Research Article

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On a class of critical Robin problems

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Abstract: We consider a nonlinear parametric Robin problem. In the reaction, there are two terms, one critical and the other locally defined. Using cut-off techniques, together with variational tools and critical groups, we show that, for all small values of the parameter, the problem has at least three nontrivial smooth solutions all with sign information, which converge to zero in $C^1(\bar{\Omega})$ as the parameter $\lambda \to 0^+$.

Keywords: Critical term, locally defined reaction, cut-off function, constant sign and nodal solutions, extremal solutions, critical groups

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper, we study the nonlinear parametric Robin problem

$$\begin{cases} -\operatorname{div}(a(Du(z))) + \xi(z)|u(z)|^{p-2}u(z) = \lambda f(z, u(z)) + k(z)|u(z)|^{p^*-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is the parameter, $1 < p < +\infty$ and

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p \end{cases}$$

is the critical Sobolev exponent related to p.

The map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator is continuous and strictly monotone (hence maximal monotone too) and satisfies certain other regularity and growth properties which are listed in hypotheses H(a) below. These conditions on $a(\cdot)$ are general enough to incorporate in our framework many nonlinear differential operators of interest and also permit the use of the nonlinear regularity theory of Lieberman [7]. The operator $u \to \operatorname{div} a((Du))$ is not in general homogeneous and thus complicates the analysis of (1.1). The potential function $\xi(z) \in L^\infty(\Omega)$.

In the reaction (right-hand side of (1.1)), we have two interesting features. One is the presence of the critical term $k(z)|u(z)|^{p^*-2}u(z)$. The second distinguishing feature of the reaction is that the parametric perturbation $\lambda f(z, u(z))$ is only locally defined in $x \in \mathbb{R}$. More precisely, the function $f(z, x)$ is Carathéodory (that is, for all $x \in \mathbb{R}$, the function $z \to f(z, x)$ is measurable, and for a.a. $z \in \Omega$, the function $x \to f(z, x)$ is continuous), and the conditions on $f(z, \cdot)$ concern only its behavior near zero.

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In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative corresponding to the map $a(\cdot)$. This normal derivative is understood through the nonlinear Green identity [2, p. 210], and if $u \in C^1(\bar{\Omega})$, then $\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N}$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(z) \geq 0$ for all $z \in \partial\Omega$, and when $\beta \equiv 0$, we have the usual Neumann problem.

Using cut-off techniques, together with variational methods based on the critical point theory and critical groups, we show that, for all $\lambda > 0$ small, problem (1.1) has at least three nontrivial smooth solutions, all with sign information: a positive solution, a negative solution and a nodal (i.e. sign-changing) solution. Moreover, we show that these solutions converge to zero in $C^1(\bar{\Omega})$ as $\lambda \to 0^+$.

In the past, problems with a reaction which is only locally defined in $\mathbb{R}$ were studied under the assumption that the function exhibits symmetry (i.e., it is odd in the x-variable). Such a hypothesis permits the use of a variant of the symmetric mountain pass theorem. This line of research was initiated with the work of Wang [21], who considered a semilinear Dirichlet problem driven by the Laplacian and a reaction of the form $\lambda|x|^{q-2}x + f(z,x)$, $\lambda > 0$ (a parameter), $1 < q < +\infty$, and $f \in C^0(\bar{\Omega}, [-\theta, \theta])$ is odd and $(q-1)$-sublinear near zero. Later, this work was extended by Li and Wang [8], who considered Schrödinger equations and produced nodal solutions. More recently, Papageorgiou and Rădulescu [13] and Papageorgiou, Rădulescu and Repovs [19] studied nonlinear Robin problems under a symmetry condition in the reaction. None of the aforementioned works had a critical term in the reaction.

For other types of operators with lower-order terms, see also [5, 6, 9, 10, 15–17]

2 Mathematical background. Hypotheses

The main spaces in the analysis of problem (1.1) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\bar{\Omega})$ and the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leq q \leq +\infty)$.

By $\|\cdot\|_{W^{1,p}(\Omega)}$, we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\|_{W^{1,p}(\Omega)} := [\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)}]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\bar{\Omega})$ is ordered by the closed convex cone

$$C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

On $\partial\Omega$, we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq +\infty$. We know that there exists a continuous linear map $\gamma_0\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u_{|\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. We know that

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \quad \text{and} \quad \ker\gamma_0 = W^{1,p}_0(\Omega).$$

Moreover, $\gamma_0(\cdot)$ is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p}[$ if $p < N$, and into $L^q(\partial\Omega)$ for all $q \in [1, +\infty[$ if $p \geq N$.

In what follows, for notational simplicity, we drop the use of trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

The hypotheses on the map $a(\cdot)$ are taken from [12]. So, let $l \in C^1(]0, +\infty[)$ with $l(t) > 0$ for all $t > 0$, and assume that

$$0 < \hat{c} \leq \frac{l'(t)t}{l(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq l(t) \leq c_2[t^{s-1} + t^{p-1}]$$

for all $t > 0$, some constants $c_1, c_2 > 0$ and $1 \leq s < p$.

Using $l(\cdot)$ we can introduce the conditions on the map $a(\cdot)$.

H(a): $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(y) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(]0, +\infty[)$, $t \to a_0(t)t$ is strictly increasing on $]0, +\infty[$, $a_0(t)t \to 0$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists a constant $c_3 > 0$ such that

$$|\nabla a(y)| \le c_3 \frac{l(|y|)}{|y|} \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\};$$

(iii) we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \frac{l(|y|)}{|y|}|\xi|^2 \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\} \text{ and all } \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t a_0(s)s\,ds$, then there exist $1 < \tau < q \le p$ and constants $c_4, c_5 > 0$ such that

$$c_4 t^p \le a_0(t)t^2 - \tau G_0(t) \quad \text{for all } t > 0,$$
$$\lim_{t\to 0^+} \frac{qG_0(t)}{t^q} = c_5 < +\infty,$$
$$t \to G_0(t^{1/q}) \text{ is convex on }]0, +\infty[.$$

Remark 2.1. Hypotheses H(a) (i), (ii), (iii) come from the nonlinear regularity theory of Lieberman [7] and the nonlinear maximum principle of Pucci and Serrin [20]. Hypothesis H(a) (iv) is motivated by the particular needs of our problem (1.1). However, this condition is not restrictive as the examples below illustrate.

The following properties of the map $a(\cdot)$ follow easily from hypotheses H(a) [12].

Lemma 1. *If hypotheses* H(a) (i), (ii), (iii) *hold, then*
(a) *the map* $y \to a(y)$ *is continuous and strictly monotone,*
(b) $|a(y)| \le c_6[|y|^{s-1} + |y|^{p-1}]$ *for all* $y \in \mathbb{R}^N$, *some constant* $c_6 > 0$,
(c) $(a(y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

It is clear (see hypothesis H(a) (i)) that the primitive $t \to G_0(t)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then we have

$$\nabla G(y) = G'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$. Therefore, $G(\cdot)$ is strictly convex (see Lemma 1 (a)), and since $G(0) = 0$, we have

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.1}$$

Then Lemma 1 and (2.1) above lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 1. *If hypotheses* H(a) (i), (ii), (iii) *hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_7[1 + |y|^p] \quad \textit{for all } y \in \mathbb{R}^N \textit{ and some constant } c_7 > 0.$$

We present some characteristic maps $a(\cdot)$ which satisfy hypotheses H(a) [12]. These examples illustrate that our framework is broad.

Examples. (a) $a(y) = |y|^{p-2}y$, $1 < p < +\infty$. This map corresponds to the p-Laplace operator Δ_p defined by

$$\Delta_p = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$, $1 < q < p$. This map corresponds to the (p, q)-Laplace operator defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such operators arise in many physical applications [12].

(c) $a(y) = [1 + |y|^2]^{\frac{p-2}{p}} y$, $1 < p < +\infty$. This map corresponds to the extended capillary differential operator defined by

$$\operatorname{div}([1 + |Du|^2]^{\frac{p-2}{p}} Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y[1 + \frac{1}{1+|y|^p}]$, $1 < p < +\infty$. This map corresponds to the differential operator

$$u \to \Delta_p u + \operatorname{div}\Big(\frac{|Du|^{p-2}}{1 + |Du|^p} Du\Big) \quad \text{for all } u \in W^{1,p}(\Omega),$$

which arises in problems of plasticity theory [1].

Next we introduce our hypotheses on the potential function $\xi(\cdot)$ and on the coefficients $k(\cdot)$ and $\beta(\cdot)$.
H_0: $\xi, k \in L^\infty(\Omega)$, and there exist two constants $c_*, m_* > 0$ such that $c_*\xi(z) - k(z) \geq m_* > 0$ for a.a. $z \in \Omega$.
$\mathrm{H}(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$, and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 1. We can have $\beta \equiv 0$, and this corresponds to the Neumann problem.

We will also need some facts about the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_r u(z) + \tilde{\xi}(z)|u|^{r-2}u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_r} + \tilde{\beta}(z)|u|^{r-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $1 < r < +\infty$ and $\frac{\partial u}{\partial n_r} = |Du|^{r-2}(Du, n)_{\mathbb{R}^N}$ for all $u \in C^1(\bar{\Omega})$. Also, $\tilde{\xi} \in L^\infty(\Omega)$ and $\tilde{\beta} \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in]0, 1[$ and $\tilde{\beta}(z) \geq 0$ for all $z \in \partial\Omega$.

We know (see [11]) that (2.2) has the smallest eigenvalue $\hat{\lambda}_1(r, \tilde{\xi}, \tilde{\beta})$, which is isolated and simple. Moreover, the eigenfunctions of $\hat{\lambda}_1(r, \tilde{\xi}, \tilde{\beta})$ have fixed sign. By $\hat{u}_1(r, \tilde{\xi}, \tilde{\beta})$, we denote the positive, L^r-normalized (i.e., $\|\hat{u}_1(r, \tilde{\xi}, \tilde{\beta})\|_{L^r(\Omega)} = 1$) eigenfunction corresponding to $\hat{\lambda}_1(r, \tilde{\xi}, \tilde{\beta})$. The nonlinear regularity theory [7] and the nonlinear Hopf theorem [20, p. 120] imply $\hat{u}_1(r, \tilde{\xi}, \tilde{\beta}) \in D_+$.

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From [3, Problem 2.192], we have the following result.

Proposition 1. *If hypotheses* H(a) (i), (ii), (iii) *hold, then* $A(\cdot)$ *is bounded, continuous, monotone and of type* $(S)_+$, *that is, the following implication holds: "if* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$, *then this implies* $u_n \to u$ *in* $W^{1,p}(\Omega)$*".*

Next let us introduce the notion of critical group. So, let X be a Banach space and $\varphi \in C^1(X; \mathbb{R})$, $c \in \mathbb{R}$, and let us define the sets

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(critical set of } \varphi\text{)},$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

If (Y_1, Y_2) is a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$, by $H_k(Y_1, Y_2)$ ($k \in \mathbb{N}_0$), we denote the k-th relative singular homology group with integer coefficients. Recall that $H_k(Y_1, Y_2) = 0$ for $k \in \mathbb{N}^-$. Let $u \in K_\varphi$ be isolated, and set $c = \varphi(u)$. Then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of a singular homology group implies that the above definition of critical groups is independent of the choice of the isolating neighborhood U.

We also present the notation which we will use in this paper. For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also, if $u, v \in W^{1,p}(\Omega)$ and $u(z) \le v(z)$ for a.a. $z \in \Omega$, then we define

$$[u, v] = \{h \in W^{1,p}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\}.$$

By $|\cdot|_N$, we denote the Lebesgue measure on $\mathbb{R}^N$.

Finally, let us state our hypotheses on the reaction $f(z, x)$.

H(f): Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists a positive function $\alpha_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \le \alpha_\rho(z) \quad \text{for a.a. } z \in \Omega \text{ and all } |x| \le \rho,$$

(ii) if $1 < \tau_0 < q < p$ (see hypothesis H(a) (iv)), we have

$$\lim_{x\to 0} \frac{f(z, x)}{|x|^{q-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

$$\lim_{x\to 0} \frac{f(z, x)}{|x|^{\tau_0-2}x} = 0 \quad \text{uniformly for a.a. } z \in \Omega,$$

(iii) if $F(z, x) = \int_0^x f(z, x)\,ds$ and $\tau_0 \le \tau < q$ (see hypothesis H(f) (ii) above), then

$$0 \le \liminf_{x\to 0} \frac{\tau F(z, x) - f(z, x)x}{|x|^p} \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 2. We stress that no global growth conditions are imposed on $f(z, \cdot)$, only conditions concerning its behavior near zero. Moreover, we do not impose any sign on condition $f(z, \cdot)$. The conditions on $f(z, \cdot)$ are minimal.

Given any $\hat{\eta}_0 > 0$, we can find $\hat{\delta}_0 \in \,]0, 1[$ such that

$$|f(z, x)| \le |x|^{\tau_0-1} \quad \text{for all } z \in \Omega, \text{ all } |x| \le \hat{\delta},$$

$$f(z, x)x \ge \hat{\eta}_0|x|^q \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \hat{\delta}_0, \tag{2.3}$$

$$|x|^p \ge c_*|x|^{p^*} \quad \text{for all } |x| \le \hat{\delta}_0 \quad (\text{see hypothesis } H_0, \text{ and recall that } p < p^*). \tag{2.4}$$

Now let $\theta \in \,]0, \hat{\delta}_0]$, and consider a cut-off function $\eta \in C_c(\mathbb{R})$ satisfying

$$\operatorname{supp}\eta \subseteq [-\theta, \theta], \quad \eta|_{[-\theta/2,\theta/2]} \equiv 1 \quad \text{and} \quad 0 < \eta \le 1 \text{ on }]-\theta, \theta[. \tag{2.5}$$

Using this cut-off function, we introduce the following modification of the reaction of problem (1.1):

$$\hat{f}_\lambda(z, x) = \eta(x)[\lambda f(z, x) + k(z)|x|^{p^*-2}x] + (1 - \eta(x))(\xi(z) - \varepsilon)|x|^{p-2}x \tag{2.6}$$

with $\varepsilon > 0$.

Evidently, $\hat{f}_\lambda$ is a Carathéodory function, and we have

$$|\hat{f}_\lambda(z, x)| \le c_8[1 + |x|^{p-1}] \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}, \text{ with } c_8 > 0 \text{ constant.} \tag{2.7}$$

We will use this modification to introduce a new nonlinear parametric Robin problem for which we will generate three nontrivial solutions with sign information and show that, for $\lambda > 0$ small, these are also solutions of (1.1). To this end, we consider the parametric Robin problem

$$\begin{cases} -\operatorname{div}(a(Du(z))) + \xi(z)|u(z)|^{p-2}u(z) = \hat{f}_\lambda(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

In what follows, we work on (2.8) and at the end pass to our problem (1.1).

3 Constant sign solutions for problem (2.8)

In this section, we study the existence and properties of constant sign solutions for problem (2.8), $\lambda > 0$.

Proposition 2. *If hypotheses* $\mathrm{H}(a)$, H_0, $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold, then, for every* $\lambda > 0$, *problem* (2.8) *has two nontrivial constant sign solutions* $u_\lambda \in [0, \theta] \cap D_+$ *and* $v_\lambda \in [-\theta, 0] \cap (-D_+)$.

Proof. From (2.5) and (2.6), we see that

$$\hat{f}_\lambda(z, \pm\theta)(\pm\theta) = [\xi(z) - \varepsilon]\theta^p < \xi(z)\theta^p \quad \text{for a.a. } z \in \Omega. \tag{3.1}$$

Then we consider the following truncation of $\hat{f}_\lambda(z, \cdot)$:

$$\hat{d}_\lambda^+(z, x) = \begin{cases} \hat{f}_\lambda(z, x^+) & \text{if } x \leq \theta, \\ \hat{f}_\lambda(z, \theta) & \text{if } \theta < x. \end{cases} \tag{3.2}$$

Clearly, $\hat{d}_\lambda^+(\cdot, \cdot)$ is a Carathéodory function. We set $\hat{D}_\lambda^+(z, x) = \int_0^\lambda \hat{d}_\lambda^+(z, s)\, ds$ and consider the C^1-functional $\hat{\varphi}_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda^+(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \hat{D}_\lambda^+(z, u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega),$$

with $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_p(u) = \int_\Omega pG(Du)\, dz + \int_\Omega \xi(z)|u|^p\, dz + \int_{\partial\Omega} \beta(z)|u|^p\, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Corollary 1, (2.4), (3.2) and hypotheses H_0 and $\mathrm{H}(\beta)$ imply that $\hat{\varphi}_\lambda^+(\cdot)$ is coercive.

Also, using the Sobolev embedding theorem and the compactness of the trace map, we infer that $\hat{\varphi}_\lambda^+(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda^+(u_\lambda) = \inf[\hat{\varphi}_\lambda^+(u) : u \in W^{1,p}(\Omega)]. \tag{3.3}$$

Let $\mu \in\,]0, \frac{\theta}{2}]$. We have

$$\begin{aligned} \hat{\varphi}_\lambda^+(\mu) &\leq \frac{\mu^p}{p}\|\xi\|_{L^\infty(\Omega)}|\Omega|_N + \frac{\mu^p}{p}\|\beta\|_{L^\infty(\partial\Omega)}\sigma(\partial\Omega) - \lambda\frac{\hat{\eta}_0}{q}\mu^q \quad \text{(see (2.5), (2.6), (2.3) and hypothesis } \mathrm{H}_0) \\ &= c_9\mu^p - c_{10}\lambda\hat{\eta}_0\mu^q \quad \text{for some constants } c_9, c_{10} > 0. \end{aligned}$$

Since $\hat{\eta}_0 > 0$ is arbitrary, we choose it big so that

$$\begin{aligned} \hat{\varphi}_\lambda^+(\mu) < 0 &\implies \hat{\varphi}_\lambda^+(u_\lambda) < 0 = \hat{\varphi}_\lambda^+(0) \quad \text{(see (3.3))} \\ &\implies u_\lambda \neq 0. \end{aligned}$$

From (3.3), we have

$$(\hat{\varphi}_\lambda^+)'(u_\lambda) = 0 \implies \langle A(\hat{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\hat{u}_\lambda|^{p-2}\hat{u}_\lambda h\, dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h\, d\sigma = \int_\Omega \hat{f}_\lambda(z, u_\lambda)h\, dz \tag{3.4}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.4), we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\frac{c_1}{p-1}\|Du_\lambda^-\|_{L^p(\Omega)}^p + \int_\Omega \xi(z)(u_\lambda^-)^p\, dz \leq 0 \quad \text{(see (3.2), Lemma 2.1 (c) and hypothesis } \mathrm{H}(\beta)) \\ &\implies c_{11}\|u_\lambda^-\|_{W^{1,p}(\Omega)} \leq 0 \quad \text{for some constant } c_{11} > 0 \quad \text{(see hypothesis } \mathrm{H}_0) \\ &\implies u_\lambda \geq 0, \quad u_\lambda \neq 0. \end{aligned}$$

Also in (3.4), we choose $h = (u_\lambda - \theta)^+ \in W^{1,p}(\Omega)$. Then we have

$$\begin{aligned}
\langle A(u_\lambda), (u_\lambda - \theta)^+\rangle &+ \int_\Omega \xi(z) u_\lambda^{p-1}(u_\lambda - \theta)^+\,dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\lambda - \theta)^+\,d\sigma \\
&= \int_\Omega \hat f_\lambda(z,\theta) h\,dz \quad \text{(see (3.2))} \\
&\leq \int_\Omega \xi(z)\theta^{p-1} h\,dz \quad \text{(see (3.1))} \\
&\leq \langle A(\theta), (u_\lambda - \theta)^+\rangle + \int_\Omega \xi(z)\theta^{p-1}(u_\lambda - \theta)^+\,dz + \int_{\partial\Omega} \beta(z)\theta^{p-1}(u_\lambda - \theta)^+\,d\sigma \\
&\qquad \text{(since } A(\theta) = 0 \text{ and } \beta \geq 0;\ \text{see hypothesis H}(\beta)) \\
&\Longrightarrow u_\lambda \leq \theta \quad \text{(see Proposition 1).}
\end{aligned}$$

We have proved (see (3.1))

$$u_\lambda \in [0,\theta], \quad u_\lambda \neq 0, \quad u_\lambda \neq \theta. \tag{3.5}$$

From (3.2), (3.4) and (3.5), we infer that

$$\begin{cases} -\operatorname{div}(a(Du_\lambda(z))) + \xi(z) u_\lambda(z)^{p-1} = \hat f_\lambda(z, u_\lambda(z)) & \text{in } \Omega, \\ \dfrac{\partial u_\lambda}{\partial n_a} + \beta(z) u_\lambda^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

From (2.7) and [14, Proposition 2.10], we have $u_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [7, p. 320] implies $u_\lambda \in C_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_{L^\infty(\Omega)}$. On account of (2.3) and (2.4), we can find $\hat\xi_\rho > 0$ big enough so that

$$\hat f_\lambda(z,x) + \hat\xi_\rho x^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\rho]. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}
&\operatorname{div} a(Du_\lambda(z)) \leq [\|\xi\|_{L^\infty(\Omega)} + \hat\xi_\rho] u_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega \\
&\Longrightarrow u_\lambda \in D_+ \quad \text{(see [20, pp. 111, 120]).}
\end{aligned}$$

To produce a negative solution, we consider the Carathéodory function

$$\hat d_\lambda^-(z,x) = \begin{cases} \hat h_\lambda(z,-\theta) & \text{if } x \leq -\theta, \\ \hat f(z,-x^-) & \text{if } x > -\theta. \end{cases}$$

We set $\hat D_\lambda^-(z,x) = \int_0^x \hat d_\lambda^-(z,s)\,ds$ and consider the C^1-functional $\hat\varphi_\lambda^- : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat\varphi_\lambda^-(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \hat D_\lambda^-(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Reasoning as above, via the direct method of the calculus of variations, we produce a solution v_λ of problem (2.8) such that $v_\lambda \in [-\theta, 0] \cap (-D_+)$. □

Next we show that these solutions converge to zero in $C^1(\bar\Omega)$ as $\lambda \to 0^+$.

Proposition 3. *If hypotheses* H(a), H$_0$, H(β) *and* H(f) *hold, then* $u_\lambda \to 0$ *and* $v_\lambda \to 0$ *in* $C^1(\Omega)$ *as* $\lambda \to 0^+$.

Proof. From the proof of Proposition 2, we know that

$$u_\lambda \in [0,\theta] \cap D_+ \quad \text{and} \quad v_\lambda \in [-\theta, 0] \cap (-D_+) \quad \text{for all } \lambda > 0. \tag{3.8}$$

The nonlinear regularity theory of Lieberman [7] implies that we can find $\alpha \in \,]0,1[$ and a constant $c_{12} > 0$ such that $u_\lambda \in C^{1,\alpha}(\bar\Omega)$ and $\|u_\lambda\|_{C^{1,\alpha}(\Omega)} \leq c_{12}$ for all $\lambda > 0$. Let $\lambda_n \to 0^+$, and set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$. Then the compact embedding of $C^{1,\alpha}(\bar\Omega)$ into $C^1(\bar\Omega)$ implies that, at least for a subsequence, we have (see (3.8))

$$u_n = u_{\lambda_n} \to \tilde u_0 \quad \text{in } C^1(\bar\Omega), \quad \tilde u_0 \in [0,\theta] \cap C_+. \tag{3.9}$$

For every $n \in \mathbb{N}$, we have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\,d\sigma = \int_\Omega \hat{f}_{\lambda_n}(z, u_n)h\,dz$$

for all $h \in W^{1,p}(\Omega)$ (see (3.2) and (3.8)).

Passing to the limit as $n \to +\infty$ and using (3.9) and (3.1), we obtain

$$\langle A(u_0), h\rangle + \int_\Omega \xi(z)u_0^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}h\,d\sigma = \int_\Omega \eta(\tilde{u}_0)[k(z)\tilde{u}_0^{p-1} - \xi(z)\tilde{u}_0^{p-1}]h\,dz \tag{3.10}$$

for all $h \in W^{1,p}(\Omega)$ (recall that $\lambda_n \to 0^+$).

Note that $\tilde{u}_0 \not\equiv \theta$. Indeed, if $\tilde{u}_0 \equiv \theta$, then we choose $h = \tilde{u}_0$ in (3.10) and obtain $\varepsilon\theta^p|\Omega|_N \le 0$ (see (2.5)), a contradiction.

Also, suppose that $\tilde{u}_0 \not\equiv 0$. Then we can find an open ball $B \subseteq \Omega$ such that $\tilde{u}_0(z) > 0$ for all $z \in B$. We have

$$\begin{aligned}
&(\tilde{u}_0(z))^p \ge c_*(\tilde{u}_0(z))^{p^*} \quad \text{for all } z \in B \quad (\text{recall that } \theta \le \delta_0, \text{ and see (2.4)}),\\
\Longrightarrow\ &\xi(z)(\tilde{u}_0(z))^p \ge \xi(z)c_*(\tilde{u}_0(z))^{p^*} > k(z)(\tilde{u}_0(z))^{p^*} \quad \text{for all } z \in B \quad (\text{see hypothesis } \mathrm{H}_0),\\
\Longrightarrow\ &\int_\Omega \eta(u_0)[k(z)(\tilde{u}_0(z))^{p^*} - \xi(z)(\tilde{u}_0(z))^p]\,dz < 0 \quad (\text{see hypothesis } \mathrm{H}_0 \text{ and (2.5), and recall that } \tilde{u}_0 \not\equiv \theta).
\end{aligned}$$

In (3.10), we choose $h = \tilde{u}_0(z) \in W^{1,p}(\Omega)$. Using Lemma 2.1 (c) and hypothesis $\mathrm{H}(\beta)$, we obtain

$$\frac{c_1}{p-1}\|D\tilde{u}_0\|_{L^p(\Omega)}^p \le \int_\Omega \eta(\tilde{u}_0)[k(z)(\tilde{u}_0(z))^{p^*} - \xi(z)(\tilde{u}_0(z))^p]\,dz < 0,$$

a contradiction. Therefore, we infer that $\tilde{u}_0 \equiv 0$, so finally we have $u_\lambda \to 0$ in $C^1(\bar{\Omega})$ as $\lambda \to 0^+$. Similarly, we show that $v_\lambda \to 0$ in $C^1(\bar{\Omega})$ as $\lambda \to 0^+$. □

Now we will show that problem (2.8) has extremal constant sign solutions; that is, there is the smallest positive solution $u_\lambda^* \in D_+$ and the biggest negative solution $v_\lambda^* \in D_+$ for problem (2.8), $\lambda > 0$.

We introduce the two sets

$$\begin{aligned}
S_\lambda^+ &= \text{set of positive solutions of (2.8) in } [0, \theta],\\
S_\lambda^- &= \text{set of negative solutions of (2.8) in } [-\theta, 0].
\end{aligned}$$

Fix $\lambda > 0$, $\mu_0 > \hat{\lambda}_1(q, \tilde{\xi}, \tilde{\beta}) > 0$ with $\tilde{\xi} = \frac{1}{c_5}\xi$, $\tilde{\beta} = \frac{1}{c_5}\beta$ (see hypothesis $\mathrm{H}(a)$ (iv)) and $r \in\,]\beta, \beta^*[$. Then, on account of hypothesis $\mathrm{H}(f)$ (i), (ii), we can find $c_{13} = c_{13}(\lambda, r) > 0$ such that

$$\lambda f(z, x)x + k(z)|x|^{p^*} \ge \mu_0|x|^q - c_{13}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \theta. \tag{3.11}$$

We now consider the nonlinear Robin problem

$$\begin{cases} -\operatorname{div}(a(Du(z))) + \xi(z)u(z)^{p-2}u(z) = \mu_0|u(z)|^{q-2}u(z) - c_{13}|u(z)|^{r-2}u(z) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Proposition 4. *If hypotheses* $\mathrm{H}(a)$ *and* $\mathrm{H}(\beta)$ *hold,* $\xi \in L^\infty(\Omega)$, $\xi \ge 0$, *then problem* (3.12) *admits a unique positive solution* $\bar{u}_\lambda \in D_+$, *and since the problem is odd, then* $\bar{v}_\lambda = -\bar{u}_\lambda \in (-D_+)$ *is the unique negative solution of* (3.12).

Proof. Consider the C^1-functional $\hat{e}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{e}_\lambda(u) = \frac{1}{p}\gamma_p(u) - \frac{\mu_0}{q}\|u^+\|_{L^q(\Omega)}^q + \frac{c_{12}}{r}\|u^+\|_{L^r(\Omega)}^r \quad \text{for all } u \in W^{1,p}(\Omega).$$

Since $q \le p < r$, we see that $\hat{e}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{e}_\lambda(\bar{u}_\lambda) = \inf\{\hat{e}_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{3.13}$$

Hypothesis H(a) (iv) implies that, given $\tilde{c} > c_5$, we can find $\delta \in]0,1[$ such that

$$G(y) \leq \frac{\tilde{c}}{q}|y|^q \quad \text{for all } |y| \leq \delta. \tag{3.14}$$

Let $t \in]0,1[$ be small such that, for $\hat{u}_1 = \hat{u}_1(q, \tilde{\xi}, \tilde{\beta}) \in D_+$, we have $0 < t\hat{u}_1(z) \leq \delta$ and $|tD\hat{u}_1(z)| \leq \delta$ for all $z \in \bar{\Omega}$. Since $q \leq p$ and $\delta < 1$, we have

$$\begin{aligned}
\hat{e}_\lambda(t\hat{u}_1) &\leq \frac{\tilde{c}}{q}\|D(t\hat{u}_1)\|^q_{L^q(\Omega)} + \frac{1}{q}\int_\Omega \xi(z)(t\hat{u}_1)^q\,dz + \frac{1}{q}\int_{\partial\Omega} \beta(z)(t\hat{u}_1)^q\,d\sigma + \frac{c_{13}}{r}\|t\hat{u}_1\|_r^r - \frac{\mu_0}{q}\|t\hat{u}_1\|_q^q \quad (\text{see } (3.14)) \\
&= \frac{\tilde{c}t^q}{q}\Big[\|D\hat{u}_1\|^q_{L^q(\Omega)} + \int_\Omega \xi(z)\hat{u}_1^q\,dz + \int_{\partial\Omega} \tilde{\beta}(z)\hat{u}_1^q\,d\sigma - \mu_0\Big] + \frac{c_{13}}{r}t^r\|\hat{u}_1\|^r_{L^r(\Omega)} \\
&= \frac{\tilde{c}t^q}{q}[\hat{\lambda}_1(q,\tilde{\xi},\tilde{\beta}) - \mu_0] + \frac{c_{13}}{r}t^r\|\hat{u}_1\|^r_{L^r(\Omega)} \\
&= c_{14}t^r - c_{15}t^q \quad \text{for some constants } c_{14}, c_{15} > 0.
\end{aligned}$$

Recall that $q \leq p < r$. So, choosing $t \in]0,1[$ even smaller if necessary, we have

$$\begin{aligned}
\hat{e}_\lambda(t\hat{u}_1) < 0 &\implies \hat{e}_\lambda(\bar{u}_\lambda) < 0 = \hat{e}_\lambda(0) \quad (\text{see } (3.13)) \\
&\implies \bar{u}_\lambda \neq 0.
\end{aligned}$$

From (3.13), we have

$$\begin{aligned}
\hat{e}'_\lambda(\bar{u}_\lambda) = 0 \implies \langle A(\bar{u}_\lambda), h\rangle &+ \int_\Omega \xi(z)|\bar{u}_\lambda|^{p-2}\bar{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|\bar{u}_\lambda|^{p-2}\bar{u}_\lambda h\,d\sigma \\
&= \int_\Omega [\mu_0(\bar{u}_\lambda^+)^{q-1} - c_{13}(\bar{u}_\lambda^+)^{r-1}]h\,dz \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned} \tag{3.15}$$

Choosing $h = -\bar{u}_\lambda^- \in W^{1,p}(\Omega)$ in (3.15), we obtain $\bar{u}_\lambda \geq 0$, $\bar{u}_\lambda \neq 0$.

So, $\bar{u}_\lambda$ is a positive solution of problem (3.12). Moreover, the nonlinear regularity theory [7] and the nonlinear maximum principle [20] imply $\bar{u}_\lambda \in D_+$.

Finally, using hypothesis H(a) (iv) (in particular, the convexity of $t \to G_0(t^{1/q})$) and reasoning as in the proof of [12, Proposition 2.7], we show that $\bar{u}_\lambda \in D_+$ is the unique solution of problem (3.12).

Since problem (3.12) is odd, then $\bar{v}_\lambda = -\bar{u}_\lambda \in (-D_+)$ is the unique negative solution of (3.12). □

We are ready to produce extremal constant sign solutions for problem (2.8), $\lambda > 0$.

Proposition 5. *If hypotheses* H(a), H_0, H(β) *and* H(f) *hold and* $\lambda > 0$, *then problem* (2.8) *admits the smallest positive solution* $u_\lambda^* \in D_+$ *and the biggest negative solution* $v_\lambda^* \in (-D_+)$.

Proof. We know that $\emptyset \neq S_\lambda^+ \subseteq [0,\theta] \cap D_+$ (see Proposition 2).

First we show that

$$\bar{u}_\lambda \leq u \quad \text{for all } u \in S_\lambda^+. \tag{3.16}$$

To this end, let $u \in S_\lambda^+ \subseteq [0,\theta] \cap D_+$, and consider the Carathéodory function $i_\lambda : \Omega \to \mathbb{R}$ defined by (see (3.11))

$$i_\lambda(z,x) = \begin{cases} \mu_0(x^+)^{q-1} - c_{13}(x^+)^{r-1} & \text{if } x \leq u(z), \\ \mu_0(u(z))^{q-1} - c_{13}(u(z))^{r-1} & \text{if } u(z) < x. \end{cases} \tag{3.17}$$

We set $I_\lambda(z,x) = \int_0^x i_\lambda(z,s)\,ds$ and consider the C^1-functional $j_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$j_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega I_\lambda(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From 3.17 and hypothesis H_0, we see that $j_\lambda(\cdot)$ is coercive. Also, it his sequentially weakly semicontinuous. So, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$j_\lambda(\tilde{u}_\lambda) = \inf\{j_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{3.18}$$

As in the proof of Proposition 4, using hypothesis H(a)(iv), we show that, for $t \in]0,1[$ small and with $\hat{u}_1 = \hat{u}_1(q, \tilde{\xi}, \tilde{\beta}) \in D_+$, we have

$$\begin{aligned} j_\lambda(t\hat{u}_1) < 0 &\implies j_\lambda(\tilde{u}_\lambda) < 0 = j_\lambda(0) \quad \text{(see (3.18))} \\ &\implies \tilde{u}_\lambda \neq 0. \end{aligned} \tag{3.19}$$

From (3.18), we have

$$j'_\lambda(\tilde{u}_\lambda) = 0 \implies \langle A(\tilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h\,d\sigma = \int_\Omega i_\lambda(z,\tilde{u}_\lambda)h\,dz \tag{3.20}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.20), first we choose $h = -\tilde{u}_\lambda^- \in W^{1,p}(\Omega)$. Using Lemma 1 (c) and hypotheses H_0 and H(β), we obtain

$$\begin{aligned} &c_{16}\|\tilde{u}_\lambda^-\|^p \le 0 \quad \text{for some constant } c_{16} > 0 \\ &\implies \tilde{u}_\lambda \ge 0, \quad \tilde{u}_\lambda \neq 0 \quad \text{(see (3.19))}. \end{aligned}$$

Next, in (3.20), we choose $h = (\tilde{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned} &\langle A(\tilde{u}_\lambda, (\tilde{u}_\lambda - u)^+)\rangle + \int_\Omega \xi(z)\tilde{u}_\lambda^{p-1}(\tilde{u}_\lambda - u)^+\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_\lambda^{p-1}(\tilde{u}_\lambda - u)^+\,d\sigma \\ &= \int_\Omega [\mu_0 u^{q-1} - c_{13}u^{r-1}](\tilde{u}_\lambda - u)^+\,d\sigma \quad \text{(see (3.17))} \\ &\le \int_\Omega [\lambda f(z,u) + k(z)u^{p^*-1}](\tilde{u}_\lambda - u)^+ \quad \text{(see (3.11))} \\ &= \langle A(u), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega \xi(z)u(\tilde{u}_\lambda - u)^+\,dz + \int_{\partial\Omega} \beta(z)u(\tilde{u}_\lambda - u)^+\,d\sigma \quad (\text{since } u \in S_\lambda^+), \\ &\implies \tilde{u}_\lambda \le u. \end{aligned}$$

So, we have proved that

$$\tilde{u}_\lambda \in [0,u], \quad \tilde{u}_\lambda \neq 0. \tag{3.21}$$

From (3.17), (3.20) and (3.21), it follows that $\tilde{u}_\lambda$ is a positive solution of (3.12), which implies $\tilde{u}_\lambda = \bar{u}_\lambda \in D_+$ (see Proposition 4). Invoking [4, Lemma 3.10, p. 178], we can find a sequence $\{u_n\} \subseteq S_\lambda^+ \subseteq [0, \theta \cap D_+]$ such that $\inf_{n\ge1} u_n = \inf S_\lambda^+$. For every $n \in \mathbb{N}$, we have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\,d\sigma = \int_\Omega \hat{f}_\lambda(z,u_n)h\,dz \tag{3.22}$$

for all $h \in W^{1,p}(\Omega)$.

From Lieberman [7], we know that there exist $\alpha \in]0,1[$ and a constant $c_{17} > 0$ such that $u_n \in C^{1,\alpha}(\bar{\Omega})$ and $\|u_n\|_{C^{1,\alpha}(\Omega)} \le c_{17}$ for all $n \in \mathbb{N}$. The compact embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, implies that, at least for a subsequence, we have $u_n \to u_\lambda^*$ in $C_0^1(\bar{\Omega})$. So, if we pass to the limit as $n \to +\infty$ in (3.22), we obtain

$$\langle A(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1}h\,d\sigma = \int_\Omega \hat{f}_\lambda(z,u_\lambda^*)h\,dz \tag{3.23}$$

for all $h \in W^{1,p}(\Omega)$.

Also, from (3.16), we deduce

$$\bar{u}_\lambda \le u_\lambda^* \implies u_\lambda^* \neq 0. \tag{3.24}$$

From (3.23) and (3.24), it follows that $u_\lambda^* \in S_\lambda^+$ and $u_\lambda^* = \inf S_\lambda^+$. Similarly, working with S_λ^-, we obtain $v_\lambda^* \in S_\lambda^-$ and $v_\lambda^* = \sup S_\lambda^-$. In this case, for $\bar{v}_\lambda = -\bar{u}_\lambda \in [-\theta, 0] \cap (-D_+)$, we have $v \le \bar{v}_\lambda$ for all $v \in S_\lambda^-$. □

4 Nodal solutions of problem (2.8)

In this section, using the extremal constant sign solutions produced in Proposition 5, we will prove the existence of nodal solutions.

So, let $u_\lambda^* \in [0, \theta] \cap D_+$ and $v_\lambda^* \in [0, \theta] \cap (-D_+)$ be two extremal constant sign solutions for problem (2.8), $\lambda > 0$ from Proposition 5. Using these solutions, we truncate the reaction $\hat{f}_\lambda(z, \cdot)$. So, we introduce the Carathéodory function

$$\hat{\tau}_\lambda(z, x) = \begin{cases} \hat{f}_\lambda(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*(z), \\ \hat{f}_\lambda(z, x) & \text{if } v_\lambda^*(z) \le x \le u_\lambda^*(z), \\ \hat{f}_\lambda(z, u_\lambda^*(z)) & \text{if } x > u_\lambda^*(z). \end{cases} \tag{4.1}$$

We also consider the positive and negative truncations of $\tau_\lambda(z, \cdot)$, namely the Carathéodory functions

$$\hat{\tau}_\lambda^\pm(z, x) = \hat{\tau}_\lambda(z, \pm x^\pm). \tag{4.2}$$

We set

$$\hat{T}_\lambda(z, x) = \int_0^x \hat{\tau}_\lambda(z, s)\, ds, \quad \hat{T}_\lambda^\pm(z, x) = \int_0^x \hat{\tau}_\lambda(z, s)\, ds$$

and consider the C^1-functionals $\hat{\psi}_\lambda, \hat{\psi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \hat{T}_\lambda(z, u)\, dz, \quad \hat{\psi}_\lambda^\pm(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \hat{T}_\lambda^\pm(z, u)\, dz$$

for all $u \in W^{1,p}(\Omega)$.

In our method of proof, we will use critical groups in order to distinguish between solutions of (2.8). For this reason, we will need the following result which improves [12, Proposition 3.7].

Proposition 6. *If hypotheses* H(α), H$_0$, H(β) *and* H(f) *hold and* $\lambda > 0$, *then* $C_k(\hat{\psi}_\lambda, 0) = 0$ *for all* $k \in \mathbb{N}_0$

Proof. Given $\hat{\eta}_0 > 0$ and $r > \rho$, on account of hypotheses H(f) (i), (ii), of (2.6) and of (4.1), we can find a constant $c_{18} = c_{18}(\eta)$ such that

$$\hat{T}_\lambda(z, x) \ge \hat{\eta}_0 |x|^q - c_{18}|x|^r \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{4.3}$$

Corollary 1 and hypothesis H(a) (iv) imply that

$$G(y) \le c_{19}[|y|^q + |y|^p] \quad \text{for some constant } c_{19} > 0, \text{ all } y \in \mathbb{R}^N. \tag{4.4}$$

Then, for every $u \in W^{1,p}(\Omega)$ and every $t > 0$, using (4.3) and (4.4), we have

$$\hat{\psi}_\lambda(tu) \le c_{20}[t^q \|u\|^q_{W^{1,p}(\Omega)} + t^p \|u\|^p_{W^{1,p}(\Omega)} + t^r \|u\|^r_{W^{1,p}(\Omega)}] - \eta_0 t^q \|u\|^q_{L^q((\Omega)}. \tag{4.5}$$

Recall that $\eta_0 > 0$ is arbitrary and $q \le p < r$. Hence, from (4.5), we see that we can find $t^* > 0$ such that $\hat{\psi}_\lambda(tu) < 0$ for all $t \in]0, t^*[$. We introduce the numbers

$$\begin{aligned} \hat{t}_1 &= \sup\{t \in [0, 1] : \hat{\psi}_\lambda(tu) < 0\}, \\ \hat{t}_2 &= \begin{cases} \inf\{t \in [0, 1] : \hat{\psi}_\lambda(tu) \ge 0\} & \text{if } \{t \in [0, 1] : \hat{\psi}_\lambda(tu) \ge 0\} \ne \emptyset, \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.6}$$

Claim: $\hat{t}_1 \le \hat{t}_2$. We argue by contradiction. So, suppose that $\hat{t}_2 < \hat{t}_1$. From hypothesis H(f) (iii), (2.6) and (4.1), we see that, given $\varepsilon_0 > 0$, we can find a constant $c_{21} = c_{21}(\varepsilon_0) > 0$ such that

$$\tau \hat{T}_\lambda(z, x) - \hat{\tau}_\lambda(z, x)x \ge -\varepsilon_0 |x|^p - c_{21}|x|^{p^*} \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}.$$

Since $\hat{t}_2 < \hat{t}_1$, we can find $\hat{t} \in\,]0,1[$ such that $\hat{\psi}_\lambda(tu) = 0$. Then we have

$$\begin{aligned}\hat{t}\Big(\frac{d}{dt}\hat{\psi}_\lambda(tu)\Big)_{t=\hat{t}} &= \langle \hat{\psi}_\lambda'(\hat{t}u), \hat{t}u\rangle \quad \text{(by the chain rule)}\\ &= \langle \hat{\psi}_\lambda'(\hat{t}u), \hat{t}u\rangle - \tau\hat{\psi}_\lambda(\hat{t}u) \quad (\text{since } \hat{\psi}_\lambda(\hat{t}u) = 0)\\ &= \langle \hat{\psi}_\lambda(y), y\rangle - \tau\hat{\psi}_\lambda(y) \quad (\text{setting } y = \hat{t}u)\\ &\geq [c_{22} - \varepsilon_0]\|u\|^p_{W^{1,p}(\Omega)} - c_{23}\|u\|^{p^*}_{W^{1,p}(\Omega)}\end{aligned}$$

for some constants $c_{22}, c_{23} > 0$ (recall that $\tau < p$).

Recalling that $\varepsilon_0 > 0$ is arbitrary, we can choose $\varepsilon_0 \in\,]0, c_{22}[$ and obtain

$$\hat{t}\Big(\frac{d}{dt}\hat{\psi}_\lambda(tu)\Big)_{t=\hat{t}} \geq c_{24}\|y\|^p_{W^{1,p}(\Omega)} - c_{23}\|y\|^{p^*}_{W^{1,p}(\Omega)}$$

with $y = \hat{t}u$, $c_{24} = c_{22} - \varepsilon_0 > 0$.

Since $p < p^*$, for some $\rho \in\,]0,1[$ small and for $0 < \|u\|_{L^p(\Omega)} \leq \rho$, we have

$$\hat{t}\Big(\frac{d}{dt}\hat{\psi}_\lambda(tu)\Big)_{t=\hat{t}} > 0 \tag{4.7}$$

for $\hat{t} \in\,]0,1[$ with $\hat{\psi}_\lambda(\hat{t}u) = 0$.

From (4.6), we have

$$\hat{\psi}_\lambda(\hat{t}_2u) = 0 \implies \hat{\psi}_\lambda(tu) > 0 \quad \text{for all } t \in\,]\hat{t}_2, \hat{t}_2 + \delta] \text{ with } \delta \in\,]0, \hat{t}_1 - \hat{t}_2[. \tag{4.8}$$

We introduce the set $E_\lambda = \{t \in\,]\hat{t}_2 + \delta, \hat{t}_1] : \hat{\psi}_\lambda(tu) = 0\}$ and

$$t^* = \begin{cases}\inf E_\lambda & \text{if } E_\lambda \neq \emptyset,\\ 1 & \text{if } E_\lambda = \emptyset.\end{cases} \tag{4.9}$$

Evidently, $t^* > \hat{t}_2 + \delta$ (see (4.8), (4.9)). Since $\hat{\psi}_\lambda(t^*u) = 0$, from (4.7), we can find $\delta' \in\,]0, \hat{t}_1 - \hat{t}_2 - \delta[$ such that

$$\hat{\psi}_\lambda(tu) < 0 \quad \text{for all } t \in\,]t^* - \delta', t^*[. \tag{4.10}$$

From (4.8) and (4.10), we see that there exists $t_0 \in\,]\hat{t}_2 + \delta, t^* - \delta'[$ such that $\hat{\psi}_\lambda(t_0u) = 0$ (Bolzano's theorem). This contradicts (4.9), so the claim is true.

If $\hat{t}_1 < \hat{t}_2$, then we have $\hat{\psi}_\lambda(tu) = 0$ for all $t \in\,]\hat{t}_1, \hat{t}_2[$, which contradicts (4.7). It follows that $\hat{t}_1 = \hat{t}_2$. Let $\hat{t}(u) = \hat{t}_1 = \hat{t}_2$. We have

$$\begin{aligned}\hat{\psi}_\lambda(tu) &< 0 \quad \text{for all } t \in\,]0, \hat{t}(u)[,\\ \hat{\psi}_\lambda(\hat{t}(u)u) &= 0,\\ \hat{\psi}_\lambda(tu) &> 0 \quad \text{for all } t \in\,]\hat{t}(u), 1].\end{aligned}$$

Therefore, the map $u \to \hat{r}(u) = \hat{t}(u)u$, $u \in \bar{B}_\rho \setminus \{0\}$ ($\bar{B}_\rho = \{u \in W^{1,p}(\Omega) : \|u\|_{W^{1,p}(\Omega)} \leq \rho\}$, $\rho > 0$) is continuous. Also, we have

$$\hat{r}(\bar{B}_\rho \setminus \{0\}) \subseteq (\hat{\psi}^0_\lambda \cap \bar{B}_\rho) \setminus \{0\}, \quad \hat{r}|_{(\hat{\psi}^0_\lambda \cap \bar{B}_\rho)\setminus\{0\}} = \mathrm{id}|_{(\hat{\psi}^0_\lambda \cap \bar{B}_\rho)\setminus\{0\}}.$$

It follows that $(\hat{\psi}^0_\lambda \cap \bar{B}_\rho) \setminus \{0\}$ is a retract of $\bar{B}_\rho \setminus \{0\}$.

The set $\bar{B}_\rho \setminus \{0\}$, contractible and a retract of a contractible space, is itself a contractible. Therefore,

$$(\hat{\psi}^0_\lambda \cap \bar{B}_\rho) \setminus \{0\} \text{ is contractible.} \tag{4.11}$$

Moreover, using the deformation $h\colon [0,1] \times (\hat{\psi}^0_\lambda \cap \bar{B}_\rho) \to \hat{\psi}^0_\lambda \cap \bar{B}_\rho$ defined by $h(t,u) = (1-t)u$, we see that

$$\hat{\psi}^0_\lambda \cap \bar{B}_\rho \text{ is contractible.} \tag{4.12}$$

Then (4.11), (4.12) and [18, Propositions 6.1.30 and 6.1.31] imply

$$\begin{aligned}&H_k(\hat{\psi}^0_\lambda \cap \bar{B}_\rho, (\hat{\psi}^0_\lambda \cap \bar{B}_\rho) \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0\\ &\implies C_k(\hat{\psi}_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0.\end{aligned}$$

□

Now we are ready to generate nodal solutions for problem (2.8).

Proposition 7. *If hypotheses* H(a), H$_0$, H(β) *and* H(f) *hold and* $\lambda > 0$, *then problem* (2.8) *admits a nodal solution* $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega})$.

Proof. Using (4.1), (4.2) and the nonlinear regularity theory, we check easily that

$$K_{\hat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega}), \quad K_{\hat{\psi}_\lambda^+} \subseteq [0, u_\lambda^*] \cap C^1(\bar{\Omega}), \quad K_{\hat{\psi}_\lambda^-} \subseteq [v_\lambda^*, 0] \cap C^1(\bar{\Omega}).$$

Then the extremity of u_λ^* and v_λ^* implies

$$K_{\hat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega}), \quad K_{\hat{\psi}_\lambda^+} = \{0, u_\lambda^*\}, \quad K_{\hat{\psi}_\lambda^-} = \{0, v_\lambda^*\}. \tag{4.13}$$

Clearly, $\hat{\psi}_\lambda^+(\cdot)$ is coercive (see (4.1), (4.2) and hypothesis H$_0$). Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda^+(\tilde{u}_\lambda^*) = \inf\{\hat{\psi}_\lambda^+(u) : u \in W^{1,p}(\Omega)\}. \tag{4.14}$$

Using (3.11) and reasoning as in the proof of Proposition 4, we show that

$$\hat{\psi}_\lambda^+(\tilde{u}_\lambda^*) < 0 = \hat{\psi}_\lambda^+(0) \implies \tilde{u}_\lambda^+ \neq 0. \tag{4.15}$$

From (4.14), we have $\tilde{u}_\lambda^* \in K_{\hat{\psi}_\lambda^+} = \{0, u_\lambda^*\}$ (see (4.13)). Therefore (see (4.15)),

$$\tilde{u}_\lambda^* = u_\lambda^* \in [0, \theta] \cap D_+. \tag{4.16}$$

Note that

$$\begin{aligned}
&\hat{\psi}_\lambda|_{C_+} = \hat{\psi}_\lambda^+|_{C_+} \quad \text{(see (4.1), (4.2))}\\
&\implies u_\lambda^* \text{ is a local } C^1\text{-minimizer of } \hat{\psi}_\lambda \quad \text{(see (4.14) and (4.16))}\\
&\implies u_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\psi}_\lambda
\end{aligned} \tag{4.17}$$

(see [14, Proposition 2.12]). In a similar fashion, using this time the functional $\hat{\psi}_\lambda^-$, we show that

$$v_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\psi}_\lambda. \tag{4.18}$$

We may assume that $\hat{\psi}_\lambda(v_\lambda^*) \leq \hat{\psi}_\lambda(u_\lambda^*)$. The reasoning is similar if the opposite inequality holds, using this time (4.18) instead of (4.17).

From (4.13), it is clear that we may assume that $K_{\hat{\psi}_\lambda}$ is finite (otherwise, we already have an infinity of nodal solutions, and so we are done). Then, using (4.17) and [18, Theorem 5.7.6], we can find $\rho \in\,]0, 1[$ small such that

$$\hat{\psi}_\lambda(v_\lambda^*) \leq \hat{\psi}_\lambda(u_\lambda^*) < \inf\{\hat{\psi}_\lambda(u) : \|u - u_\lambda^*\|_{W^{1,p}(\Omega)} = \rho\} = \hat{m}_\lambda, \quad \|v_\lambda^* - u_\lambda^*\|_{W^{1,p}(\Omega)} > \rho. \tag{4.19}$$

Recall that $\hat{\psi}_\lambda(\cdot)$ is coercive. Hence

$$\hat{\psi}_\lambda \text{ satisfies the Palais–Smale condition} \tag{4.20}$$

(see [18, Proposition 5.1.15]). Then (4.19) and (4.20) permit the use of the mountain pass theorem. So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\hat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega}) \quad \text{(see (4.13))}, \quad \hat{m}_\lambda \leq \hat{\psi}_\lambda(y_\lambda). \tag{4.21}$$

From (4.19) and (4.21), it follows that

$$y_\lambda \notin \{u_\lambda^*, v_\lambda^*\}. \tag{4.22}$$

Theorem 6.5.8 of Papageorgiou, Rădulescu and Repovs [18] implies

$$C_1(\hat{\psi}_\lambda, y_\lambda) \neq 0. \tag{4.23}$$

On the other hand, from Proposition 6, we have

$$C_k(\hat{\psi}_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.24}$$

Comparing (4.23) and (4.24), we will infer that

$$y_\lambda \neq 0. \tag{4.25}$$

Then, from (4.21), (4.22) and (4.25), we conclude that $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega})$ is a nodal solution of (2.8). □

5 Multiplicity theorem for problem (1.1)

In this section, using the analysis for problem (2.8) conducted in the previous two sections, we prove a multiplicity theorem for problem (1.1) when $\lambda > 0$ is small. Moreover, we provide sign information for all the solutions produced.

Let $u_\lambda^* \in D_+$ and $v_\lambda^* \in (-D_+)$ be the extremal constant sign solutions and $y_\lambda \in [u_\lambda^*, v_\lambda^*] \cap C^1(\bar{\Omega})$ the nodal solution for problem (2.8), $\lambda > 0$ (see Propositions 5 and 7). Then, on account of Proposition 3, we have the following result.

Proposition 8. *If hypotheses* $\mathrm{H}(a)$, H_0, $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold, then* $u_\lambda^*, v_\lambda^*, y_\lambda \to 0$ *in* $C^1(\bar{\Omega})$ *as* $\lambda \to 0^+$.

Proposition 8 and (2.6) lead at once to the following multiplicity theorem for problem (1.1).

Theorem 5.1. *If hypotheses* $\mathrm{H}(a)$, H_0, $\mathrm{H}(\beta)$ *and* $\mathrm{H}(f)$ *hold, then, for all* $\lambda > 0$ *small, problem* (1.1) *has at least three nontrivial smooth solutions*

$$u_\lambda \in D_+, \quad v_\lambda \in (-D_+), \quad y_\lambda \in [v_\lambda, u_\lambda] \cap C^1(\bar{\Omega}) \quad \textit{nodal};$$

moreover, $u_\lambda, v_\lambda, y_\lambda \to 0$ *in* $C^1(\bar{\Omega})$ *as* $\lambda \to 0^+$.

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