) hold, then $\lambda^* \in \mathscr{L}$.*

Proof Consider a sequence $\{\lambda_n\} \subseteq]0, \lambda^*[$ such that $\lambda_n \to (\lambda^*)^-$. Let $u_n \in S(\lambda_n) \subseteq D_+$, $\forall n \in \mathbb{N}$. From the proof of Proposition 3.5, we see that we can have that the sequence $\{u_n\}$ is increasing. Thus we get

$$\begin{aligned} &< A(u_n), h > + \int_\Omega \xi(z) u_n^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, d\sigma = \int_\Omega [\lambda_n f(z, u_n) \\ &+ g(z, u_n)] h\, dz, \end{aligned} \tag{3.54}$$

$\forall h \in W^{1,p}(\Omega), \ \forall n \in \mathbb{N}$.

Reasoning as in the claim in the proof of Proposition 3.1, we show that

$$\{u_n\} \subseteq W^{1,p}(\Omega) \ \text{ is bounded.}$$

So we may assume that

$$u_n \xrightarrow{w} u_* \ \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.55}$$

In (3.54) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, we pass to the limit as $n \to +\infty$ and we use (3.55). Then

$$\begin{aligned}&\lim_{n\to} < A(u_n), u_n - u_* >= 0\\ &\quad \Rightarrow u_n \to u_* \text{ in } W^{1,p}(\Omega), u_* \neq 0 (\text{since } u_1 \leq u_n \text{ for all } n \in \mathbb{N}).\end{aligned} \tag{3.56}$$

Passing to the limit as $n \to +\infty$ in (3.54) and using (3.56), we conclude that

$$u_* \in S(\lambda^*) \subseteq D_+ \Rightarrow \lambda^* \in \mathscr{L}.$$

□

Now we turn our attention to the existence of minimal positive solutions (that is, a function $\bar{u}_\lambda \in S(\lambda) \subseteq D_+$ such that $\bar{u}_\lambda \leq u$ for all $u \in S(\lambda)$). After establishing the existence of such a minimal positive solution $\bar{u}_\lambda$, we will examine the monotonicity and continuity properties of the map $\lambda \to \bar{u}_\lambda$.

Hypotheses (f1)–(f4) and (g1)–(g4) imply that we can find constants $c_{14} > 0$ and $c_{15} > \|\xi\|_{L^\infty}$ such that

$$\lambda f(z,x) + g(z,x) \geq \lambda c_{14} x^{q-1} - c_{15} x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{all } \lambda \in \mathscr{L}. \tag{3.57}$$

This unilateral growth restriction on the reaction of (1.1) suggests the following auxiliary Robin problem

$$\begin{cases} -\Delta_p u(z) + \xi(z) u(z)^{p-1} = \lambda c_{14} u(z)^{q-1} - c_{15} u(z)^{p-1}, \quad u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.58}$$

Proposition 3.7 *If hypotheses (ξ), (β) hold and $\lambda > 0$, then problem* (3.58) *admits a unique positive solution $\tilde{u}_\lambda \in D_+$.*

Proof We consider the C^1-functional $\hat{a}_\lambda : \| \to \mathbb{R}$ defined by

$$\hat{a}_\lambda(u) := \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u^-\|_{L^p}^p + c_{15}\|u^+\|_{L^p}^p - \frac{\lambda c_{14}}{q}\|u^+\|_{L^q}^q, \quad \forall u \in W^{1,p}(\Omega),$$

with $\mu > \|\xi\|_{L^\infty}$.

Since $\mu, c_{15} > \|\xi\|_{L^\infty}$, we see that $\hat{a}_\lambda(\cdot)$ is coercive. Also, it is sequentially lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{a}_\lambda(\tilde{u}_\lambda) = \inf\{\hat{a}_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{3.59}$$

Since $q < p$, for $u \in D_+$ and $t \in]0,1[$ small enough, we will have $\hat{a}_\lambda(tu) < 0$, hence

$$\begin{aligned}&\hat{a}_\lambda(\tilde{u}_\lambda) < 0 \quad [\text{see } (3.59)]\\ &\Rightarrow \tilde{u}_\lambda \neq 0.\end{aligned}$$

From (3.59) we have

$$\begin{aligned}
&\hat{a}_\lambda(\tilde{u}_\lambda) = 0 \\
&\quad \Rightarrow < A(\tilde{u}_\lambda), h > + \int_\Omega \xi(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h\,d\sigma \\
&\qquad -\mu \int_\Omega (\tilde{u}_\lambda^-)^{p-1} h\,dz = \int_\Omega [\lambda c_{14}(\tilde{u}_\lambda^+)^{q-1} - c_{15}(\tilde{u}_\lambda^+)^{p-1}] h\,dz, \quad \forall h \in W^{1,p}(\Omega).
\end{aligned} \tag{3.60}$$

In (3.60) we choose $h = -\tilde{u}_\lambda^- - \varepsilon \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned}
&\gamma_p(\tilde{u}_\lambda^-) + \mu\|\tilde{u}_\lambda^-\|_{L^p}^p = 0 \\
&\quad \Rightarrow c_{16}\|\tilde{u}_\lambda^-\|^p \le 0 \quad \text{for some constant } c_{16} > 0 \\
&\quad \Rightarrow \tilde{u}_\lambda \ge 0,\ \tilde{u}_\lambda \ne 0.
\end{aligned}$$

Then from (3.60), the nonlinear regularity theory and the strong maximum principle , we infer that $\tilde{u}_\lambda \in D_+$ is a positive solution of (3.58).

Next we show the uniqueness of the positive solution of (3.58).

So, we suppose that $\tilde{v}_\lambda$ is another positive solution of (3.58). Again, we have that $\tilde{v}_\lambda \in D_+$. Let $t > 0$ be the biggest positive real number such that

$$t\tilde{v}_\lambda \le \tilde{u}_\lambda. \tag{3.61}$$

Suppose that $t < 1$ and let $c_{17} > c_{15} > \|\xi\|_{L^\infty}$. We have

$$\begin{aligned}
&-\Delta_p(t\tilde{v}_\lambda) + [\xi(z) + c_{17}](t\tilde{v}_\lambda)^{p-1} \\
&\quad = t^{p-1}\left[-\Delta_p\tilde{v}_\lambda + (\xi(z) + c_{17})\tilde{v}_\lambda^{p-1}\right] \\
&\quad = t^{p-1}\left[\lambda c_{14}\tilde{v}_\lambda^{q-1} + (c_{17} - c_{15})\tilde{v}_\lambda^{p-1}\right] \\
&\quad < \lambda c_{14}(t\tilde{v}_\lambda)^{q-1} + (c_{17} - c_{15})(t\tilde{v}_\lambda)^{p-1} \quad (\text{since } t < 1,\ q < p) \\
&\quad \le \lambda c_{14}\tilde{u}_\lambda^{q-1} + (c_{17} - c_{15})\tilde{u}_\lambda^{p-1} \quad (\text{see (3.61) and recall that } c_{17} > c_{15}) \\
&\quad = -\Delta_p\tilde{u}_\lambda + [\xi(z) + c_{17}]\tilde{u}_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega.
\end{aligned} \tag{3.62}$$

Since $\tilde{v}_\lambda \in D_+$, from (3.62) and Proposition 2.5 it follows that

$$\tilde{u}_\lambda - t\tilde{v}_\lambda \in int\hat{C}_+ \quad [\text{see (3.61)}].$$

This contradicts the maximality of $t > 0$. Therefore $t \ge 0$ and so

$$\tilde{v}_\lambda \le \tilde{u}_\lambda \quad [\text{see (3.61)}].$$

Interchanging the roles of $\tilde{u}_\lambda$ and $\tilde{v}_\lambda$ in the above argument, we obtain

$$\tilde{u}_\lambda \le \tilde{v}_\lambda \Rightarrow \tilde{u}_\lambda = \tilde{v}_\lambda.$$

$\square$

This unique solution $\tilde{u}_\lambda \in D_+$ of problem (3.58) provides a lower bound for the elements of $S(\lambda)$, for all $\lambda \in \mathscr{L} =]0, \lambda^*]$.

Proposition 3.8 *If hypotheses* (ξ), (β), *(f1)–(f4), (g1)–(g4) and* (2.8) *hold and* $\lambda \in \mathscr{L}$, *then* $\tilde{u} \leq u$ *for all* $u \in S(\lambda)$.

Proof Let $u \in S(\lambda) \subseteq D_+$. We introduce the following Carathéodory function

$$\hat{\eta}_\lambda(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda c_{14} x^{q-1} + (\mu - c_{15}) x^{p-1} & \text{if } 0 \leq x \leq u(z) \\ \lambda c_{14} u(z)^{q-1} + (\mu - c_{15}) u(z)^{p-1} & \text{if } x > u(z). \end{cases} \tag{3.63}$$

We set $\hat{H}_\lambda(z, x) = \displaystyle\int_0^x \hat{\eta}_\lambda(z, s)\, ds$ and consider the C^1-functional $\hat{\tau}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defund by

$$\hat{\tau}_\lambda(u) := \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}\|u^-\|_{L^p}^p - \int_\Omega \hat{H}_\lambda(z, u)\, dz, \quad \forall u \in W^{1,p}(\Omega).$$

The functional $\hat{\tau}_\lambda(\cdot)$ is coercive [see (3.63)] and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\tau}_\lambda(\tilde{u}_0) = \inf\{\hat{\tau}_\lambda(u) : u \in W^{1,p}(\Omega)\} < 0 = \hat{\tau}_\lambda(0) \quad (\text{since } q < p) \\ &\Rightarrow \tilde{u}_0 \neq 0 \ \text{ and } \hat{\tau}'_\lambda(\tilde{u}_0) = 0. \end{aligned} \tag{3.64}$$

From the equality in (3.64), we have

$$\begin{aligned} &< A(\tilde{u}_0), h > + \int_\Omega [\xi(z) + \mu]|\tilde{u}_0|^{p-2}\tilde{u}_0 h\, dz + \int_{\partial\Omega} \beta(z)|\tilde{u}_0|^{p-2}\tilde{u}_0 h\, d\sigma \\ &= \int_\Omega \hat{\eta}_\lambda(z, \tilde{u}_0) h\, dz, \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{3.65}$$

In (3.65) first we choose $h = -\tilde{u}_0^- \in W^{1,p}(\Omega)$ then we have

$$\begin{aligned} &\gamma_p(\tilde{u}_0) + \mu\|\tilde{u}_0^-\|_{L^p}^p = 0 \quad [\text{see (3.63)}] \\ &\Rightarrow \tilde{u}_0 \geq 0,\ \tilde{u}_0 \neq 0. \end{aligned}$$

Also in (3.65) we choose $h = (\tilde{u}_0 - u)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} &< A(\tilde{u}_0), (\tilde{u}_0 - u)^+ > + \int_\Omega [\xi(z) + \mu]\tilde{u}_0^{p-1}(\tilde{u}_0 - u)^+\, dz \\ &\quad + \int_{\partial\Omega} \beta(z)\tilde{u}_0^{p-1}(\tilde{u}_0 - u)^+\, d\sigma \\ &= \int_\Omega [\lambda c_{14} u^{q-1} + (\mu - c_{15} u^{p-1})](\tilde{u}_0 - u)^+\, dz \quad [\text{see (3.63)}] \\ &\leq \int_\Omega [\lambda f(z, u) + g(z, u) + \mu u^{p-1})](\tilde{u}_0 - u)^+\, dz \quad [\text{see (3.57)}] \end{aligned}$$

$$
\begin{aligned}
&=< A(u), (\tilde{u}_0 - u)^+ > + \int_\Omega [\xi(z) + \mu] u^{p-1} (\tilde{u}_0 - u)^+ \, dz \\
&\quad + \int_{\partial\Omega} \beta(z) u^{p-1} (\tilde{u}_0 - u)^+ \, d\sigma \\
&\Rightarrow \tilde{u}_0 \le u \quad (\text{since } \mu > \|\xi\|_{L^\infty}).
\end{aligned}
$$

So, we have proved that

$$
\begin{aligned}
&\tilde{u}_0 \in [0, u],\ \tilde{u}_0 \ne 0 \\
&\quad \Rightarrow \tilde{u}_0 = \tilde{u}_\lambda \quad (\text{see (3.63), (3.65) and Proposition 3.7}) \\
&\quad \Rightarrow \tilde{u}_\lambda \le u \quad \text{for all } u \in S(\lambda).
\end{aligned}
$$

□

From Papageorgiou–Radulescu–Repovs [18] (see the proof of Proposition 7 in [18]), we have that the solution set $S(\lambda)$ is downward directed, that is, if $u, \hat{u} \in S(\lambda)$, then we can find $y \in S(\lambda)$ such that $y \le u$, $y \le \hat{u}$. Using this fact, we can show that $S(\lambda)$ admits a minimal element.

Proposition 3.9 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and (2.8) hold and $\lambda \in \mathscr{L}$, then problem (1.1) admits a smallest positive solution $\bar{u}_\lambda \in S(\lambda) \subseteq D_+$ (that is, $\bar{u}_\lambda \le u$ for all $u \in S(\lambda)$).*

Proof On account of Lemma 3.10, p. 178 of Hu–Papageorgiou [9], we can find $\{u_n\} \subseteq S(\lambda)$ decreasing such that

$$
\inf S(\lambda) = \inf_{n\in\mathbb{N}} u_n.
$$

We then have

$$
\begin{aligned}
&< A(u_n), h > + \int_\Omega \xi(z) u_n^{p-1} h \, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h \, d\sigma \\
&\quad = \int_\Omega [\lambda f(z, u_n) + g(z, u_n)] h \, dz,
\end{aligned} \tag{3.66}
$$

$\forall h \in W^{1,p}(\Omega)$, $\forall n \in \mathbb{N}$.

Since $0 \le u_n \le u_1$, for all $n \in \mathbb{N}$, if in (3.66) we choose $h = u_n \in W^{1,p}(\Omega)$, then we see that

$$
\{u_n\} \subseteq W^{1,p}(\Omega) \ \text{ is bounded.}
$$

So, we may assume that

$$
u_n \xrightarrow{w} \bar{u}_\lambda \ \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to \bar{u}_\lambda \text{ in } L^p(\Omega) \text{and in } L^p(\partial\Omega). \tag{3.67}
$$

In (3.66) we choose $h = u_n - \bar{u}_\lambda \in W^{1,p}(\Omega)$, we pass to the limit as $n \to +\infty$ and use (3.67). We deduce

$$\begin{aligned} &\lim_{n\to+\infty} < A(u_n), u_n - \bar{u}_\lambda >= 0 \\ &\Rightarrow u_n \to \bar{u}_\lambda \quad \text{in } W^{1,p}(\Omega)(\text{see Proposition 2.2}). \end{aligned} \tag{3.68}$$

From Proposition 3.8 we have

$$\begin{aligned} &\tilde{u}_\lambda \le u_n \quad \forall n \in \mathbb{N} \\ &\Rightarrow \tilde{u}_\lambda \le \bar{u}_\lambda. \end{aligned} \tag{3.69}$$

Therefore, if in (3.66) we pass to the limit as $n \to +\infty$ and we use (3.68) and (3.69) then we conclude that

$$\bar{u}_\lambda \in S(\lambda) \subseteq D_+ \text{ and } \bar{u}_\lambda = \inf S(\lambda).$$

□

Next we examine the monotonicity and continuity properties of the map $\lambda \to \bar{u}_\lambda$ from $\mathscr{L} =]0, \lambda^*]$ into $C^1(\bar{\Omega})$.

Proposition 3.10 *If hypotheses (ξ), (β), (f1)–(f4), (g1)–(g4) and (2.8) hold, then the map $\lambda \to \bar{u}_\lambda$ from $\mathscr{L} =]0, \lambda^*]$ into $C^1(\bar{\Omega})$ satisfies*

(a) *it is strictly increasing that is,*

$$0 < \lambda < \theta \le \lambda^* \Rightarrow \bar{u}_\theta - \bar{u}_\lambda \in int\hat{C}_+;$$

(b) *it is left continuous.*

Proof (a) Let $0 < \lambda < \theta \le \lambda^*$. From Proposition 3.4 we know that we can find $u_\lambda \in S(\lambda) \subseteq D_+$ such that

$$\begin{aligned} &\bar{u}_\theta - u_\lambda \in int\hat{C}_+ \\ &\Rightarrow \bar{u}_\theta - \bar{u}_\lambda \in int\hat{C}_+ \quad (\text{since } \bar{u}_\theta \le u_\lambda). \end{aligned}$$

This proves the strictly monotonicity of the map $\lambda \to \bar{u}_\lambda$.
(a) Let $\{\lambda_n\} \subseteq \mathscr{L}$ and assume that $\lambda_n \to \lambda^-$. Evidently $\lambda \in \mathscr{L}$. From (a) we have

$$\begin{aligned} &\bar{u}_{\lambda_n} \le \bar{u}_\lambda \quad \forall n \in \mathbb{N} \\ &\Rightarrow \{\bar{u}_{\lambda_n}\} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.} \end{aligned} \tag{3.70}$$

From (3.62) and Proposition 3.2 of Papageorgiou–Radulescu [17] we know that we can find a constant $c_{18} > 0$ such that

$$\|\bar{u}_{\lambda_n}\|_{L^\infty} \le c_{18}, \quad \forall n \in \mathbb{N}.$$

Then, Theorem 2 of Lieberman [13] implies that we can find $\theta \in]0, 1[$ and a constant $c_{19} > 0$ such that

$$\bar{u}_{\lambda_n} \in C^{1,\theta}(\bar{\Omega}), \ \|\bar{u}_{\lambda_n}\|_{C^{1,\theta}(\bar{\Omega})} \le c_{19}, \quad \forall n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\theta}(\bar{\Omega})$ into $C^1(\bar{\Omega})$ and the monotonicity of $\{\bar{u}_{\lambda_n}\}$ (see (a)), we have

$$\bar{u}_{\lambda_n} \to \tilde{u}_\lambda \quad \text{in } C^1(\bar{\Omega}). \tag{3.71}$$

Suppose that $\tilde{u}_\lambda \neq \bar{u}_\lambda$. Then we can find $z_0 \in \bar{\Omega}$ such that

$$\begin{aligned} &\bar{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0) \\ &\quad \Rightarrow \bar{u}_\lambda(z_0) < \bar{u}_{\lambda_n}(z_0) \quad \text{for all } n > n_0. \end{aligned}$$

This contradicts (a). Therefore $\tilde{u}_\lambda = \bar{u}_\lambda$ and so we can conclude that the map $\lambda \to \bar{u}_\lambda$ is left continuous. □

Remark 3.11 A similar proof can show that the map $\lambda \to \tilde{u}_\lambda$ from $\mathbb{R}$ into $C^1(\bar{\Omega})$ (see Proposition 3.7) is strictly increasing. This fact can be used to provide an alternative proof that $\lambda^* \in \mathscr{L}$ (see Proposition 3.6) .

We can state the following theorem which summarizes the dependence of the set of positive solutions of (1.1) on the parameter λ.

Theorem 3.12 *If hypotheses* (ξ), (β), *(f1)–(f4), (g1)–(g4) and* (2.8) *hold, then there exists* $\lambda^* > 0$ *such that*

(a) *for all* $\lambda \in]0, \lambda^*[$ *problem* (1.1) *has at least two positive solutions*

$$u_0,\ \bar{u} \in D_+,\ u - 0 \neq \bar{u};$$

(b) *for* $\lambda = \lambda^*$ *problem* (1.1) *has at least one positive solution* $u_* \in D_+$*;*
(c) *for all* $\lambda > \lambda^*$ *problem* (1.1) *has no positive solutions;*
(d) *for every* $\lambda \in]0, \lambda^*]$ *problem* (1.1) *has a smallest positive solution* $\bar{u}_\lambda$ *and the map* $\lambda \to \bar{u}_\lambda$ *from* $\mathscr{L} =]0, \lambda^*]$ *into* $C^1(\bar{\Omega})$ *is*

- *strictly increasing*
- *left continuous.*

Remark 3.13 Hypotheses $(f3)$ and $(g3)$ imply that the reaction of (1.1), asymptotically at $+\infty$, is uniformly nonresonant with resect to $\hat{\lambda}_1$. It is an interesting open problem whether Theorem 3.12 above remains valid if we can have resonance with respect to $\hat{\lambda}_1$ or even nonuniform nonresonance with respect to $\hat{\lambda}_1$, that is,

$$\hat{\eta}(z) \leq \lim_{x \to +\infty} \frac{g(z,x)}{x^{p-1}} \quad \text{uniformly for a.a. } z \in \Omega$$

with $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \geq \hat{\lambda}_1$, for a.a. $z \in \Omega$, and the inequality is strict on a set of positive Lebesgue measure.

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References

1. Aizicovici, S., Papageorgiou, N.S., Staicu, V.: Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints, Memoirs of the American Mathematical Society, vol. 196, no. 915 (2018)
2. Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
3. Brezis, H., Nirenberg, L.: H^1 versus C^1 local minimizers. CRAS Paris Ser. I Math. **317**, 465–472 (1993)
4. Candito, P., Livrea, R., Papageorgiou, N.S.: Nonlinear nonhomogeneous Neumann eigenvalue problems. Electron. J. Qual. Theory Differ. Equ. **46**, 1 (2015)
5. Garcia Azorero, J.P., Manfredi, J., Peral Alonso, J.: Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations. Commun. Contemp. Math. **2**, 385–404 (2000)
6. Gasinski, L., Papageorgiou, N.S.: Nonlinear Analysis. Chapman & Hall/CRC, Boca Raton (2006)
7. Gasinski, L., Papageorgiou, N.S.: Exercises in Analysis. Part 2: Nonlinear Analysis. Springer, Berlin (2016)
8. Guo, Z., Zhang, Z.: $W^{1,p}(\Omega)$ local minimizers and multiplicity results for quasilinear elliptic equations. J. Math. Anal. Appl. **286**, 32–50 (2003)
9. Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis, Vol. I: Theory. Kluwer Academic Publishers, Dordrecht (1997)
10. Hu, S., Papageorgiou, N.S.: Multiplicity of solutions for parametric p-Laplacian equations with nonlinearity concave near the origin. Tohoku Math. J. **62**, 137–162 (2010)
11. Leonardi, S.: Morrey estimates for some classes of elliptic equations with a lower order term. Nonlinear Anal. **177**(part B), 611–627 (2018). https://doi.org/10.1016/j.na.2018.05.010
12. Leonardi, S., Papageorgiou, N.S.: Existence and multiplicity of positive solutions for parametric nonlinear nonhomogeneous singular Robin problems, Preprint (2018)
13. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988)
14. Marano, S.A., Papageorgiou, N.S.: Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter. Commun. Pure Appl. Anal. **12**, 815–829 (2013)
15. Mugnai, D., Papageorgiou, N.S.: Resonant nonlinear Neumann problems with indefinite weight. Ann. Scu. Norm. Sup. Pisa (5) **12**, 729–788 (2012)
16. Papageorgiou, N.S., Radulescu, V.D.: Multiple solutions with precise sigh information for nonlinear parametric Robin problem. J. Differ. Equ. **256**, 393–430 (2014)
17. Papageorgiou, N.S., Radulescu, V.D.: Nonlinear nonhomogeneous Robin problem with superlinear reaction term. Adv. Nonlinear Stud. **16**, 737–764 (2016)
18. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.: Positive solutions for perturbations of the Robin eigenvalue problem plus an indefinite potential. Discrete Contin. Dyn. Syst. **37**, 2589–2618 (2017)
19. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.: Positive solutions for nonlinear nonhomogeneous parametric Robin problems. Forum Math. **30**(3), 553–580 (2017). https://doi.org/10.1515/forum-2017-0124
20. Papageorgiou, N.S., Radulescu, V.D., Repovs, D.: Robin problems with degenerate indefinite linear part and competition phenomena. Commun. Pure Appl. Anal. **16**, 1293–1314 (2017)
21. Pucci, P., Serrin, J.: The Maximun Principle. Birkhäuser, Basel (2007)