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Functional Analysis — *Weighted extrapolation in grand Morrey spaces and applications to partial differential equations*, by Vakhtang Kokilashvili, Alexander Meskhi and Maria Alessandra Ragusa, communicated on November 9, 2018.

Dedicated to Prof. Carlo Sbordone on the occasion of his 70th anniversary

Abstract. — In this paper we derive weighted extrapolation results in grand Morrey spaces. In particular, the main statements say that if for a class of pairs of measurable functions the one-weight inequality holds in the classical weighted $L_w^{p_0}$ space for some p_0 and for all Muckenhoupt A_{p_0} weights, then the one-weight inequality is also true in weighted grand Morrey space $L_w^{p),r,\theta}$ with A_p weights for all p. The spaces under consideration are defined on quasi-metric measure spaces. The obtained results are applied to derive one-weight estimates for some operators of Harmonic Analysis and to study regularity properties of solutions of second order partial differential equations with discontinuous coefficients.

Key words: Extrapolation, grand Morrey spaces, spaces of homogeneous type, weighted inequality, singular integrals, fractional integrals, commutators

Mathematics Subject Classification (primary; secondary): 42B20; 42B25, 42B35, 47B38

1. Introduction

Rubio de Francia's extrapolation result (see [39]) dealing with classical weighted Lebesgue spaces is one of the most powerful tool in modern Harmonic Analysis. In this paper extrapolation theorems in weighted grand Morrey spaces are established. From these results we obtain new one-weight inequalities in these spaces for operators of Harmonic Analysis for which the L_w^p bounddeness holds under the Muckenhoupt A_p condition for w. The derived results for commutators of Harmonic Analysis are applied to study regularity properties of solutions of second order partial differential equations (PDEs for short) with discontinuous coefficients. Integral transforms and function spaces are defined on quasi-metric measure spaces (X, d, μ) with doubling measure μ. We are interested in the weighted grand Morrey space $L_w^{p),r,\theta}(X)$ with a weight function w defined by the norm:

$$(1.1)\quad \|f\|_{L_w^{p),r,\theta}(X)} := \sup_{0<\varepsilon<p-1} \sup_B \frac{\varepsilon^\theta}{(w(B))^{\frac{1}{p-\varepsilon}+r}} \|f\|_{L_w^{p-\varepsilon}(B)} := \sup_{0<\varepsilon<p-1} \varepsilon^\theta \|f\|_{L_w^{p-\varepsilon,r}(X)},$$

where $1 < p < \infty$, $-1/p < r < 0$ and $\theta > 0$. We are stimulated to investigate the extrapolation problem in such a type of grand Morrey space because of the paper [11], where the extrapolation problem was studied in the classical weighted Morrey spaces $L_w^{p,r}(\mathbb{R}^n)$. The study of the one-weight problem for integral operators in weighted classical Morrey spaces with Muckenhoupt weights defined on $\mathbb{R}^n$ was initiated in the paper [28]. The same problem for sublinear operators involving maximal, Calderón–Zygmund and fractional integrals, etc in classical weighted Morrey spaces with A_p weights was investigated in [34], [42], [20], [35] (see also references cited therein and [40] for related topics). It should be emphasize that the one-weight boundedness problem for sublinear operators involving their commutators in grand Morrey spaces were explored in [25] and [24]. In those papers the authors deal with weighted grand Morrey spaces $\mathscr{M}_w^{p),\theta,\lambda}(X)$ and $M_w^{p),\theta,\lambda}(X)$ defined with respect to the norms:

$$\|f\|_{\mathscr{M}_w^{p),\theta,\lambda}(X)} := \sup_{0<\varepsilon<p-1} \sup_{B\subset X} \left(\frac{\varepsilon^\theta}{(w(B))^\lambda} \int_B |f(x)|^{p-\varepsilon} w(x)\mu(x) \right)^{\frac{1}{p-\varepsilon}}, \tag{1.2}$$

where $1 < p < \infty$, $0 < \lambda < 1/p$, $\theta > 0$, and

$$\|f\|_{M_w^{p),\theta,\lambda}(X)} := \sup_{0<\varepsilon<p-1} \sup_{B\subset X} \frac{1}{(w(B))^\lambda} \left(\varepsilon^\theta \int_B |f(x)|^{p-\varepsilon} w(x)\mu(x) \right)^{\frac{1}{p-\varepsilon}}, \tag{1.3}$$

where $1 < p < \infty$, $0 < \lambda < 1$, $\theta > 0$.

Grand Morrey spaces were introduced in the paper [30]. In that paper the author obtained the appropriate boundedness for operators of Harmonic Analysis. The spaces introduced and studied in [30] are defined as follows:

$$\mathscr{M}^{p),\theta,\lambda}(X) = \left\{ f : X \mapsto \mathbb{R} : \|f\|_{\mathscr{M}^{p),\theta,\lambda}(X)} \right.$$
$$\left. := \sup_{0<\varepsilon<p-1} \sup_{B} \left(\frac{\varepsilon^\theta}{(\mu(B))^\lambda} \int_B |f(y)|^{p-\varepsilon} \, d\mu(y) \right)^{1/(p-\varepsilon)} \right\} < \infty.$$

Later, H. Rafeiro [37] introduced generalized grand Morrey space defined by the norm including the "grandification" taken not only with respect to p but also for λ. Those spaces are defined with respect to the norm:

$$\|f\|_{\mathscr{M}_{\theta,A}^{p),\lambda)}(X)} := \sup_{0<\varepsilon<s_{\max}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon,\lambda-A(\varepsilon)}(X)}, \quad s_{\max} = \min\{p-1, a\},$$

where A be a non-decreasing real-valued non-negative function with $\lim_{x\to 0+} A(x) = 0$ and $a = \sup\{x > 0 : A(x) \le \lambda\}$. We refer also to the paper [25] for mapping properties of sublinear operators in $\mathscr{M}_{\theta,A}^{p),\lambda)}(X)$.

Weighted extrapolation results in grand Lebesgue spaces were proved in [21], [22]. For the classical Morrey spaces defined on $\mathbb{R}^n$ such results were derived by

J. Duoandikoetxea and M. Rosental [11] (see also the paper by M. Rosental and H.-J. Schmeisser [38] for related topics).

One of the aims of this paper is to obtain regularity properties of solutions of second order PDEs with discontinuous coefficients in $M_w^{p),r,\theta}(X)$ spaces. We consider PDEs of elliptic type having coefficients that can be discontinuous and show that if the known term belongs to weighted grand Morrey spaces, then the highest order derivatives of the solutions of the equations are in the same class.

Finally we emphasize the regularity and inner estimates for the solution of second order elliptic PDEs in grand Morrey spaces $M_{\theta,A}^{p),\lambda)}(\Omega)$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$ were studied in Chapter 16 of [27].

Let (X, d, μ) be a quasi-metric measure space (QMMS briefly) with a quasi-metric d and measure μ. A quasi-metric d is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$.
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(c) There is a constant $\kappa > 0$ such that $d(x, y) \le \kappa(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

If μ satisfies the doubling condition, i.e., if there is a positive constant C_{dc} such that for all $x \in X$ and $r > 0$,

$$\mu B(x, 2r) \le C_{dc}\mu B(x, r), \tag{1.4}$$

then QMMS (X, d, μ) is called a space of homogeneous type (SHT briefly).

It is known (see [29]) that for any quasi-metric space (X, d), there is a continuous quasi-metric ρ on X which is equivalent to d such that all balls corresponding to ρ are open in the topology induced by ρ, and there exist constants C and $\theta \in (0, 1)$ such that for all $x, y, z \in X$,

$$|\rho(x, z) - \rho(y, z)| \le C\rho^{\theta}(x, y)(\rho(x, z) + \rho(y, z))^{1-\theta}.$$

Without loss of generality we assume that d is continuous and all balls are open with respect to d.

For the definition, examples and some properties of an *SHT* see, e.g., monograph [43].

If C_{dc} is the doubling constant, then the constant

$$D_\mu := \log_2 C_{dc} \tag{1.5}$$

is called the *doubling order of* μ.

Let $\ell := \operatorname{diam}(X) = \sup_{x,y\in X} d(x, y)$. Notice that the condition $\ell < \infty$ implies that $\mu(X) < \infty$.

Definition 1.1. The triple (X, d, μ) is called an *RD*-space if it is an *SHT* and μ satisfies the reverse doubling condition (*RDC* briefly): there exist constants

$a, b > 1$ such that for all $x \in X$ and $0 < r < \ell/a$,

$$b\mu(B(x,r)) \leq \mu B(x, ar).$$

Throughout the paper we assume that (X, d, μ) is an *RD*-space and that $\mu(X) < \infty$.

REMARK 1.2. It is known that (X, d, μ) is an *RD*-space if and only if it is an *SHT* and there is a constant $\bar{c}$ such that for all $x \in X$ and $0 < r < \frac{\ell}{\bar{c}}$,

$$B(x, \bar{c}r) \backslash B(x, r) \neq \emptyset, \quad x \in X, \tag{1.6}$$

(for the proof we refer to see, e.g., [43, p. 11, Lemma 20]).

There are many interesting and useful for applications examples of an *SHT*. Among them it is a bounded domain Ω in $\mathbb{R}^n$ together with induced Lebesgue measure satisfying so called $\mathscr{A}$ condition: there is a positive constant C such that for all $x \in \overline{\Omega}$ and $\rho \in (0, \ell)$,

$$\mu(\tilde{B}(x, \rho)) \geq C\rho^n, \tag{1.7}$$

where ℓ is a diameter of Ω and $\tilde{B}(x, \rho) := \Omega \cap B(x, \rho)$.

In 1992 T. Iwaniec and C. Sbordone [18], in their studies related with the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of them, $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [13].

Harmonic Analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during last years due to various applications, we mention e.g. the papers by G. Anatriello, C. Capone, M. R. Formica, G. Di Fratta, A. Fiorenza, T. Futamura, B. Gupta, T. Iwaniec, P. Jain, G. E. Karadzhov, R. E. Castillo, V. Kokilashvili, P. Koskela, M. Krbec, A. Mercaldo, A. Meskhi, M. Milman, Y. Mizuta, T. Ohno, J. M. Rakotoson, H. Rafeiro, C. Sbordone, X. Zhong, S. Samko, Y. Sawano, X. Ye, etc (see also the monographs [26], [27] and references cited therein).

Throughout the paper, we will say that a constant $C_{\mu,\kappa} > 0$ is a structural constant if it depends only on the quasi-metric constants κ, and the doubling constant C_{dc}. The results of this manuscript are true for any quasi-metric space constants but for simplicity, sometimes in the proofs we will assume that $\kappa = 1$. In this case the structural constant $C_{\kappa,\mu}$ will be denoted by C_μ.

Morrey spaces $L^{p,\lambda}$ were introduced in 1938 by C. Morrey [33] in relation to regularity problems of solutions to PDEs, and provided a useful tool in the regularity theory of PDEs.

Let $\mu(X) < \infty$, $1 < p < \infty$, $-\frac{1}{p} < r < 0$, $\theta > 0$. Suppose that w is a weight function on X, i.e. w is μ- a.e. positive integrable function on X. We denote by $L^{p),r,\theta}(X)$ the space defined with respect to the norm (1.1). The symbol $L^s_w(X)$

$(1 \le s < \infty)$ denotes the classical weighted Lebesgue space defined by:

$$\|f\|_{L^s_w(X)} = \left(\int_X |f(x)|^s w(x)\, d\mu(x) \right)^{1/s} < \infty.$$

If $\theta = 0$, then $L^{p),r,\theta}(X)$ is the classical weighted Morrey space defined on an SHT which is denoted by $L^{p,r}(X)$ (see [11] for the definition of this type of the Morrey spaces norm). It is easy to see that $L^{p,r}_w(X) \hookrightarrow L^{p),r,\theta}_w(X)$.

We say that a weight function w belongs to the Muckenhoupt class $A_s(X)$ (or A_s) $1 < s < \infty$, if

$$[w]_{A_s} := \sup_B \left(\frac{1}{\mu(B)} \int_B w(x)\, d\mu(x) \right) \left(\frac{1}{\mu(B)} \int_B w^{1-s'}(x)\, d\mu(x) \right)^{s-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$. The symbol $[w]_{A_s}$ is called the characteristic of w. Further, a weight w belongs to $A_1(X)$ if $Mw(x) \le Cw(x)$ a.e., where

$$Mw(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B w(y)\, d\mu(y). \tag{1.8}$$

The characteristic $[w]_{A_1(X)}$ is defined as the essential supremum of Mw/w.

Since the $A_s(X)$ classes are increasing with respect to s one can define the class $A_\infty(X)$ in the natural way $A_\infty(X) = \bigcup_{s>1} A_s$. Further (see [16] and [17]),

$$[w]_{A_\infty} := \sup_B \left(\frac{1}{\mu(B)} w\, d\mu \right) \exp\left(\frac{1}{\mu(B)} \int_B \log w^{-1}\, d\mu \right).$$

There exists also another A_∞ characteristic due to [12]:

$$[w]^W_{A_\infty} := \sup_B \frac{1}{w(B)} \int_B M(w\chi_B)\, d\mu.$$

It can be checked (see also [17]) that $[w]^W_{A_\infty} \le C_{\kappa,\mu}[w]_{A_\infty}$ with some structural constant $C_{\kappa,\mu}$.

Lemma 1.3. *Let $1 < p < \infty$ and let w be a weight such that $w \in A_p(X)$. Then the measure $E \mapsto w(E)$ is doubling with doubling constant $C_{\kappa,\mu,w,p} := C^p_{dc}[w]_{A_p(X)}$.*

The proof is straightforward; therefore it is omitted (see also [15], Proposition 7.1.5).

To prove the main results of the paper we need some auxiliary statements. The following statement is known as Kolmogorov's theorem which we need because of the value of constant there (see [15], Exercise 2.1.5 for Euclidean spaces). The proof is similar to that in the case of Euclidean spaces but we mention the main idea of the proof for completeness.

Lemma 1.4. *Let $0 < \gamma < 1$ and let E be a μ measurable set with finite measure. Suppose that S is a sublinear operator of weak $(1,1)$ type with the operator norm $\|S\| := \|S\|_{L^1 \mapsto L^{1,\infty}}$. Then the following inequality holds:*

$$\Big(\frac{1}{\mu(E)} \int_E |Sf(x)|^\gamma \, d\mu\Big)^{1/\gamma} \le \frac{\|S\|}{1-\gamma} \Big(\frac{1}{\mu(E)} \int_E |f(x)| \, d\mu(x)\Big).$$

Proof. The proof follows from the representation

$$\int_E |S(f)|^\gamma \, d\mu = \int_0^\infty \gamma \lambda^{\gamma-1} \mu\{x \in X : (Sf)(x) > \lambda\} \, d\lambda$$
$$= \int_0^{\frac{\|S\| \, \|f\|_{L^1}}{\mu(E)}} (\cdots) \, d\lambda + \int_{\frac{\|S\| \, \|f\|_{L^1}}{\mu(E)}}^\infty (\cdots) \, d\lambda$$

and the weak $(1,1)$ type inequality for S in the second integral. $\square$

The next statement probably is known but since we deal with quasi-metric measure space and are interested in quantitative estimates we give the proof for completeness.

Lemma 1.5. *Let $0 < \gamma < 1$ and let f be a μ- a.e. positive locally integrable function. Then $(Mf)^\gamma \in A_1(X)$. Moreover,*

$$[(Mf)^\gamma]_{A_1} \le \frac{C_{\kappa,\mu}}{1-\gamma},$$

where $C_{\kappa,\mu}$ is a structural constant.

Proof. We follow the proof of [15], Theorem 9.2.7. Taking into account the definition of the A_1 class, it is enough to prove that

$$\frac{1}{\mu(B)} \int_B Mf(y)^\gamma \, d\mu(t) \le \frac{C_{\kappa,\mu}}{1-\gamma} (Mf)^\gamma(x) \tag{1.9}$$

for μ- a.e. $x \in B$. To prove (1.9) we use the representation $f = f_{1,B} + f_{2,B}$, where $f_{1,B} = f\chi_{5\kappa B}$ and $f_{2,B} = f\chi_{(5\kappa B)^c}$.

By using Lemma 1.4 and the fact that $0 < \gamma < 1$ we find that

$$\frac{1}{\mu(B)} \int_B (Mf_{1,B}(y))^\gamma \, d\mu(y) \le \Big(\frac{\|M\|_{L^1 \mapsto L^{1,\infty}}}{1-\gamma}\Big)^\gamma \Big(\frac{1}{\mu(B)} \int_X f_{1,B}(y) \, d\mu(y)\Big)^\gamma$$
$$\le \frac{C^{(1)}_{\kappa,\mu}}{1-\gamma} (Mf(x))^\gamma,$$

for a.e. $x \in B$ with positive structural constant $C^{(1)}_{\kappa,\mu}$. Further, due to simple geometric observations we have that for all $y, x \in B$,

$$M(f_{2,B})(y) \le C^{(2)}_{\kappa,\mu} (Mf)(x),$$

with another structural positive constant $C^{(2)}_{\kappa,\mu}$. Combining these estimate we get the desired result. □

LEMMA 1.6. *Let $1 \le \gamma < p < \infty$ and let $w \in A_{p/\gamma}$. Then there is $q_0 \in (\gamma, p)$, such that for all $q \in [\gamma, q_0]$ and all $s \in (1, s_0(q, w))$ with constant $s_0(q, w)$ depending on q and w the inequality*

$$\sup_{0<\varepsilon<\sigma} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}} \le C_{p,\sigma,q_0,k,\mu}[w]^{p'_\sigma-1}_{A_{p'_\varepsilon/\gamma}}$$

holds, where $p_\varepsilon := \frac{p-\varepsilon}{q}$, σ is a constant such that $0 < \sigma < p - q_0$ and the constant $C_{p,\sigma,q_0,\kappa,\mu}$ is defined by

$$C_{p,\sigma,q_0,\kappa,\mu} := 2^{(\frac{p-\sigma}{q_0})'-1}(4\kappa)^{(\frac{p-\sigma}{q_0})'C_{dc}}[2^{p/\gamma-1}(4\kappa)^{(p/\gamma)C_{dc}}]^{(p_\sigma)'-1}. \tag{1.10}$$

PROOF. Let $w \in A_{p/\gamma}(X)$. Then (see Theorem 1.2 of [17])

$$[w]_{A_{(p/\gamma)-\eta}(X)} \le 2^{(p/\gamma)-1}(4\kappa)^{(p/\gamma)C_{dc}}[w]_{A_{p/\gamma}(X)},$$

where $\eta := \frac{(p/\gamma)-1}{1+\tau_{\kappa,\mu}[w^{1-(p/\gamma)'}]_{A_\infty}}$,

$$\tau_{\kappa,\mu} = 6(32\kappa^2(4\kappa^2+\kappa)^2)^{C_{dc}}. \tag{1.11}$$

Let $\eta_1 = \frac{(p/\gamma)-1}{1+\tau_{\kappa,\mu}[w^{1-(p/\gamma)'}]_{A_{(p/\gamma)'}(X)}}$. Observe that $\eta_1 \le \eta$. Consequently,

$$[w]_{A_{(p/\gamma)-\eta_1} \le [w]_{A_{(p/\gamma)-\eta}}} \le 2^{(p/\gamma)-1}(4\kappa)^{(p/\gamma)C_{dc}}[w]_{A_{p/\gamma}(X)}. \tag{1.12}$$

Further, let us choose $q \le \gamma$ and $0 < \sigma < p - q_0$ so that $\frac{p}{\gamma} - \eta_1 = \frac{p-\sigma}{q_0}$. Then $\frac{p-\sigma}{q_0} < p_\varepsilon$ for all q and ε satisfying the conditions $\gamma \le q \le q_0$, $\varepsilon \le \sigma$. Moreover, by (1.12) we have that

$$[w]_{A_{p_\varepsilon}(X)} \le [w]_{A_{(p-\sigma)/q_0}} \le 2^{(p/\gamma)-1}(4\kappa)^{(pC_{dc})/\gamma}[w]_{A_{p/\gamma}(X)}. \tag{1.13}$$

Further, taking (1.13) and simple observations into account we find that

$$\begin{aligned}[w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon}(X)} = [w]^{p_\varepsilon-1}_{A_{p_\varepsilon(X)}} &\le [2^{\frac{p}{\gamma}-1}(4\kappa)^{\frac{pC_{dc}}{\gamma}}[w]_{A_{\frac{p}{\gamma}}(X)}]^{p'_\varepsilon-1} \\ &\le [2^{\frac{p}{\gamma}-1}(4\kappa)^{\frac{pC_{dc}}{\gamma}}]^{p'_\varepsilon-1}[w]^{p'_\sigma-1}_{A_{\frac{p}{\gamma}}}.\end{aligned} \tag{1.14}$$

Applying again Theorem 1.2 of [17] we get the inequality

$$\begin{aligned}[w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon-\eta_2}(X)} &\le 2^{p'_\varepsilon-1}(4\kappa)^{p'_\varepsilon C_{dc}}[w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon}(X)} \\ &\le 2^{(\frac{p-\sigma}{q_0})'-1}(4\kappa)^{(\frac{p-\sigma}{q_0})'C_{dc}}[w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon}(X)}\end{aligned} \tag{1.15}$$

holds, where

$$\eta_2 = \frac{p_\varepsilon' - 1}{1 + \tau_{\kappa,\mu}[w]_{A_{p\varepsilon}(X)}}$$

and $\tau_{\kappa,\mu}$ is defined by (1.11). Further, due to (1.13) we have that

$$[w]_{A_{p_\varepsilon}(X)} \le [w]_{A_{(p/\gamma-\sigma)/q_0}(X)} \le 2^{p-1}(4\kappa)^{(pC_{dc})/\gamma}[w]_{A_{p\sigma}(X)}. \tag{1.16}$$

That is why,

$$\eta_2 \ge \frac{p' - 1}{1 + \tau_{\kappa,\mu} 2^{p/\gamma-1}(4\kappa)^{(pC_{dc})/\gamma}[w]_{A_{p\sigma}(X)}} =: \eta_0.$$

Consequently, summarazing (1.14)–(1.16) we find that

$$[w^{1-p_\varepsilon'}]_{A_{p_\varepsilon'-\eta_0}(X)} \le [w^{1-p_\varepsilon'}]_{A_{p_\varepsilon'-\eta_2}(X)} \le C_{p,\sigma,q_0,\kappa,\mu}[w]^{p_\sigma'-1}_{A_{p/\gamma}(X)}.$$

Let us choose $1 < s_0 < s_0 = (q_0, \sigma, w)$ so that $\eta_0 > \frac{p_\sigma'}{s_0'}$. Then for all $0 < \varepsilon < \sigma$ and $1 < s < s_0$, $\eta_0 > p_\varepsilon'/s'$. Consequently, for all such ε and s we get $\frac{p_\varepsilon'}{s} > p_\varepsilon' - \eta_0$. Hence,

$$[w^{1-p_\varepsilon'}]_{A_{p_\varepsilon'/s}(X)} \le [w^{1-p_\varepsilon'}]_{A_{p_\varepsilon'-\eta_0}(X)} \le c_{p,\sigma,q_0,\kappa,\mu}[w]^{p_\sigma'-1}_{A_{p/\gamma}(X)}. \qquad \square$$

Lemma 1.7. *Let conditions of Lemma* 1.6 *be satisfied for* γ, p *and* w. *Then for all balls* B *and all measurable sets* $E \subset B$,

$$\frac{w(E)}{w(B)} \le \bar{C}_{RH}\left[\frac{\mu(E)}{\mu(B)}\right]^{\frac{\eta}{\eta+1}}, \tag{1.17}$$

where

$$\bar{C}_{RH} = 2(4\kappa)^{C_{dc}}[C_{dc}^{p/\gamma}[w]_{A_{p/\gamma}}]^{2+\log_2\kappa}, \tag{1.18}$$

$$\eta = \frac{1}{\tau_{\kappa,\mu}[w]_{A_{p/\gamma}(X)}} \tag{1.19}$$

and $\tau_{\kappa,\mu}$ *is defined by* (1.11).

Proof. By using Theorem 1.1 in [17] we have that for all balls B,

$$\left(\frac{1}{\mu(B)}\int_B w^{1+\bar{\eta}}\,d\mu\right)^{1/(1+\bar{\eta})} \le 2(4\kappa)^{C_{dc}}\left(\frac{1}{\mu(2\kappa B)}\int_{2\kappa B} w\,d\mu\right), \tag{1.20}$$

where $\bar{\eta} := \frac{1}{\tau_{\kappa,\mu}[w]_{A_\infty(X)}}$. Since $[w]_{A_\infty(X)} \le [w]_{A_{p/\gamma}(X)}$, we find that (1.20) holds for $\bar{\eta}$ replaced by η defined by (1.19). Hence,

$$\begin{aligned}
w(E) &= \int_B \chi_E w\, d\mu \le \Big(\int_B w^{1+\eta}\, d\mu\Big)^{1/(1+\eta)} \mu(E)^{\eta/(1+\eta)} \\
&= \Big(\frac{1}{\mu(B)}\int_B w^{1+\eta}\, d\mu\Big)^{1/(1+\eta)} \mu(B)^{1/(1+\eta)} \mu(E)^{\eta/(1+\eta)} \\
&\le 2(4\kappa)^{C_{dc}} \Big(\frac{1}{\mu(2\kappa B)}\int_{2\kappa B} w\, d\mu\Big) \mu(B)^{1/(1+\eta)} \mu(E)^{\eta/(1+\eta)} \\
&\le 2(4\kappa)^{C_{dc}} w(2\kappa B)\mu(B)^{-\eta/(1+\eta)} \mu(E)^{\eta/(1+\eta)} \\
&\le 2(4\kappa)^{C_{dc}} [C_{dc}^{p/\gamma}[w]_{p/\gamma}]^{2+\log_2\kappa} w(B)[\mu(E)/\mu(B)]^{\eta/(1+\eta)}.
\end{aligned}$$

In the last inequality we used Lemma 1.3. □

It is well-known the Muckenhoupt's theorem for an SHT (see e.g., [43]) stating that M is bounded in $L^p_w(X)$ if and only if $w \in A_p(X)$, where $1 < p < \infty$. There exists a sharper result of Buckley [3] type:

THEOREM A [17]. *Let $1 < p < \infty$. Then there is a structural constant $C_{\mu,\kappa}$ such that the following inequality holds:*

$$\|M\|_{L^p_w(X)} \le C_{\mu,\kappa} p' [w]_{A_p(X)}^{1/(p-1)}$$

with the structural constant $C_{\mu,\kappa}$.

2. MAIN RESULTS

X. Duoandikoetxea and M. Rosental in their recent paper [11] proved the following extrapolation results:

THEOREM A. *Let $1 \le p_0 < \infty$ and let $\mathscr{F}(X)$ be a collection of non-negative measurable pairs of functions defined on X. Suppose that for all $(f,g) \in \mathscr{F}(X)$ and for all $w \in A_{p_0}(X)$, the inequality*

$$\|g\|_{L^{p_0}_w(X)} \le C\|f\|_{L^{p_0}_w(X)} \tag{2.1}$$

holds, where the constant C does not depend on w. Then for every $1 < p < \infty$, $-1/p \le r < 0$, $\theta > 0$ and $w \in A_p(X)$, we have

$$\|g\|_{L^{p,r}_w(X)} \le \overline{C}\|f\|_{L^{p,r}_w(X)}, \quad (f,g) \in \mathscr{F}(X), \tag{2.2}$$

where $\overline{C}$ is the positive constant independent of $(f,g) \in \mathscr{F}(X)$.

THEOREM B. *Let $\mathscr{F}(X)$ be a family of pairs of functions (f,g), where f and g are defined on X. Suppose that for some $p_0 \in (0,\infty)$, and for all $w \in A_\infty$, we have*

$$\|g\|_{L^{p_0}_w(X)} \le C\|f\|_{L^{p_0}_w(X)}, \quad (f,g) \in \mathscr{F}(X) \tag{2.3}$$

for some positive constant C independent of w and (f, g). *Then for every* $1 < p < \infty$, $-1/p \le r < 0$, $\theta > 0$ *and* $w \in A_\infty(X)$, *we have,*

$$\|g\|_{L_w^{p,r}(X)} \le C\|f\|_{L_w^{p,r}(X)}, \quad (f, g) \in \mathscr{F}(X), \tag{2.4}$$

where the positive constant C is independent of (f, g).

One of the main results of this paper states:

Theorem 2.1. *Let* $1 \le p_0 < \infty$ *and let* $\mathscr{F}(X)$ *be a collection of non-negative measurable pairs of functions defined on X. Suppose that for all* $(f, g) \in \mathscr{F}(X)$ *and for all* $w \in A_{p_0}(X)$, *the inequality*

$$\|f\|_{L_w^{p_0}(X)} \le CN([w]_{A_{p_0}(X)})\|f\|_{L_w^{p_0}(X)} \tag{2.5}$$

holds, where N is a non-decreasing function and the constant C does not depend on (f, g) *and w. Then for every* $1 < p < \infty$, $-1/p \le r < 0$, $\theta > 0$ *and* $w \in A_p(X)$ *we have,*

$$\|g\|_{L_w^{p),r,\theta}(X)} \le C\bar{C}\|f\|_{L_w^{p),r,\theta}(X)}, \quad (f, g) \in \mathscr{F}(X), \tag{2.6}$$

where C is the constant from (2.5), *and the constant* $\bar{C}$ *is independent of* (f, g).

Extrapolation statement regarding A_∞ class of weights reads as follows:

Theorem 2.2. *Let* $\mathscr{F}(X)$ *be a family of pairs of functions* (f, g), *where f and g are defined on X. Suppose that for some* $p_0 \in (0, \infty)$, *and for all* $w \in A_\infty$ *we have*

$$\|g\|_{L_w^{p_0}(X)} \le CN([w]_{A_l(X)})\|f\|_{L_w^{p_0}(X)}, \quad (f, g) \in \mathscr{F}(X), \tag{2.7}$$

for some $l \ge 1$, *where N is non-decreasing function, and the constant C does not depend on w and* (f, g). *Then for every* $1 < p < \infty$, $-1/p \le r < 0$, $\theta > 0$ *and* $w \in A_\infty(X)$ *we have*

$$\|g\|_{L_w^{p),r,\theta}(X)} \le C\bar{C}\|f\|_{L_w^{p),r,\theta}(X)}, \quad (f, g) \in \mathscr{F}(X), \tag{2.8}$$

where C is the same constant as in (2.7) *and* $\bar{C}$ *is independent of* (f, g).

3. Proofs of the main results

In this section we prove the main results of this paper. First we formulate and prove the following statement:

Proposition 3.1. *Let* $1 \le \gamma < p < \infty$ *and let* $w \in A_{p/\gamma}(X)$. *Then there is* $q_0 \in (\gamma, p)$ *such that for all* $q \in [\gamma, q_0]$, *all* $s \in (1, s_0(q, w))$, *where* $s_0(q, w)$ *is the constant depending on q and w, all balls B and functions* $h \in L_w^{(p/q)'}(B)$ *with* $\|h\|_{L_w^{(p/q)'}(B)} = 1$, *the inequality*

$$\|f\|_{L^q_{(HW)_{s,B}}(X)} \le C(\mu(B))^{1/(p-\varepsilon)+r}\|f\|_{L_w^{p-\varepsilon,r}(X)} \tag{3.1}$$

holds, where

$$(HW)_{s,B}(x) := M(h^s w^s \chi_B)^{1/s}(x),$$

and the constant C is independent of f, B and ε.

PROOF. For simplicity assume that $\kappa = 1$. Let $w \in A_{p/\gamma}(X)$. Then by the openness property of the A_p class, there are $q_0 \in (1, p)$ and $\sigma \in (0, p - q_0)$ such that $w \in A_{(p-\sigma)/q_0}$. Hence, $w \in A_{(p-\sigma)/q}$ for all $q \in [\gamma, q_0]$. Fix q and set $p_\varepsilon := \frac{p-\varepsilon}{q}$, where $0 < \varepsilon < \sigma$. Let s be such a number that $1 < s < p_0'$ and that $w^{1-p_\varepsilon'} \in A_{p_\varepsilon'/s}(X)$. Such an s exists because $w \in A_{p_\varepsilon}(X)$ and, consequently, $w^{1-p_\varepsilon'} \in A_{p_\varepsilon'(X)}$. Let h be a function such that $\|h\|_{L_w^{p_\varepsilon'}(B)} = 1$. Observe that the weight $(HW)_{s,B}(x) := M(h^s w^s \chi_B)^{1/s}$ is well defined because $\int_X h^s w^s \chi_B \, d\mu < \infty$ and, consequently, $(HW)_{s,B}(x) < \infty$ a.e.. Let us check that $h^s w^s \chi_B \in L^1(X)$. We will see that the inequality holds:

$$\left(\int_X h^s w^s \chi_B \, d\mu\right)^{1/s} \le [w]_{A_{p_\varepsilon'/s}}^{1/p_\varepsilon'} w(B)^{1/p_\varepsilon} \mu(B)^{-1/s'} < \infty. \tag{3.2}$$

Indeed, by Hölder's inequality with respect to the exponent p_ε'/s and the fact that $w^{1-p_\varepsilon'} \in A_{p_\varepsilon'/s}$ we find that

$$\begin{aligned}
\left(\int_X h^s w^s \chi_B \, d\mu\right)^{1/s} &\le \left(\int_B h^{p_\varepsilon'} w \, d\mu\right)^{1/p_\varepsilon'} \left(\int_B w^{(s-1)\frac{p_\varepsilon'}{p_\varepsilon'-s}+1} \, d\mu\right)^{\frac{p_\varepsilon'-s}{p_\varepsilon' s}} \\
&\le [w]_{A_{p_\varepsilon'/\gamma}(X)}^{1/p_\varepsilon'} \mu(B)^{1/s} \left(\int_B w^{1-p_\varepsilon'} \, d\mu\right)^{-1/p_\varepsilon'} \\
&\le [w]_{A_{p_\varepsilon'}}^{1/p_\varepsilon'/\gamma} \left(\int_B w^{-1} w \, d\mu\right)\left(\int_B w^{1-p_\varepsilon'} \, d\mu\right)^{-1/p_\varepsilon'} \mu(B)^{-1/s'} \\
&\le [w]_{A_{p_\varepsilon'/\gamma}}^{1/p_\varepsilon'} \left(\int_B w^{1-p_\varepsilon'} \, d\mu\right)^{1/p_\varepsilon'} \left(\int_B w^{1-p_\varepsilon'} \, d\mu\right)^{-1/p_\varepsilon'} w(B)^{1/p_\varepsilon} \mu(B)^{-1/s'} \\
&= [w]_{A_{p_\varepsilon'/\gamma}}^{1/p_\varepsilon'} w(B)^{1/p_\varepsilon} \mu(B)^{-1/s'} < \infty.
\end{aligned}$$

Let now $f \in L_w^{p-\varepsilon, r}(X)$ and let f is non-negative. We have to estimate the norm of f in $L^q_{(HW)_{s,B}}(X)$. Using the representation $f = f_{1,B} + f_{2,B}$, where $f_{1,B} = f\chi_{2B}$, $f_{2,B} = f - f_{1,B}$ and B has sufficiently small radius, we get:

$$\begin{aligned}
\left(\int_X f^q(y)(HW)_{s,B}(y) \, d\mu(y)\right)^{1/q} &\le \left(\int_X f_{1,B}^q(y)(HW)_{s,B}(y) \, d\mu(y)\right)^{1/q} \\
&\quad + \left(\int_X f_{2,B}^q(y)(HW)_{s,B}(y) \, d\mu(y)\right)^{1/q} \\
&=: I_1 + I_2.
\end{aligned}$$

Due to Hölder's inequality with respect to the exponents p_ε and p'_ε, the doubling condition, Theorem A, Lemma 1.6 for $\kappa = 1$ we find that

$$
\begin{aligned}
I_1 &\le \Big(\int_{2B} f^{p-\varepsilon}(y)w(y)\,d\mu(y)\Big)^{1/(p-\varepsilon)} \Big(\int_{2B} (HW)^{p'_\varepsilon}_{s,B}(y)w^{1-p'_\varepsilon}\,d\mu(y)\Big)^{1/(qp'_\varepsilon)} \\
&\le w(2B)^{\frac{1}{p-\varepsilon}+r} w(2B)^{-\frac{1}{p-\varepsilon}+r}\Big(\int_{2B} f^{p-\varepsilon}(y)w(y)\,d\mu(y)\Big)^{\frac{1}{p-\varepsilon}} \\
&\quad\times \Big(\int_{2B} (HW)^{p'_\varepsilon}_{s,B}(y)w^{1-p'_\varepsilon}\,d\mu(y)\Big)^{\frac{1}{(qp'_\varepsilon)}} \\
&\le \|f\|_{L^{p-\varepsilon,r}_w(X)} w(2B)^{1/(p-\varepsilon)+r}\Big(\int_X (HW)^{p'_\varepsilon}_{s,B}(y)w^{1-p'_\varepsilon}\,d\mu(y)\Big)^{1/(p'_\varepsilon q)} \\
&\le C_\mu^{1/q}\Big(\frac{p'_\varepsilon}{s}\Big)'[w^{1-p'_\varepsilon}]^{[\frac{s}{p'_\varepsilon-s}-1]/q}_{A_{p'_\varepsilon/s}(X)}\Big(\int_B (h^s w^s)^{p'_\varepsilon/s} w^{1-p'_\varepsilon}\,d\mu\Big)^{1/(qp'_\varepsilon)} \\
&\quad\times [C_{dc}^{p/\gamma}[w]_{A_{p/\gamma}(X)}]^{1/(p-\varepsilon)+r} w(B)^{1/(p-\varepsilon)+r}\|f\|_{L^{p-\varepsilon,r}_w(X)} \\
&\le C_\mu^{1/q}(C_{p,\sigma,q_0,\mu}[w]^{p'_\sigma-1}_{A_{p/\gamma}(X)})^{\frac{s}{p'_\varepsilon-s}-1}[C_{dc}^{p/\gamma}[w]_{A_{p/\gamma}(X)}]^{1/(p-\varepsilon)+r} \\
&\quad\times \|f\|_{L^{p-\varepsilon,r}_w(X)}\|h\|^{1/q}_{L^{p'_\varepsilon(B)}_w}\mu(B)^{1/(p-\varepsilon)+r} \\
&\le C_\mu^{1/q}(C_{p,\sigma,q_0,\mu}[w]^{p'_\sigma-1}_{A_{p/\gamma}(X)})^{\frac{s}{p'_\varepsilon-s}-1}[C_{dc}^{p/\gamma}[w]_{A_{p/\gamma}(X)}]^{1/(p-\sigma)}\|f\|_{L^{p-\varepsilon,r}_w(X)},
\end{aligned}
$$

where $C_{p,\sigma,q_0,\mu}$ is defined by (1.10). Thus, $I_1 \le \overline{C}^{(1)}_{p,\sigma,w}$, with constant $\overline{C}^{(1)}_{p,\sigma,w}$ defined by

$$
\overline{C}^{(1)}_{p,\sigma,w} := C_\mu^{1/q}(C_{p,\sigma,q_0,\mu}[w]^{p'_\sigma-1}_{A_{p/\gamma}(X)})^{\frac{s}{p'_\varepsilon-s}-1}[C_{dc}^{p/\gamma}[w]_{A_{p/\gamma}(X)}]^{1/(p-\sigma)}. \tag{3.3}
$$

Now we estimate I_2. First observe that if $x \in 2^{j+1}B\backslash 2^jB$, $j \in \mathbb{Z}$, then

$$
Mg(x) \le \frac{C_\mu}{\mu(2^jB)}\int_{2^jB}|g|\,d\mu,
$$

with a structural constant C_μ.

Using this observation, (3.2), Hölder's inequality, the fact that $w \in A_{p_\varepsilon}(X)$ and Lemma 1.7 we find that

$$
\begin{aligned}
&\int_{X\backslash 2B} f^q (HW)_{s,B}\,d\mu \\
&\quad\le C_\mu^{1/s}\Big(\sum_j \int_{2^{j+1}B\backslash 2^{j+1}B} f^q\,d\mu\Big)\Big(\frac{1}{\mu(2^jB)}\int_{2^jB} h^s w^s\,d\mu\Big)^{1/s} \\
&\quad\le C_\mu^{1/s}\Big[\sum_j \int_{2^{j+1}B\backslash 2^{j+1}B} f^q w^{\frac{1}{p_\varepsilon}} w^{-\frac{1}{p_\varepsilon}}\,d\mu\Big][w^{1-p'_\varepsilon}]^{1/p'_\varepsilon}_{A_{p'_\varepsilon/s}}
\end{aligned}
$$

$$\times \frac{1}{\mu(B)^{1/s}} w(B)^{1/p_\varepsilon} \mu(B)^{-1/s'} \mu(2^j B)^{-1/s}$$

$$\leq C_\mu^{1/s} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} \left[\sum_j \int_{2^{j+1}B} f^{p-\varepsilon} w \, d\mu \right]^{1/p_\varepsilon}$$

$$\times (w^{1-p_\varepsilon}(2^{j+1}B))^{1/p'_\varepsilon} \mu(B)^{-1/s} w(B)^{1/p_\varepsilon} \mu(B)^{-1/s'} \mu(2^j B)^{-1/s}$$

$$\leq C_\mu^{1/s} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} \|f\|^q_{L_w^{p-\varepsilon, r}(X)} \sum_j w(2^{j+1}B)^{(1/(p-\varepsilon)+r)q}$$

$$\times [w^{1-p'_\varepsilon}(2^{j+1}B)]^{1/p'_\varepsilon} w(B)^{1/p_\varepsilon} \mu(2^j B)^{-1/s} \mu(B)^{-1/s'}$$

$$\leq C_\mu^{1/s} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} [w]_{A_{p_\varepsilon}}^{1/p_\varepsilon} \|f\|^q_{L_w^{p-\varepsilon, r}(X)}$$

$$\times \sum_j w(2^{j+1}B)^{rq} \mu(2^{j+1}B) \mu(2^{j+1}B)^{-1/s} \mu(B)^{-1/s'} w(B)^{1/p_\varepsilon}$$

$$\leq C_\mu^{1/s} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} [w]_{A_{p_\varepsilon}}^{1/p_\varepsilon} \|f\|^q_{L_w^{p-\varepsilon, r}(X)} w(B)^{1/p_\varepsilon}$$

$$\times \sum_j w(2^{j+1}B)^{rq} \mu(2^{j+1}B) \mu(2^{j+1}B)^{-1/s} \mu(B)^{-1/s'} w(B)^{1/p_\varepsilon}$$

$$\leq C_\mu^{1/s} C_{dc} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} [w]_{A_{p_\varepsilon}}^{1/p_\varepsilon} \|f\|^q_{L_w^{p-\varepsilon, r}(X)} \sum_j w(2^{j+1}B)^{rq} C_{dc}^{j/s'} (w(B)^{1/p_\varepsilon})$$

$$\leq C_\mu^{1/s} C_{dc} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} [w]_{A_{p_\varepsilon}}^{1/p_\varepsilon} \|f\|^q_{L_w^{p-\varepsilon, r}(X)} (\overline{C}_{RH})^{rq} \sum_j (C_{dc}^{j\frac{rq\eta}{\eta+1} + j/s'}) w(B)^{rq+1/p_\varepsilon}$$

$$\leq C_\mu^{1/s} C_{dc} C_{p,\sigma,q_0,\mu} [w]_{A_{p/\gamma}}^{\frac{p'\sigma-1}{p'_\varepsilon}} [w^{1-p'_\varepsilon}]_{A_{p'_\varepsilon/s}}^{1/p'_\varepsilon} [w]_{A_{p_\varepsilon}}^{1/p_\varepsilon} (\overline{C}_{RH})^{rq}$$

$$\times \frac{C_{dc}^{\frac{1}{s'} + \frac{rq\eta}{\eta+1}}}{1 + C_{dc}^{\frac{1}{s'} + \frac{rq\eta}{\eta+1}}} w(B)^{rq+1/p_\varepsilon} \|f\|^q_{L_w^{p-\varepsilon, r}(X)},$$

where $C_{p,\sigma,q_0,\mu}$ is defined by (1.10). Here we assumed that s can be chosen so small that the sum/series with respect to j is convergent. Thus, we conclude that $I_2 \leq \overline{C}^{(2)}_{p,\varepsilon,w}$, where

$$\overline{C}^{(2)}_{p,\sigma,w} := \left[C_\mu^{1/s} C_{dc} (\overline{C}_{RH})^{rq} c_{p,\sigma,q_0,\mu} [w]_{A_{p'_\varepsilon/\gamma}}^{\frac{p'_\sigma - 1}{p'_\varepsilon}} [w]_{A_{p_\varepsilon}}^{1/p_\varepsilon} \frac{C_{dc}^{\frac{1}{s'} + \frac{rq\eta}{\eta+1}}}{1 + C_{dc}^{\frac{1}{s'} + \frac{rq\eta}{\eta+1}}} \right]^{1/q}. \tag{3.4}$$

Finally we conclude that (3.3) and (3.3) imply

$$I_1 + I_2 \leq \overline{C}^{(1)}_{p,\sigma,w} + \overline{C}^{(2)}_{p,\sigma,w} := \overline{C}_{p,\sigma,w}. \tag{3.5}$$

This completes the proof of the proposition. □

Proof of Theorem 2.1. For simplicity assume that $\kappa = 1$. It is enough to show that

$$\sup_{0<\varepsilon<\sigma} \varepsilon^{\theta}(\mu(B))^{-(1/(p-\varepsilon)+r)}\Big(\int_B g^{p-\varepsilon} w\, d\mu\Big)^{1/(p-\varepsilon)} \le C_{\mu,\varepsilon,w}\|f\|_{L_w^{p),r,\theta}(X)}$$

for some sufficiently small positive σ because by Hölder's inequality we have that for $\sigma < \varepsilon < p-1$,

$$(\mu(B))^{-(1/(p-\varepsilon)+r)}\Big(\int_B g^{p-\varepsilon} w\, d\mu\Big)^{1/(p-\varepsilon)} \le (\mu(B))^{-(1/(p-\sigma)+r)}\Big(\int_B g^{p-\sigma} w\, d\mu\Big)^{1/(p-\sigma)}.$$

As before, we denote: $(MH)_{B,s} := (Mh^s w^s \chi_B)^{1/s}$. Suppose that $1 < p < \infty$. By the classical extrapolation theorem (see [10] for Euclidean spaces and [21] for an SHT) we have that

$$\|g\|_{L_w^p(X)} \le C\varphi([w]_{A_p(X)})\|f\|_{L_w^p(X)}, \quad w \in A_p(X), \tag{3.6}$$

for all $(f,g) \in \mathscr{F}$, where C is the constant independent of (f,g) and w, and $x \mapsto \varphi(x)$ is an increasing function. Fix $w \in A_p(X)$. Choose $q \in (1,p)$ and $s > 1$ so that inequality (3.1) holds. We set $p_\varepsilon := \frac{p-\varepsilon}{q}$. Let us fix a ball $B \subset X$. It is obvious that

$$\Big(\int_B g^{p-\varepsilon} w\, d\mu\Big)^{1/(p-\varepsilon)} = \Big(\int_B g^{p_\varepsilon q} w\, d\mu\Big)^{1/(p_\varepsilon q)} = \sup_{\|h\|_{L_w^{p'_\varepsilon}(X)}=1}\Big(\int_B g^q h w\, d\mu\Big)^{1/q}$$

Let us fix such an h. By Lemma 1.5 we have that $[(MH)_{B,s}]_{A_q} \le [(MH)_{B,s}]_{A_1} \le \frac{C_\mu}{1-s^{-1}}$. Further, observe that (3.6) implies that

$$\Big(\int_X g^q w\, d\mu\Big)^{1/q} \le C_\mu \varphi([w]_{A_q(X)})\Big(\int_X f^q w\, d\mu\Big)^{1/q}$$

for all $(f,g) \in \mathscr{F}$ and all $w \in A_q(X)$, where $x \mapsto \varphi(x)$ is non-decreasing function. Consequently, by Proposition 3.1 and Lemma 1.5,

$$\begin{aligned}
\Big(\int_X g^q h w \chi_B\, d\mu\Big)^{1/q} &\le \Big(\int_X g^q (MH)_{B,s}\, d\mu\Big)^{1/q} \\
&\le C\varphi([(MH)_{B,s}]_{A_q(X)})\Big(\int_X f^q (MH)_{B,s}\, d\mu\Big)^{1/q} \\
&\le C\overline{C}_{p,\varepsilon,w}\varphi([(MH)_{B,s}]_{A_q(X)})\mu(B)^{1/(p-\varepsilon)+r}\|f\|_{L_w^{p-\varepsilon,r}(X)} \\
&\le C\overline{C}_{p,\varepsilon,w}\varphi([(MH)_{B,s}]_{A_1(X)})\mu(B)^{1/(p-\varepsilon)+r}\varepsilon^{-\theta}\|f\|_{L_w^{p),r,\theta}(X)} \\
&\le C\overline{C}_{p,\sigma,w}\varphi\Big(\frac{C_\mu}{1-s^{-1}}\Big)\mu(B)^{1/(p-\varepsilon)+r}\varepsilon^{-\theta}\|f\|_{L_w^{p),r,\theta}(X)},
\end{aligned}$$

where $\overline{C}_{p,\sigma,w}$ is defined by (3.5).

Thus,

$$\varepsilon^{\theta}(\mu(B))^{-(1/(p-\varepsilon)+r)}\Big(\int_B g^{p-\varepsilon}w\,d\mu\Big)^{1/(p-\varepsilon)} \le C\|f\|_{L_w^{p),r,\theta}(X)}.$$

The theorem has been proved. □

PROOF OF THEOREM 2.2. Let (2.7) hold for some $p_0 > 0$. Then by the classical A_∞ extrapolation result (see [9] for Euclidean spaces and [23], Theorem A, for an SHT) we have that

$$\|g\|_{L_w^p(X)} \le C_p\psi([w]_{A_p})\|f\|_{L_w^p(X)} \tag{3.7}$$

for all $1 < p < \infty$, all $w \in A_p$, for some constant C_p and increasing function ψ.

Let $1 < p < \infty$ and let $w \in A_\infty$. We will show that (2.8) holds for this w and all $(f, g) \in \mathscr{F}$. If $p \ge q$, then $A_q \subset A_p$, and by (3.7) and Theorem 2.1 we get that (2.8) holds for that w and all $(f, g) \in \mathscr{F}$.

Let now $p < q$. Since $w \in A_q$ we have that $w \in A_{q-\sigma}$ for some small positive σ. Consequently, $w \in A_{q-\eta}$ for all η satisfying $0 < \eta < \sigma$. Hence, by (3.7) we find that

$$\|\,|g|^{\frac{p-\varepsilon}{q-\eta}}\|_{L^{q-\eta,\frac{r(p-\varepsilon)}{q-\eta}}(X)} \le C_{p,q,\varepsilon,\eta}\psi([w]_{A_{q-\eta}})\|\,|f|^{\frac{p-\varepsilon}{q-\eta}}\|_{L^{q-\eta,\frac{r(p-\varepsilon)}{q-\eta}}(X)} \tag{3.8}$$

with the positive constant $C_{p,q,\varepsilon,\eta}$ and for some increasing function ψ. Since $[w]_{A_{q-\eta}} \le [w]_{A_{q-\sigma}}$ and $\sup_{\varepsilon,\eta} C_{p,q,\varepsilon,\eta} < \infty$ (see the proof of Theorem 2.1 for the latter fact) we have that

$$\sup_{\varepsilon,\eta} C_{p,q,\varepsilon,\eta}\psi([w]_{A_{q-\eta}}) < \infty,$$

where the supremum is taken over all sufficiently small η and ε. From (3.8) we derive

$$\|g\|^{\frac{p-\varepsilon}{q-\eta}}_{L^{p-\varepsilon,r}(X)} \le C_{p,q,\varepsilon,\eta}\psi([w]_{A_{q-\eta}})\|f\|^{\frac{p-\varepsilon}{q-\eta}}_{L^{p-\varepsilon,r}(X)}. \tag{3.9}$$

Raising both sides of (3.9) to the power $\frac{q-\eta}{p-\eta}$ and multiplying on ε^θ we get the desired result. □

4. APPLICATIONS TO ONE-WEIGHT INEQUALITIES FOR COMMUTATORS OF SINGULAR INTEGRALS

Theorems proved above give the one-weight boundedness in $L_w^{p),r,\theta}(X)$ spaces for operators of Harmonic Analysis satisfying one-weight inequality in the classical Lebesgue spaces under the Muckenhoupt condition. Such operators are, e.g., maximal, Calderón–Zygmund, fractional integral operators and their commutators. Here we are focused on some of these operators.

Below we denote by $\mathscr{D}(f)$ the class of all essentially bounded functions on X.

Let us recall the definition of the Calderón–Zygmund kernel on quasi-metric measure spaces.

Let $k : X \times X \backslash \{(x,x) : x \in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \leq \frac{c}{\mu B(x, d(x,y))}, \quad x, y \in X,\ x \neq y;$$

$$|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \leq c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big) \frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2,y) > d(x_1,x_2)$, where ω is a positive, non-decreasing function on $(0,\infty)$ satisfying Δ_2 condition ($\omega(2t) \leq c\omega(t)$, $t > 0$) and the Dini condition $\int_0^1 \omega(t)/t\,dt < \infty$.

We also assume that for some p_0, $1 < p_0 < \infty$, and all $f \in L^{p_0}(X,\mu)$ the limit

$$(Tf)(x) = \lim_{\varepsilon \to 0} \int_{X \backslash B(x,\varepsilon)} k(x,y) f(y)\, d\mu(y)$$

exists almost everywhere on X and that T is bounded in $L^{p_0}(X,\mu)$.

It is known (see [36]) that there is a constant $\widetilde{c_0} := \widetilde{c_0}([w]_{A_\infty})$ independent of f and depending on $[w]_{A_\infty}$ such that for all $f \in \mathscr{D}(X)$.

$$\|Tf\|_{L^{p_0}_w(X,\mu)} \leq \widetilde{c_0}([w]_{A_l}) \|Mf\|_{L^{p_0}_w(X,\mu)}, \quad f \in \mathscr{D}(X),\ w \in A_\infty(X),$$

where $l \geq 1$, T is the Calderón–Zygmund singular integral defined on X and the mapping $x \to \widetilde{c_0}(x)$ is non-decreasing on $(1,\infty)$.

Taking extrapolation Theorem 3.1 into account we have the next statement:

Theorem 4.1. *Let $1 < p < \infty$, $-1/p \leq r < 0$ and let $\theta > 0$. Then there is a positive constant C such that for all $f \in \mathscr{D}(X)$ and all $w \in A_\infty(X)$,*

$$\|Tf\|_{L^{p),r,\theta}_w(X)} \leq C\|Mf\|_{L^{p),r,\theta}_w(X)}.$$

We say that a function b defined on X belongs to BMO if

$$\|b\|_{BMO} = \frac{1}{\mu(B)} \int_B |b(x) - b_B|\, d\mu(x) < \infty,$$

where $b_B = \frac{1}{\mu(B)} b(y)\, d\mu(y)$.

Let $b \in BMO(X)$, $m \in \mathbb{N} \cup \{0\}$ and let

$$T_b^m f(x) = \int_X [b(x) - b(y)]^m k(x,y) f(y)\, d\mu(y),$$

where k is the Calderón–Zygmund kernel.

It is known (see [36]) that if $1 < r < \infty$ and $w \in A_\infty$, then the one-weight inequality

$$\|T_b^m f\|_{L_w^r(X)} \le C\|b\|_{BMO(X)}^m \|M^{m+1} f\|_{L_w^r(X)}, \quad f \in \mathscr{D}(X),$$

where M^{m+1} is the the Hardy–Littlewood maximal operator iterated $m+1$ times. Thus we have

Theorem 4.2. *Let $1 < p < \infty$, $-1/p \le r < 0$ and let $\theta > 0$. Then there is a positive constant C such that for all $f \in \mathscr{D}(X)$ and all $w \in A_\infty(X)$,*

$$\|T_b^m f\|_{L_w^{p),r,\theta}(X)} \le C\|b\|_{BMO(X)}^m \|M^{m+1} f\|_{L_w^{p),r,\theta}(X)}, \quad f \in \mathscr{D}(X).$$

Corollary 4.3. *Let $1 < p < \infty$, $-1/p \le r < 0$ and let $\theta > 0$. Then there is a positive constant C such that for all $f \in \mathscr{D}(X)$ and all $w \in A_p(X)$,*

$$\|Tf\|_{L_w^{p),r,\theta}(X)} \le C\|f\|_{L_w^{p),r,\theta}(X)}.$$

Corollary 4.4. *Let $1 < p < \infty$, $-1/p \le r < 0$ and let $\theta > 0$. Then there is a positive constant C such that for all $f \in \mathscr{D}(X)$ and all $w \in A_p(X)$,*

$$\|T_b^m f\|_{L_w^{p),r,\theta}(X)} \le C\|b\|_{BMO(X)}^m \|f\|_{L_w^{p),r,\theta}(X)}, \quad f \in \mathscr{D}(X).$$

Let

$$I_\alpha f(x) = \int_X K_\alpha(x, y) f(y)\, d\mu(y), \quad x \in X,$$

be fractional integral operator defined on (X, d, μ), where

$$K_\alpha(x, y) = \begin{cases} \mu(B_{xy})^{\alpha-1}, & x \ne y, \\ \mu\{x\}, & x = y,\ \mu\{x\} > 0, \end{cases}$$

$0 < \alpha < 1$ and $B_{xy} := B(x, d(x, y))$.

Suppose that

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y), \quad 0 < \alpha < 1.$$

It is obvious the pointwise inequality $I_\alpha f(x) \le C_{\alpha,\kappa,\mu} Mf(x)$ for all non-negative f.

It is known that (see [2]) if $0 < r < \infty$ and $w \in A_\infty(X)$, then the inequality

$$\|I_\alpha f\|_{L_w^r(X)} \le CN([w]_{A_\infty})\|M_\alpha f\|_{L_w^r(X)},$$

holds for some positive constant $C \equiv C_{\alpha,r,\kappa,\mu}$ and increasing N. That is why based on this result and Theorem 2.2 we have the next statement

Theorem 4.5. *Let $1 < p < \infty$, $-1/p < r < 0$ and let $\theta > 0$. Let $w \in A_\infty(X)$. Then there is a positive constant C such that*

$$\|I_\alpha f\|_{L_w^{p),r,\theta}(X)} \le C\|M_\alpha f\|_{L_w^{p),r,\theta}(X)}, \quad f \in \mathscr{D}(X).$$

Further, for $b \in BMO(X)$, let

$$I^m_{\alpha,b} f(x) = \int_X [b(x) - b(y)]^m K_\alpha(x,y)\,d\mu(y), \quad 0 < \alpha < 1,$$

$$\mathscr{I}^m_{\alpha,b} f(x) = \int_X |b(x) - b(y)|^m K_\alpha(x,y)\,d\mu(y), \quad 0 < \alpha < 1.$$

It is easy to see that, for $f \ge 0$, $|I^m_{\alpha,b} f(x)| \le \mathscr{I}^m_{\alpha,b} f(x)$. In the same paper [2] the authors showed that if $0 < p < \infty$, $0 < \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $w \in A_\infty(X)$, $b \in BMO(X)$, then there is a constant $C \equiv C_{\alpha,m,p,\kappa,\mu}$ such that

$$\int_X |\mathscr{I}^m_{\alpha,b} f(x)|^p w(x)\,d\mu(x) \le CN([w]_{A_\infty})\|b\|^{mp}_{BMO(X)} \int_X [M_\alpha(M^m f)(x)]^p w(x)\,d\mu(x)$$

for some non-decreasing function N. Hence we have the following statement:

Theorem 4.6. *Let $1 < p < \infty$, $m \in \mathbb{N} \cup \{0\}$, $-1/p < r < 0$, and let $\theta > 0$. Suppose that $w \in A_\infty(X)$. Then there is a positive constant C such that*

$$\|\mathscr{I}^m_{\alpha,b} f\|_{L_w^{p),r,\theta}(X)} \le C\|b\|^m_{BMO(X)}\|M_\alpha(M^m f)\|_{L_w^{p),r,\theta}(X)}, \quad f \in \mathscr{D}(X)$$

Corollary 4.7. *Let $1 < p < \infty$, $-1/p < r < 0$ and let $\theta > 0$. Let $w \in A_p(X)$. Then there is a positive constant C such that*

$$\|I_\alpha f\|_{L_w^{p),r,\theta}(X)} \le C\|f\|_{L_w^{p),r,\theta}(X)}, \quad f \in \mathscr{D}(X).$$

Corollary 4.8. *Let $1 < p < \infty$, $m \in \mathbb{N} \cup \{0\}$, $-1/p < r < 0$ and let $\theta > 0$. Let $w \in A_p(X)$. Then there is a positive constant C such that*

$$\|\mathscr{I}^m_{\alpha,b} f\|_{L_w^{p),r,\theta}(X)} \le C\|b\|^m_{BMO(X)}\|f\|_{L_w^{p),r,\theta}(X)}, \quad f \in \mathscr{D}(X).$$

5. Applications to PDE

In the last thirty years a number of papers have been devoted to the study of local and global regularity properties of strong solutions to elliptic equations with discontinuous coefficients. To be more precise, let us consider the second order equation

$$\mathscr{L}u \equiv \sum_{i,j=1}^n a_{ij}(x) D_{x_i x_j} u = f(x) \quad \text{for almost all } x \in \Omega, \tag{5.1}$$

where $\mathscr{L}$ is a uniformly elliptic operator over the bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

We assume that a domain Ω satisfies $\mathscr{A}$ condition (see (1.7)). In this case Ω with induced Lebesgue measure and Euclidean metrics is an SHT. Hence, the statements of Section 4 are valid for such domains.

Regularizing properties of $\mathscr{L}$ in Hölder spaces (i.e. $\mathscr{L}u \in C^\alpha(\overline{\Omega})$ implies $u \in C^{2+\alpha}(\overline{\Omega})$) have been well studied in the case of Hölder continuous coefficients $a_{ij}(x)$. Also, unique classical solvability of the Dirichlet problem for (5.1) has been derived in this case (we refer to [14] and the references therein). In the case of uniformly continuous coefficients a_{ij}, an L^p-Schauder theory has been elaborated for the operator $\mathscr{L}$ ([1], [14]). In particular, $\mathscr{L}u \in L^p(\Omega)$ always implies that the strong solution to (5.1) belongs to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1, \infty)$.

However, the situation becomes rather difficult if one tries to allow discontinuity at the principal coefficients of $\mathscr{L}$. In general, it is well-known (cf. [31]) that arbitrary discontinuity of a_{ij} implies that the L^p-theory of $\mathscr{L}$ and the strong solvability of the Dirichlet problem for (5.1) break down. A notable exception of that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). It was shown by G. Talenti ([44]) that the solely condition on measurability and boundedness of the a_{ij}'s ensures isomorphic properties of $\mathscr{L}$ considered as a mapping from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ into $L^2(\Omega)$. To handle with the multidimensional case ($n \geq 3$) requires that additional properties on $a_{ij}(x)$ should be added to the uniform ellipticity in order to guarantee that $\mathscr{L}$ possesses the regularizing property in Sobolev functional scales. In particular, if $a_{ij}(x) \in W^{1,n}(\Omega)$ (cf. [32]), or if the difference between the largest and the smallest eigenvalues of $\{a_{ij}(x)\}$ is small enough (the Cordes condition), then $\mathscr{L}u \in L^2(\Omega)$ yields that $u \in W^{2,2}(\Omega)$ and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2 - \varepsilon, 2 + \varepsilon)$ with sufficiently small ε.

Later the Sarason class VMO of functions with vanishing mean oscillation was used in the study of local and global Sobolev regularity of the strong solutions to (5.1).

Next, we define the space BMO and then the smallest VMO class, where we consider coefficients a_{ij} and later that one where we consider the known term f.

In the sequel let Ω be an open bounded set in $\mathbb{R}^n$.

DEFINITION 5.1. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. We define the integral mean $f_{x,R}$ by

$$f_{x,R} := \frac{1}{|\Omega \cap B(x,R)|} \int_{\Omega \cap B(x,R)} f(y)\, dy,$$

where $B(x,R)$ ranges in the class of balls centered in x with radius R and $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$.

If we are not interested in specifying which the center is, we just use the notation f_R.

We now give the definition of Bounded Mean Oscillation functions (BMO) that appeared at first in the note by F. John and L. Nirenberg [19].

Definition 5.2. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. We say that f belongs to $BMO(\Omega)$ if the seminorm $\|f\|_*$ is finite, where

$$\|f\|_* := \sup_{B(x,R)} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y) - f_{x,R}|\, dy.$$

Next, we consider the definition of the space of Vanishing Mean Oscillation functions, given at first by D. Sarason in [41].

Definition 5.3. Let $f \in BMO(\Omega)$ and

$$\eta(f,R) := \sup_{\rho \le R} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_\rho|\, dy,$$

where $B\rho$ ranges over the class of the balls of $\mathbb{R}^n$ of radius ρ.

A function $f \in VMO(\Omega)$ if

$$\lim_{R \to 0} \eta(f,R) = 0.$$

The Sarason class is then expressed as the subspace of the functions in the John–Nirenberg class whose BMO norm over a ball vanishes as the radius of the balls tends to zero. This property implies a number of good features of VMO functions not shared by general BMO functions; in particular, they can be approximated by smooth functions.

This class of functions was considered by many others. At first, we recall the paper by F. Chiarenza, M. Frasca and P. Longo [6], where the authors answer a question raised thirty years before by C. Miranda in [32]. In his paper he considers a linear elliptic equation where the coefficients a_{ij} of the higher order derivatives are in the class $W^{1,n}(\Omega)$ and asks whether the gradient of the solution is bounded, if $p > n$. In [6] the authors suppose that $a_{ij} \in VMO$ and prove that Du is Hölder continuous for all $p \in\,]1, +\infty[$.

We observe that $W^{1,n} \subset VMO$, this fact follows by using Poincaré's inequality

$$\frac{1}{|B|}|f(x) - f_B| \le c(n)\Big(\int_B |\nabla u|\, dx\Big)^{\frac{1}{n}}$$

and the term on the right-hand side tends to zero as $|B| \to 0$.

We point out that C^0 is strictly contained in VMO.

Also, it is possible to check that bounded uniformly functions are in VMO as well as functions of fractional Sobolev spaces $W^{\theta,\frac{n}{\theta}}$, $\theta \in\,]0,1[$.

The study of Sobolev regularity of strong solutions of (5.1) was initiated in 1991 with the pioneeristic work [6]. It was obtained that, if $a_{ij}(x) \in VMO \cap L^\infty(\Omega)$ and $\mathscr{L}u \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega)$ for *each* value of p in the range $(1, \infty)$. Moreover, well-posedness of the Dirichlet problem for (5.1) in $W^{2,p}(\Omega) \cap$

$W_0^{1,p}(\Omega)$ was proved. As a consequence, Hölder continuity of the strong solution or of its gradient follows if the exponent p is sufficiently large.

Thanks to the fundamental accessibility of these two papers [6] and [7], F. Chiarenza, M. Franciosi and M. Frasca in [8] and many other authors have used this space VMO to obtain regularity results for PDEs and systems with discontinuous coefficients.

Continuing the study of regularity of PDEs, we see that Hölder continuity can be inferred for small p if one has more information on $\mathscr{L}u$, such as its belonging to suitable Morrey class $L^{p,\lambda}(\Omega)$.

We now define the Morrey space $L^{p,\lambda}(\Omega)$.

DEFINITION 5.4. Let $1 < p < +\infty$, $0 < \lambda < n$, and let f be a real measurable function on the open bounded set $\Omega \subset \mathbb{R}^n$.

If $|f|^p$ is summable in Ω and the set described by the quantity

$$\frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)} |f(y)|^p\, dy \tag{5.2}$$

when changing of ρ in $]0, diam\,\Omega[$, $x \in \Omega$, has an upper bound, then we say that f belongs to the *Morrey Space* $L^{p,\lambda}(\Omega)$.

If $f \in L^{p,\lambda}(\Omega)$, we define

$$\|f\|_{p,\lambda}^p := \sup_{\substack{x\in\Omega \\ 0<\rho<diam\,\Omega}} \frac{1}{\rho^\lambda}\int_{\Omega\cap B_\rho(x)} |f(y)|^p\, dy \tag{5.3}$$

and the vector space naturally associated to the set of functions in $L^p(\Omega)$ such that (5.3) is finite, endowed with the norm (5.3), is a normed space, which, as we will see later, is complete.

The exponent λ can take values that are not belonging to $]0, n[$ but the unique cases of real interest are that one for which $\lambda \in]0, n[$. Indeed, from the definition we immediately see that $L^{p,\lambda}(\Omega) = L^p(\Omega)$, if $\lambda \le 0$. Sometimes later we will explicitly use the fact that $L^{p,0}(\Omega) = L^p(\Omega)$.

Moreover, if $\lambda = n$, by applying the Lebesgue differentiation theorem, we find that

$$\lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{\Omega\cap B_\rho(x)} |f(y)|^p\, dy = \lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{B_\rho(x)} |f(y)|^p\, dy = C|f(x)|^p$$

for every Lebesgue point or, equivalently, almost everywhere in Ω. Then, in order that $f(x) \in L^{p,n}(\Omega)$ it is necessary and sufficient that f is bounded. It means that $L^{p,n}(\Omega) = L^\infty(\Omega)$.

If $\lambda > n$, then the set described by (5.2) in general is not upper bounded, except for $f = 0$ a.e. in Ω. This means that $L^{p,\lambda}(\Omega) = \{0\}$, for $\lambda > n$.

Using the spaces defined above a natural problem arises namely to study the regularizing properties of the operator $\mathscr{L}$ in Morrey spaces in the case of *VMO* principal coefficients. In [4], L. Caffarelli proved that each $W^{2,p}$-viscosity solution to (5.1) lies in $C^{1+\alpha}(\Omega)$ if $f(x)$ belongs to the Morrey space $L^{n,n\alpha}(\Omega)$ with $\alpha \in (0,1)$.

Main result of the paper is to obtain local regularity, in grand Morrey Spaces, for highest order derivatives of solutions of elliptic nondivergence form with coefficients, which can be discontinuous.

We recall that, in the case of continuous coefficients of the above kind of equation, the results obtained by S. Agmon, A. Douglis and L. Nirenberg in [1]. Later, discontinuous coefficients were considered by S. Campanato in [5].

Then, this paper can be regarded as a continuation of the study of L^p regularity of solutions of second order elliptic PDEs to the maximum order derivatives of the solutions to a certain class of linear elliptic equations in nondivergence form with discontinuous coefficients.

Let us consider the second order differential operator

$$\mathscr{L} \equiv a_{ij}(x)D_{ij}, \quad D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}.$$

Here we have adopted the usual summation convention on repeated indices.

In the sequel, we need the following regularity and ellipticity assumptions on the coefficients of $\mathscr{L}$, $\forall i,j = 1 \dots n$:

$$\text{(5.4)} \qquad \begin{cases} a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \\ a_{ij}(x) = a_{ji}(x), \quad \text{a.a. } x \in \Omega \\ \exists \kappa > 0 : \kappa^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \end{cases}$$

Set η_{ij} for the VMO-modulus of the function $a_{ij}(x)$ and let $\eta(r) = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{1/2}$. We denote by $\Gamma(x,t)$ the normalized fundamental solution of $\mathscr{L}$, i.e.,

$$\Gamma(x,\xi) = \frac{1}{n(2-n)\omega_n\sqrt{\det\{a_{ij}(x)\}}}\left(\sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j\right)^{(2-n)/2}$$
$$\text{for a.a. } x \text{ and all } \xi \in \mathbb{R}^n \backslash \{0\},$$

where $A_{ij}(x)$ stand for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$. We set also

$$\Gamma_i(x,\xi) = \frac{\partial}{\partial \xi_i}\Gamma(x,\xi), \quad \Gamma_{ij}(x,\xi) = \frac{\partial}{\partial \xi_i \partial \xi_j}\Gamma(x,\xi),$$

$$M = \max_{i,j=1,\dots,n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x,\xi)}{\partial \xi^\alpha} \right\|_{L^\infty(\Omega \times \Sigma)}.$$

It is well known that $\Gamma_{ij}(x,\xi)$ are Calderón–Zygmund kernels in the ξ variable.

Theorem 5.5. *Let* (5.4) *be true,* $1 < p < \infty$, $0 < \lambda < n$, $\theta > 0$. *Let* Ω *be a domain satisfying* $\mathscr{A}$ *condition* (*see* (1.7)) *and let* w *be a weight on* Ω *such that*

$w \in A_p(\Omega)$. Then there exist positive constants $c = c(n, \kappa, p, \lambda, \theta, M, w)$ and $\rho_0 = \rho_0(C, n) \in (0, r)$ such that for every ball $B_\rho \subset\subset \Omega$, $\rho < \rho_0$ and every $u \in W_0^{2,p}(B_\rho)$ such that $D_{ij}u \in L_w^{p),\lambda,\theta}(B_\rho)$, we have

$$\|D_{ij}u\|_{L_w^{p),\lambda,\theta}(B_\rho)} \le c\|\mathscr{L}u\|_{L_w^{p),\lambda,\theta}(B_\rho)}, \quad \forall i, j = 1, \dots, n. \tag{5.5}$$

PROOF. Our argument rest in two tools. The **first** one is to show the representation formula for the second order derivatives of a solution by commutators and some singular integral operators with Calderón–Zygmund kernels.

For second derivatives of functions in $W_0^{2,p}(B)$, where B is an open ball in $\mathbb{R}^n$, the representation formula is the following (see [6], Sect. 3):

$$D_{ij}u(x) = \text{P.V.} \int_B \Gamma_{ij}(x, x-y) \sum_{h,k=1}^{n} (a_{hk}(x) - a_{hk}(y)) D_{hk}u(y)\, dy \tag{5.6}$$
$$+ \text{P.V.} \int_B \Gamma_{ij}(x, x-y) \mathscr{L}u(y)\, dy + \mathscr{L}u(x) \int_{|\xi|=1} \Gamma_i(x, \xi)\xi_j \, d\sigma_\xi,$$

Second tool is to derive the estimate (5.5). In order to do it we take the $L^{p),\lambda,\theta}$-norms of the both sides of (5.6). So, let us remark that

i) The first and the second integrals appearing in (5.6) are Principal Value ones and we can use Corollary 4.4 and Corollary 4.3 respectively to obtain the appropriate weighted inequality in $L_w^{p),\lambda,\theta}(\Omega)$, where w is the weight.
ii) $\int_{|\xi|=1} \Gamma_i(x, \xi)\xi_j \, d\sigma_\xi \in L^\infty(B_\rho)$ with a bound independent of ρ.

Now, taking the $L_w^{p),\lambda,\theta}(B_\rho)$ norms of both sides in (5.6) and applying Corollaries 4.4 and 4.3 we get

$$\|D_{ij}u\|_{L_w^{p),\lambda,\theta}(B_\rho)} \le c(\eta(r)\|D_{ij}u\|_{L_w^{p),\lambda,\theta}(B_\rho)} + \|\mathscr{L}u\|_{L_w^{p),\lambda,\theta}(B_\rho)}).$$

This way, in view of the VMO assumption on the coefficients $a_{ij}(x)$, it is possible to choose ρ_0 so small that $c\eta(\rho_0) = 1/2$ and then

$$\|D_{ij}u\|_{L_w^{p),\lambda,\theta}(B_\rho)} \le c\|\mathscr{L}u\|_{L_w^{p),\lambda,\theta}(B_\rho)} \quad \text{for each } \rho < \rho_0. \qquad \square$$

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