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On a quasilinear elliptic problem with convection term and nonlinear boundary condition

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ABSTRACT

The first part of this paper deals with existence of solutions to the quasilinear elliptic problem

$$\begin{aligned} -\operatorname{div} a(x,\nabla u) &= f(x,u,\nabla u) && \text{in } \Omega,\\ a(x,\nabla u)\cdot\nu &= g(x,u)-\zeta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{P}$$

involving a general nonhomogeneous differential operator, namely $\operatorname{div} a$, and Carathéodory functions $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$. Under appropriate conditions on the perturbations, we show that (P) possesses a bounded solution. In the second part, we consider the special case when $\operatorname{div} a$ is the (p,q)-Laplacian with a parameter $\mu>0$, and study the asymptotic behavior of solutions as μ goes to zero or to infinity. A uniqueness result is also provided.

1. Introduction

Let $\Omega\subseteq\mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we are interested in the existence of solutions to the following quasilinear problem, driven by a nonhomogeneous differential operator and with nonlinear boundary condition,

$$\begin{aligned} -\operatorname{div} a(x,\nabla u) &= f(x,u,\nabla u) && \text{in } \Omega,\\ a(x,\nabla u)\cdot\nu &= g(x,u)-\zeta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x\in\partial\Omega$, $1<p<\infty$, $\zeta>0$, and $a:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$ is a continuous strictly monotone map in the second variable satisfying appropriate regularity and growth conditions, listed in hypotheses H(a) of Section 2.

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The nonlinearities $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, that is, $x \mapsto f(x,s,\xi)$, $x \mapsto g(x,s)$ are measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, while $(s,\xi) \mapsto f(x,s,\xi)$, $s \mapsto g(x,s)$ are continuous for a.e. $x \in \Omega$, respectively, for a.e. $x \in \partial\Omega$.

In the first part of this paper we prove that, under general growth conditions on the perturbations, problem (1.1) admits a bounded weak solution. This result is obtained via the classical main theorem on pseudomonotone operators. If $\operatorname{div} a$ coincides with the weighted (p,q)-Laplacian, namely

$$\Delta_p u + \mu \Delta_q u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u\right), \tag{1.2}$$

where $1 < q < p < \infty$, $\mu > 0$, and $u \in W^{1,p}(\Omega)$, then (1.1) becomes

$$\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(x,u,\nabla u) && \text{in } \Omega,\\ \left(|\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u\right)\cdot\nu &= g(x,u) - \zeta|u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

The second part treats the uniqueness of solutions to (1.3) and its asymptotic behavior as $\mu \to 0^+$ and $\mu \to \infty$, respectively.

It should be noted that the presence of a reaction depending also on the gradient of the solution prevents to apply variational methods, like critical point theory. This difficulty is overcome by adapting the approach of Averna–Motreanu–Tornatore [2], who considered problem (1.1) with a homogeneous Dirichlet boundary condition and weighted (p,q)-Laplacian as defined in (1.2). Our paper exhibits at least two novelties:

- a more general differential operator, which may also depend on $x \in \Omega$, appears;
- nonlinear Robin boundary conditions with perturbation $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are taken on.

Moreover, a bounded solution to (1.1) exists once a suitable inequality involving the first eigenvalue of the Robin eigenvalue problem for the p-Laplacian (cf. (2.7) in Section 2 and (3.6) of Section 3) holds.

For other existence results on quasilinear equations with convection term we refer to the papers of De Figueiredo–Girardi–Matzeu [3], Dupaigne–Ghergu–Rădulescu [4], Faraci–Motreanu–Puglisi [5], Faria–Miyagaki–Motreanu [6], Faria–Miyagaki–Motreanu–Tanaka [7], Motreanu–Motreanu–Moussaoui [14], Motreanu–Tanaka [16], Motreanu–Tornatore [17], Ruiz [20], Tanaka [21], and the references therein. Finally, we mention the works of Filippucci–Pucci–Rădulescu [8] and Winkert [22] concerning problems with nonlinear boundary condition and the recent monograph of Papageorgiou–Rădulescu–Repovš [18].

2. Preliminaries

For $1 \le p < \infty$, we denote by $L^p(\Omega)$ and $L^p(\Omega,\mathbb{R}^N)$ the usual Lebesgue spaces, equipped with the norm $\|\cdot\|_p$. If $1 < p < \infty$ then $W^{1,p}(\Omega)$ stands for the Sobolev space, endowed with the norm $\|\cdot\|_{1,p}$. The duality pairing between $W^{1,p}(\Omega)$ and its dual $W^{1,p}(\Omega)^*$ will be denoted by $\langle\cdot,\cdot\rangle$.

On the boundary $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ, by which we can define in the usual way the boundary Lebesgue space $L^p(\partial\Omega)$, with norm $\|\cdot\|_{p,\partial\Omega}$. It is known that there exists a unique continuous linear operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$, where $p \le q \le p_*$ (see (3.3) below), called trace map, such that

$$\gamma(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega}).$$

Henceforth, although all restrictions of Sobolev functions to $\partial\Omega$ are understood in the sense of traces, we will avoid the usage of the trace operator γ to simplify notation.

Given any $\varrho > 0$, consider the norm

$$\|u\|_{\varrho,p} = \left(\|\nabla u\|_p^p + \varrho\|u\|_{p,\partial\Omega}^p\right)^{\frac{1}{p}}, \tag{2.1}$$

which is equivalent to the standard one $\|\cdot\|_{1,p}$; see Papageorgiou–Winkert [19]. If $s > 1$ then $s' := \frac{s}{s-1}$ denotes its conjugate, $x \cdot z$ is the inner product of $x, z \in \mathbb{R}^N$, while $\mathbb{R}_+ := [0, +\infty)$. The well-known inequality

$$\left(|s_1|^{r-2}s_1 - |s_2|^{r-2}s_2\right)(s_1 - s_2) \geq 2^{2-r}|s_1 - s_2|^r \quad \forall\, s_1, s_2 \in \mathbb{R} \tag{2.2}$$

holds, where $r \geq 2$; see Lindqvist [12, p. 71, inequality I]. The Lebesgue measure is denoted by $|\cdot|$ and the same notation is used for the Hausdorff surface measure (it will be clear from the context which one is used).

Let us now introduce the hypotheses on the map $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator. Suppose $\vartheta \in C^1((0, \infty))$ satisfies

$$0 < \hat{d}_0 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq \hat{d}_1 \quad \text{and} \quad \hat{d}_2 t^{p-1} \leq \vartheta(t) \leq \hat{d}_3\left(1 + t^{p-1}\right) \tag{2.3}$$

for all $t > 0$, with $1 < p < \infty$ and appropriate constants $\hat{d}_0, \hat{d}_1, \hat{d}_2, \hat{d}_3 > 0$. The conditions on $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H(a): $a(x, \xi) := \hat{a}(x, |\xi|)\,\xi$ in $\Omega \times \mathbb{R}^N$, where $\hat{a} \in C^0(\overline{\Omega} \times \mathbb{R}_+)$ and $\hat{a}(x, t) > 0$ for every $(x, t) \in \overline{\Omega} \times (0, +\infty)$. Moreover,

(i) $\hat{a} \in C^1(\overline{\Omega} \times (0, \infty))$, $t \to t\hat{a}(x, t)$ is strictly increasing in $(0, \infty)$, and

$$\lim_{t\to 0^+} t\hat{a}(x, t) = 0, \qquad \lim_{t\to 0^+} \frac{t\hat{a}'_t(x, t)}{\hat{a}(x, t)} = \hat{d} > -1 \quad \forall\, x \in \overline{\Omega};$$

(ii) $\|\nabla_\xi a(x, \xi)\| \leq \hat{d}_4 \frac{\vartheta(|\xi|)}{|\xi|}$ for every $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$, with suitable $\hat{d}_4 > 0$;

(iii) $(\nabla_\xi a(x, \xi) y) \cdot y \geq \frac{\vartheta(|\xi|)}{|\xi|}|y|^2$ for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N \setminus \{0\}$, and $y \in \mathbb{R}^N$.

Remark 2.1. Setting

$$G_0(x, t) = \int_0^t \hat{a}(x, s)s\, ds, \quad (x, t) \in \overline{\Omega} \times \mathbb{R}_+,$$

one has $G_0 \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ as well as $t \mapsto G_0(x, t)$ increasing and strictly convex. Accordingly, the function $G(x, \xi) := G_0(x, |\xi|)$ lies in $C^1(\overline{\Omega} \times \mathbb{R}^N)$ and is convex with respect to ξ. Since an easy computation shows that

$$\nabla_\xi G(x, \xi) = (G_0)'_t(x, |\xi|)\frac{\xi}{|\xi|} = \hat{a}(x, |\xi|)\xi = a(x, \xi) \quad \forall\, \xi \in \mathbb{R}^N \setminus \{0\}$$

while $\nabla_\xi G(x, 0) = 0$, the function $G(x, \cdot)$ turns out a primitive of $a(x, \cdot)$. When combined with $G(x, 0) = 0$ and $\xi \mapsto G(x, \xi)$ convex, this entails

$$G(x, \xi) \leq a(x, \xi) \cdot \xi, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{2.4}$$

The next lemma summarizes the main properties of $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$. It immediately follows from (2.3) and H(a).

Lemma 2.2. *Let H(a) be satisfied. Then:*

(i) $a \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$. *Moreover, the map* $\xi \mapsto a(x, \xi)$, $x \in \mathbb{R}^N$, *is continuous and strictly monotone, whence maximal monotone too.*

(ii) $|a(x, \xi)| \leq \hat{d}_5\left(1 + |\xi|^{p-1}\right)$ *for every* $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$, *where* $\hat{d}_5 > 0$.

(iii) $a(x, \xi) \cdot \xi \geq \frac{\hat{d}_2}{p-1}|\xi|^p$ *for all* $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$, *with* $\hat{d}_2$ *given by* (2.3).

Gathering Lemma 2.2 and (2.4) together yields the estimates below for the primitive $G(x, \cdot)$.

Corollary 2.3. *Under assumptions H(a) one has, for appropriate $\hat{d}_6 > 0$,*

$$\frac{\hat{d}_2}{p(p-1)}|\xi|^p \le G(x,\xi) \le \hat{d}_6\,(1+|\xi|^p) \quad in\ \overline{\Omega}\times\mathbb{R}^N.$$

Example 2.4. The following functions, where we drop the x-dependence for the sake of simplicity, fulfill H(a).

(a) $a(\xi) := |\xi|^{p-2}\xi$, with $1 < p < \infty$. It corresponds to the p-Laplacian

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall\, u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}|\xi|^p$.

(b) $a(\xi) := |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$, where $1 < q < p < \infty$ and $\mu > 0$. It arises from the parametric (p,q)-Laplacian

$$u \mapsto \Delta_p u + \mu\Delta_q u \quad \forall\, u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}|\xi|^p + \frac{\mu}{q}|\xi|^q$.

(c) $a(\xi) := \left(1+|\xi|^2\right)^{\frac{p-2}{2}}\xi$, with $1 < p < \infty$, It represents the generalized p-mean curvature differential operator

$$u \mapsto \operatorname{div}\left[(1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right] \quad \forall\, u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p}(1+|\xi|^2)^{\frac{p}{2}}$.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be defined by

$$\langle A(u),\varphi\rangle := \int_\Omega a(x,\nabla u)\cdot\nabla\varphi\,dx, \quad u,\varphi \in W^{1,p}(\Omega). \tag{2.5}$$

The next proposition collects some basic properties of A; proofs can be found in Gasiński–Papageorgiou [9].

Proposition 2.5. *Let H(a) be satisfied and let A be as in (2.5). Then the operator A is bounded, continuous, monotone (hence maximal monotone), and of type* (S_+).

Evidently,

$$\langle A_p(u),\varphi\rangle := \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \ \ \forall\, u,\varphi \in W^{1,p}(\Omega) \tag{2.6}$$

represents a meaningful special case of A.

Given $\beta > 0$, consider the Robin eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\nabla u\cdot\nu &= -\beta|u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{2.7}$$

It is known (see Lê [10]) that the first eigenvalue $\lambda_{1,p,\beta}$ of (2.7) is positive, simple, and isolated. Moreover, it can be variationally characterized through

$$\lambda_{1,p,\beta} = \inf_{u\in W^{1,p}(\Omega)}\left\{\int_\Omega |\nabla u|^p dx + \beta\int_{\partial\Omega}|u|^p d\sigma : \int_\Omega |u|^p dx = 1\right\}. \tag{2.8}$$

3. Existence results

The assumptions on the perturbations $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ read as follows. To avoid unnecessary technicalities, 'for all x' will take the place of 'for almost all x'.

(H) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that:

(i) There exist $\alpha_1 \in L^{q_1'}(\Omega), \alpha_2 \in L^{q_2'}(\partial\Omega)$ and $a_1, a_2, a_3 \in \mathbb{R}_+$ satisfying

$$|f(x,s,\xi)| \le a_1|\xi|^{\frac{p}{q_1'}} + a_2|s|^{q_1-1} + \alpha_1(x) \qquad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{3.1}$$

$$|g(x,s)| \le a_3|s|^{q_2-1} + \alpha_2(x) \qquad \text{on } \partial\Omega \times \mathbb{R}, \tag{3.2}$$

where $1 < q_1 < p^*$ and $1 < q_2 < p_*$, with critical exponents

$$p^* := \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{otherwise,} \end{cases} \qquad p_* := \begin{cases} \dfrac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

(ii) There exist $\omega_f \in L^1(\Omega), \omega_g \in L^1(\partial\Omega)$ and $b_1, b_2, b_3 \in \mathbb{R}_+$ such that

$$f(x,s,\xi)s \le b_1|\xi|^p + b_2|s|^p + \omega_f(x) \qquad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{3.4}$$

$$g(x,s)s \le b_3|s|^p + \omega_g(x) \qquad \text{on } \partial\Omega \times \mathbb{R}. \tag{3.5}$$

Moreover,

$$b_1 + b_2\lambda_{1,p,\beta}^{-1} + \frac{b_3}{\zeta} < \min\left\{\frac{\hat{d}_2}{p-1}, 1\right\} \quad \text{and} \quad 0 < \beta \le \zeta. \tag{3.6}$$

Definition 3.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1.1) if it satisfies

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega f(x,u,\nabla u)\varphi\,dx + \int_{\partial\Omega}\left[g(x,u) - \zeta|u|^{p-2}u\right]\varphi\,d\sigma \tag{3.7}$$

for all test functions $\varphi \in W^{1,p}(\Omega)$.

Using the embeddings (cf. Adams [1])

$$i : W^{1,p}(\Omega) \to L^{q_1}(\Omega) \quad \text{and} \quad \gamma : W^{1,p}(\Omega) \to L^{q_2}(\partial\Omega), \tag{3.8}$$

Hölder's inequality, as well as (H)(i), one easily verifies that all the integrals involved in (3.7) are finite.

We are now ready to formulate our existence result, whose proof chiefly exploits the main theorem on pseudomonotone operators.

Theorem 3.2. *Under hypotheses H(a) and (H), problem (1.1) admits at least one weak solution* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $\hat{N}_f$ and $\hat{N}_g$ the Nemytskij operators associated with f and g, respectively, let $i^* : L^{q_1'}(\Omega) \to W^{1,p}(\Omega)^*$ and $\gamma^* : L^{q_2'}(\partial\Omega) \to W^{1,p}(\Omega)^*$ be the adjoints of embeddings (3.8), and let

$$N_f := i^* \circ \hat{N}_f, \quad N_g := \gamma^* \circ \hat{N}_g \circ \gamma, \quad N := \gamma^* \circ \left(\zeta|\cdot|^{p-2}\cdot\right) \circ \gamma. \tag{3.9}$$

Set, provided $u \in W^{1,p}(\Omega)$,

$$\mathcal{A}(u) = A(u) - N_f(u) - N_g(u) + N(u). \tag{3.10}$$

From (H)(i) it immediately follows that $\mathcal{A} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ maps bounded sets into bounded sets. Moreover, the operator $\mathcal{A}$ is pseudomonotone. Indeed, if $\{u_n\} \subseteq W^{1,p}(\Omega)$ fulfills

$$u_n \overset{\mathrm{w}}{\to} u \quad \text{in } W^{1,p}(\Omega), \quad \limsup_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u\rangle \le 0$$

then, by compactness of embeddings (3.8), one has

$$u_n \to u \quad \text{in } L^{q_1}(\Omega), \quad u_n \to u \quad \text{in } L^{q_2}(\partial\Omega).$$

When combined with (3.1)–(3.2) this entails, after using Hölder's inequality,

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,dx = 0,\\ \lim_{n\to\infty} \int_{\partial\Omega} g(x,u_n)(u_n-u)\,d\sigma = 0,\\ \lim_{n\to\infty} \int_{\partial\Omega} \zeta |u_n|^{p-2}u_n(u_n-u)\,d\sigma = 0, \end{aligned} \tag{3.11}$$

whence

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle = \limsup_{n\to\infty} \langle \mathcal{A}(u_n), u_n - u\rangle \le 0. \tag{3.12}$$

Since A enjoys property (S_+), the weak convergence of $\{u_n\}$ in $W^{1,p}(\Omega)$ and (3.12) yield $u_n \to u$. So, $\mathcal{A}(u_n) \to \mathcal{A}(u)$ in $W^{1,p}(\Omega)^*$, because $\mathcal{A}$ is continuous. Let us finally show that the operator $\mathcal{A}$ turns out coercive, i.e.,

$$\lim_{\|u\|_{\zeta,p}\to\infty} \frac{\langle \mathcal{A}u, u\rangle}{\|u\|_{\zeta,p}} = +\infty, \tag{3.13}$$

where $\|\cdot\|_{\zeta,p}$ denotes the equivalent norm on $W^{1,p}(\Omega)$ defined in (2.1), for $\varrho := \zeta$. Via (2.8) and (3.6) we have

$$\lambda_{1,p,\beta} \le \frac{\|\nabla u\|_p^p + \beta\|u\|_{p,\partial\Omega}^p}{\|u\|_p^p} \le \frac{\|u\|_{\zeta,p}^p}{\|u\|_p^p} \quad \forall\, u \in W^{1,p}(\Omega)\setminus\{0\}. \tag{3.14}$$

Exploiting Lemma 2.2(iii), (3.4), (3.5), and (3.14) leads to

$$\begin{aligned} &\langle \mathcal{A}(u), u\rangle \\ &= \int_\Omega a(x,\nabla u)\cdot\nabla u\,dx + \zeta\|u\|_{p,\partial\Omega}^p - \int_\Omega f(x,u,\nabla u)u\,dx - \int_{\partial\Omega} g(x,u)u\,dx \\ &\ge \min\left\{\frac{\hat{d}_2}{p-1}, 1\right\}\|u\|_{\zeta,p}^p - b_1\|\nabla u\|_p^p - b_2\|u\|_p^p - \|\omega_f\|_1 - b_3\|u\|_{p,\partial\Omega}^p - \|\omega_g\|_{1,\partial\Omega} \\ &\ge \left[\min\left\{\frac{\hat{d}_2}{p-1}, 1\right\} - b_1 - b_2\lambda_{1,p,\beta}^{-1} - \frac{b_3}{\zeta}\right]\|u\|_{\zeta,p}^p - \|\omega_f\|_1 - \|\omega_g\|_{1,\partial\Omega}. \end{aligned}$$

On account of (3.6), conclusion (3.13) follows at once from $p > 1$.

Summing up, $\mathcal{A} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is bounded, pseudomonotone, and coercive. Thus, the main theorem on pseudomonotone operators (see, e.g., Motreanu–Motreanu–Papageorgiou [15]) provides $u \in W^{1,p}(\Omega)$ such that $\mathcal{A}(u) = 0$. Thanks to (3.10) the function u turns out a weak solution of problem (1.1), while Theorem 3.1 in Marino–Winkert [13] gives $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. This completes the proof. □

Remark 3.3. To achieve the $C^{1,\sigma}$-regularity of the solution given by Theorem 3.2, we need an additional condition on the boundary term g, namely

$$|g(x,s) - g(y,t)| \le L\left[|x-y|^\alpha + |s-t|^\alpha\right], \quad |g(x,s)| \le L$$

for all $(x,s),(y,t) \in \partial\Omega \times [-M_0, M_0]$, with appropriate $L \in \mathbb{R}_+$, $\alpha \in (0,1]$, $M_0 > 0$. In such a case, the desired result follows from Marino–Winkert [13, Theorem 3.9], which is a direct consequence of the famous Lieberman's regularity theory [11].

4. Asymptotic behavior and uniqueness results for the (p, q)-Laplacian

Throughout this section, $a(x,\xi) := |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$, where $1 < q < p < \infty$ and $\mu \geq 0$. Hence, problem (1.1) writes

$$\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(x,u,\nabla u) && \text{in } \Omega, \\ \left(|\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u\right)\cdot\nu &= g(x,u) - \zeta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{P_μ}$$

while (3.6) becomes

$$b_1 + b_2\lambda_{1,p,\beta}^{-1} + \frac{b_3}{\zeta} < 1 \quad \text{and} \quad \beta \leq \zeta, \tag{4.1}$$

because $\hat{d}_2 = p-1$. If $\mu := 0$ then (P_μ) reduces to

$$\begin{aligned} -\Delta_p u &= f(x,u,\nabla u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u\cdot\nu &= g(x,u) - \zeta|u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{P_0}$$

Using Example 2.4 and Theorem 3.2 we directly infer the following theorem.

Theorem 4.1. *Let (H) be satisfied. Then, for each $\mu \geq 0$, problem (P_μ) possesses at least one weak solution $u_\mu \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$.*

To investigate the asymptotic behavior of (P_μ) as μ goes to zero and $+\infty$, respectively, the next elementary a priori estimate will be employed.

Proposition 4.2. *Let hypotheses (H) be satisfied and let $u_\mu \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ be the weak solution of problem (P_μ) obtained from Theorem 4.1. Then there exists a constant $C > 0$, independent of μ, such that*

$$\|u_\mu\|_{\zeta,p} \leq C \quad \forall\, \mu \in \mathbb{R}_+. \tag{4.2}$$

Proof. Note that u_μ fulfills (3.7), pick $\varphi := u_\mu$, and recall (3.14), to arrive at

$$\begin{aligned} \|u_\mu\|_{\zeta,p}^p &= \|\nabla u_\mu\|_p^p + \zeta\|u_\mu\|_{p,\partial\Omega}^p \\ &\leq \int_\Omega \left(|\nabla u_\mu|^p + \mu|\nabla u_\mu|^q\right)dx + \zeta\int_{\partial\Omega}|u_\mu|^p d\sigma \\ &= \int_\Omega f(x,u_\mu,\nabla u_\mu)u_\mu\,dx + \int_{\partial\Omega} g(x,u_\mu)u_\mu\,d\sigma \\ &\leq b_1\|\nabla u_\mu\|_p^p + b_2\|u_\mu\|_p^p + \|\omega_f\|_1 + b_3\|u_\mu\|_{p,\partial\Omega}^p + \|\omega_g\|_{1,\partial\Omega} \\ &\leq \left(b_1 + b_2\lambda_{1,p,\beta}^{-1} + \frac{b_3}{\zeta}\right)\|u\|_{\zeta,p}^p + \|\omega_f\|_1 + \|\omega_g\|_{1,\partial\Omega}. \end{aligned}$$

Therefore, by (4.1), setting

$$C := \left(\frac{\|\omega_f\|_1 + \|\omega_g\|_{1,\partial\Omega}}{1 - \left(b_1 + b_2\lambda_{1,p,\beta}^{-1} + \frac{b_3}{\zeta}\right)}\right)^{\frac{1}{p}} > 0$$

inequality (4.2) holds true for any $\mu \in \mathbb{R}_+$. □

We first treat the case when $\mu \to 0^+$.

Theorem 4.3. *Let (H) be satisfied. Then to every sequence $\mu_n \to 0^+$ there correspond a (not relabeled) subsequence $\{u_{\mu_n}\} \subseteq W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that*

(1) u_{μ_n} *is a weak solution of* (P_{μ_n}) *for all* $n \in \mathbb{N}$,
(2) $u_{\mu_n} \to u$ *in* $W^{1,p}(\Omega)$, *and*
(3) $u \in W^{1,p}(\Omega)$ *is a weak solution of* (P_0).

Proof. Fixed $\{\mu_n\}$ as above, Theorem 4.1 gives $\{u_{\mu_n}\} \subseteq W^{1,p}(\Omega) \cap L^\infty(\Omega)$ enjoying (1). Thanks to Proposition 4.2, the sequence $\{u_{\mu_n}\}$ turns out bounded. So, we may assume

$$u_{\mu_n} \stackrel{\mathrm{w}}{\to} u \text{ in } W^{1,p}(\Omega), \quad u_{\mu_n} \to u \text{ in } L^{q_1}(\Omega) \text{ and } L^{q_2}(\partial\Omega), \tag{4.3}$$

which easily produce (3.11); cf. the proof of Theorem 3.2. From (3.7) written for $u := u_{\mu_n}$, $\varphi := u_{\mu_n} - u$ it thus follows

$$\lim_{n\to+\infty} \langle A_p(u_{\mu_n}), u_{\mu_n} - u\rangle = 0,$$

with A_p given by (2.6), because $\mu_n \to 0$. The (S_+)-property of A_p now entails $u_{\mu_n} \to u$ in $W^{1,p}(\Omega)$, namely (2) holds. Finally, due to (H)(i) and standard results on the Nemytskij operator, we have

$$N_f(u_{\mu_n}) \to N_f(u), \;\; N_g(u_{\mu_n}) \to N_g(u), \;\; N(u_{\mu_n}) \to N(u) \tag{4.4}$$

in $W^{1,p}(\Omega)^*$; see (3.9) for the meaning of symbols N_f, N_g, N. Since

$$\begin{aligned} &\int_\Omega |\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n} \cdot \nabla\varphi\, dx - \mu_n \int_\Omega |\nabla u_{\mu_n}|^{q-2}\nabla u_{\mu_n} \cdot \nabla\varphi\, dx \\ &= \int_\Omega f(x, u_{\mu_n}, \nabla u_{\mu_n})\varphi\, dx + \int_{\partial\Omega} \left[g(x, u_{\mu_n}) - \zeta |u_{\mu_n}|^{p-2} u_{\mu_n}\right]\varphi\, d\sigma, \quad n \in \mathbb{N}, \end{aligned} \tag{4.5}$$

and, moreover,

$$\langle A_p(u_{\mu_n}), \varphi\rangle \to \langle A_p(u), \varphi\rangle, \;\; \langle A_q(u_{\mu_n}), \varphi\rangle \to \langle A_q(u), \varphi\rangle$$

whatever $\varphi \in W^{1,p}(\Omega)$, while $\mu_n \to 0$, letting $n \to +\infty$ in (4.5) shows (3). □

We now come to the case when $\mu \to +\infty$.

Theorem 4.4. *If hypotheses (H) hold,* $\mu_n \to +\infty$, *and* $\{u_{\mu_n}\} \subseteq W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *fulfills conclusion (1) of Theorem 4.3 then* $u_{\mu_n} \to c$ *in* $W^{1,q}(\Omega)$ *for some* $c \in \mathbb{R}$.

Proof. The same arguments employed in the previous proof yield here (4.3), (4.4), as well as

$$\begin{aligned} &\frac{1}{\mu_n}\int_\Omega |\nabla u_{\mu_n}|^{p-2}\nabla u_{\mu_n} \cdot \nabla\varphi\, dx + \int_\Omega |\nabla u_{\mu_n}|^{q-2}\nabla u_{\mu_n} \cdot \nabla\varphi\, dx \\ &= \frac{1}{\mu_n}\int_\Omega f(x, u_{\mu_n}, \nabla u_{\mu_n})\varphi\, dx + \frac{1}{\mu_n}\int_{\partial\Omega} \left[g(x, u_{\mu_n}) - \zeta |u_{\mu_n}|^{p-2} u_{\mu_n}\right]\varphi\, d\sigma, \end{aligned} \tag{4.6}$$

with arbitrary $\varphi \in W^{1,p}(\Omega)$. Put $\varphi =: u_{\mu_n} - u$ and recall that $\mu_n \to +\infty$ to achieve

$$\lim_{n\to+\infty} \langle A_q(u_{\mu_n}), u_{\mu_n} - u\rangle = 0,$$

i.e., $u_{\mu_n} \to u$ in $W^{1,q}(\Omega)$ by the (S_+)-property of A_q. From (4.6) it next follows, after letting $n \to +\infty$,

$$\int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi\, dx = 0, \quad \varphi \in W^{1,p}(\Omega),$$

which clearly means

$$\Delta_q u = 0 \text{ in } \Omega, \quad |\nabla u|^{q-2}\nabla u \cdot \nu = 0 \text{ on } \partial\Omega.$$

Hence, $u = c$ for some $c \in \mathbb{R}$. Since these arguments apply to every subsequence of $\{u_{\mu_n}\}$, the proof is complete. □

Remark 4.5. Theorems 4.3–4.4 remain valid for the more general problem

$$\begin{aligned} -\operatorname{div}[a_1(x,\nabla u)+\mu a_2(x,\nabla u)] &= f(x,u,\nabla u) && \text{in } \Omega, \\ [a_1(x,\nabla u)+\mu a_2(x,\nabla u)]\cdot\nu &= g(x,u)-\zeta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned}$$

where $a_1, a_2 : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfy assumptions H(a).

The last part of this section addresses uniqueness of weak solutions to problem (P_μ). Adapting the approach of Averna–Motreanu–Tornatore [2], we will treat the cases $p = 2$ or $q = 2$ under the following assumptions.

(U1) There exist $c_1, c_2 \in \mathbb{R}_+$ such that

$$\begin{aligned} &(f(x,s,\xi)-f(x,t,\xi))(s-t) \le c_1|s-t|^2 \ \ \forall x\in\Omega,\ s,t\in\mathbb{R},\ \xi\in\mathbb{R}^N, \\ &(g(x,s)-g(x,t))(s-t) \le c_2|s-t|^2 \ \ \forall x\in\partial\Omega,\ s,t\in\mathbb{R}. \end{aligned}$$

(U2) With appropriate $\rho \in L^{r'}(\Omega)$, where $1 < r' < p^*$, and $c_3 \in \mathbb{R}_+$ one has both $\xi \mapsto f(x,s,\xi)-\rho(x)$ linear for every $(x,s)\in\Omega\times\mathbb{R}$ and

$$|f(x,s,\xi)-\rho(x)| \le c_3|\xi| \ \text{ in } \ \Omega\times\mathbb{R}\times\mathbb{R}^N.$$

Theorem 4.6. *Let (H), (U1), and (U2) be satisfied.*

(a) If $p := 2 > q > 1$ *and* $c_1\lambda_{1,2,\beta}^{-1} + c_3\lambda_{1,2,\beta}^{-\frac{1}{2}} + c_2\zeta^{-1} < 1$ *then* (P_μ) *admits a unique weak solution for every* $\mu > 0$.

(b) If $p > q := 2$ *then* (P_μ) *possesses only one weak solution provided*

$$\min\left\{\mu, \frac{2^{2-p}}{1+|\partial\Omega|^{\frac{p-2}{p}}}\right\} > c_1\lambda_{1,2,\beta}^{-1} + c_3\lambda_{1,2,\beta}^{-\frac{1}{2}} + \frac{c_2}{\zeta}. \tag{4.7}$$

Proof. Fix $\mu > 0$. Theorem 4.1 gives a weak solution $u_\mu \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ of (P_μ). Suppose $v_\mu \in W^{1,p}(\Omega)$ enjoys the same property. Using (3.7) with $\varphi := u_\mu - v_\mu$ easily leads to

$$\begin{aligned} &\langle A_p(u_\mu)-A_p(v_\mu), u_\mu-v_\mu\rangle + \mu\langle A_q(u_\mu)-A_q(v_\mu), u_\mu-v_\mu\rangle \\ &\quad + \zeta\int_{\partial\Omega}(|u_\mu|^{p-2}u_\mu - |v_\mu|^{p-2}v_\mu)(u_\mu-v_\mu)\,d\sigma \\ &= \int_\Omega (f(x,u_\mu,\nabla u_\mu)-f(x,v_\mu,\nabla u_\mu))(u_\mu-v_\mu)\,dx \\ &\quad + \int_\Omega (f(x,v_\mu,\nabla u_\mu)-f(x,v_\mu,\nabla v_\mu))(u_\mu-v_\mu)\,dx \\ &\quad + \int_{\partial\Omega}(g(x,u_\mu)-g(x,v_\mu))(u_\mu-v_\mu)\,d\sigma. \end{aligned} \tag{4.8}$$

(a) Let $p := 2 > q > 1$. By monotonicity of A_q, the left-hand side in (4.8) can be estimated through

$$\begin{aligned} &\langle A_2(u_\mu)-A_2(v_\mu), u_\mu-v_\mu\rangle + \mu\langle A_q(u_\mu)-A_q(v_\mu), u_\mu-v_\mu\rangle \\ &\quad + \zeta\int_{\partial\Omega}(u_\mu-v_\mu)(u_\mu-v_\mu)\,d\sigma \\ &\ge \|\nabla(u_\mu-v_\mu)\|_2^2 + \zeta\|u_\mu-v_\mu\|_{2,\partial\Omega}^2 = \|u_\mu-v_\mu\|_{\zeta,2}^2, \end{aligned} \tag{4.9}$$

where $\|\cdot\|_{\zeta,2}$ denotes the equivalent norm (2.1). As regards the right-hand side, due to (U1), (U2), Hölder's inequality, and (3.14), we have

$$\begin{aligned}
&\int_\Omega (f(x,u_\mu,\nabla u_\mu)-f(x,v_\mu,\nabla u_\mu))(u_\mu-v_\mu)\,dx\\
&\quad+\int_\Omega (f(x,v_\mu,\nabla u_\mu)-f(x,v_\mu,\nabla v_\mu))(u_\mu-v_\mu)\,dx\\
&\quad+\int_{\partial\Omega} (g(x,u_\mu)-g(x,v_\mu))(u_\mu-v_\mu)\,d\sigma\\
&\le c_1\|u_\mu-v_\mu\|_2^2+\int_\Omega\left(f\left(x,v_\mu,\nabla\left(\frac{1}{2}(u_\mu-v_\mu)^2\right)\right)-\rho(x)\right)dx\\
&\quad+c_2\|u_\mu-v_\mu\|_{2,\partial\Omega}^2\\
&\le c_1\|u_\mu-v_\mu\|_2^2+c_3\int_\Omega |u_\mu-v_\mu||\nabla(u_\mu-v_\mu)|\,dx+c_2\|u_\mu-v_\mu\|_{2,\partial\Omega}^2\\
&\le\left(c_1\lambda_{1,2,\beta}^{-1}+c_3\lambda_{1,2,\beta}^{-\frac{1}{2}}+c_2\zeta^{-1}\right)\|u_\mu-v_\mu\|_{\zeta,2}^2.
\end{aligned}\tag{4.10}$$

Gathering (4.8)–(4.10) together now yields

$$\|u_\mu-v_\mu\|_{\zeta,2}^2\le\left(c_1\lambda_{1,2,\beta}^{-1}+c_3\lambda_{1,2,\beta}^{-\frac{1}{2}}+c_2\zeta^{-1}\right)\|u_\mu-v_\mu\|_{\zeta,2}^2,$$

which implies $u_\mu=v_\mu$, because $c_1\lambda_{1,2,\beta}^{-1}+c_3\lambda_{1,2,\beta}^{-\frac{1}{2}}+\frac{c_2}{\zeta}<1$.

(b) Let $p>q:=2$. Likewise before, the left-hand side of (4.8) becomes

$$\begin{aligned}
&\langle A_p(u_\mu)-A_p(v_\mu),u_\mu-v_\mu\rangle+\mu\langle A_2(u_\mu)-A_2(v_\mu),u_\mu-v_\mu\rangle\\
&\quad+\zeta\int_{\partial\Omega}\left(|u_\mu|^{p-2}u_\mu-|v_\mu|^{p-2}v_\mu\right)(u_\mu-v_\mu)\,d\sigma\\
&\ge\mu\|\nabla(u_\mu-v_\mu)\|_2^2+\zeta\int_{\partial\Omega}\left(|u_\mu|^{p-2}u_\mu-|v_\mu|^{p-2}v_\mu\right)(u_\mu-v_\mu)\,d\sigma,
\end{aligned}\tag{4.11}$$

while (2.2) entails

$$\int_{\partial\Omega}\left(|u_\mu|^{p-2}u_\mu-|v_\mu|^{p-2}v_\mu\right)(u_\mu-v_\mu)\,d\sigma\ge 2^{2-p}\|u_\mu-v_\mu\|_{p,\partial\Omega}^p.\tag{4.12}$$

Via Hölder's inequality we then get

$$\|u_\mu-v_\mu\|_{2,\partial\Omega}^2\le\|u_\mu-v_\mu\|_{p,\partial\Omega}^2\,|\partial\Omega|^{\frac{p-2}{p}}\le\|u_\mu-v_\mu\|_{p,\partial\Omega}^p\left(1+|\partial\Omega|^{\frac{p-2}{p}}\right).\tag{4.13}$$

Thus, from (4.11)–(4.13) it follows

$$\begin{aligned}
&\langle A_p(u_\mu)+A_p(v_\mu),u_\mu-v_\mu\rangle+\mu\langle A_2(u_\mu)+A_2(v_\mu),u_\mu-v_\mu\rangle\\
&\quad+\zeta\int_{\partial\Omega}\left(|u_\mu|^{p-2}u_\mu-|v_\mu|^{p-2}v_\mu\right)(u_\mu-v_\mu)\,d\sigma\\
&\ge\mu\|\nabla(u_\mu-v_\mu)\|_2^2+\zeta\frac{2^{2-p}}{1+|\partial\Omega|^{\frac{p-2}{p}}}\|u_\mu-v_\mu\|_{2,\partial\Omega}^2\\
&\ge\min\left\{\mu,\frac{2^{2-p}}{1+|\partial\Omega|^{\frac{p-2}{p}}}\right\}\left(\|\nabla(u_\mu-v_\mu)\|_2^2+\zeta\|u_\mu-v_\mu\|_{2,\partial\Omega}^2\right)\\
&=\min\left\{\mu,\frac{2^{2-p}}{1+|\partial\Omega|^{\frac{p-2}{p}}}\right\}\|u_\mu-v_\mu\|_{\zeta,2}^2.
\end{aligned}\tag{4.14}$$

Combining (4.8) with (4.14) and (4.10) yields

$$\min\left\{\mu, \frac{2^{2-p}}{1+|\partial\Omega|^{\frac{p-2}{p}}}\right\}\|u_\mu - v_\mu\|_{\zeta,2}^2 \le \left(c_1\lambda_{1,2,\beta}^{-1} + c_3\lambda_{1,2,\beta}^{-\frac{1}{2}} + c_2\zeta^{-1}\right)\|u_\mu - v_\mu\|_{\zeta,2}^2.$$

Therefore, if μ satisfies (4.7) then $u_\mu = v_\mu$. □

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