ON A DIRICHLET PROBLEM WITH (p, q)-LAPLACIAN AND PARAMETRIC CONCAVE-CONVEX NONLINEARITY

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ABSTRACT. A homogeneous Dirichlet problem with (p, q)-Laplace differential operator and reaction given by a parametric p-convex term plus a q-concave one is investigated. A bifurcation-type result, describing changes in the set of positive solutions as the parameter $\lambda > 0$ varies, is proven. Since for every admissible λ the problem has a smallest positive solution $\bar{u}_\lambda$, both monotonicity and continuity of the map $\lambda \mapsto \bar{u}_\lambda$ are studied.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$, let $1 < \tau < q < p < +\infty$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Consider the Dirichlet problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta_p u - \Delta_q u = u^{\tau-1} + \lambda f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter while Δ_r, $r > 1$, denotes the r-Laplacian, namely

$$\Delta_r u := \mathrm{div}(|\nabla u|^{r-2}\nabla u) \quad \forall\, u \in W_0^{1,r}(\Omega).$$

The nonhomogeneous differential operator $Au := \Delta_p u + \Delta_q u$ that drives (P_λ) is usually called (p, q)-Laplacian. It stems from a wide range of important applications, including models of elementary particles [8], biophysics [9], plasma physics [26], reaction-diffusion equations [7], elasticity theory [27], etc. That's why the relevant literature looks daily increasing and numerous meaningful works on this subject are by now available; see the survey paper [19] for a larger bibliography.

Since $\tau < q < p$, the function $\xi \mapsto \xi^{\tau-1}$ grows $(q-1)$-sublinearly at $+\infty$, whereas $\xi \mapsto f(x, \xi)$ is assumed to be $(p-1)$-superlinear near $+\infty$, although it need not satisfy the usual (in such cases) Ambrosetti-Rabinowitz condition. So, the reaction in (P_λ) exhibits the competing effects of concave and convex terms, with the latter multiplied by a positive parameter.

The aim of this paper is to investigate how the solution set of (P_λ) changes as λ varies. In particular, we prove that there exists a critical parameter value $\lambda^* > 0$ for which problem (P_λ) admits

- at least two solutions if $\lambda \in (0, \lambda^*)$,
- at least one solution when $\lambda = \lambda^*$, and
- no solution provided $\lambda > \lambda^*$.

Moreover, we detect a smallest positive solution $\bar{u}_\lambda$ for each $\lambda \in (0, \lambda^*]$ and show that the map $\lambda \mapsto \bar{u}_\lambda$ turns out left-continuous, besides increasing.

The first bifurcation result for semilinear Dirichlet problems driven by the Laplace operator was established, more than twenty years ago, in the seminal paper [2] and then extended to the p-Laplacian in [11, 16]. These works treat the reaction

$$\xi \mapsto \lambda \xi^{s-1} + \xi^{r-1}, \quad \xi \geq 0,$$

where $1 < s < p < r < p^*$, $\lambda > 0$, and p^* denotes the critical Sobolev exponent. A wider class of nonlinearities has recently been investigated in [22], while [24] deals with Robin boundary conditions. It should be noted that, unlike our case, λ always multiplies the concave term, which changes the

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analysis of the problem. Finally, [4, 14, 23] contain analogous bifurcation theorems for problems of a different kind, whereas [20, 21] study (p,q)-Laplace equations having merely concave right-hand side.

Our approach is based on the critical point theory, combined with appropriate truncation and comparison techniques.

2. Mathematical background and hypotheses

Let $(X, \|\cdot\|)$ be a real Banach space. Given a set $V \subseteq X$, write $\overline{V}$ for the closure of V, ∂V for the boundary of V, and $\mathrm{int}_X(V)$ or simply $\mathrm{int}(V)$, when no confusion can arise, for the interior of V. If $x \in X$ and $\delta > 0$ then

$$B_\delta(x) := \{z \in X : \|z - x\| < \delta\}, \quad B_\delta := B_\delta(0).$$

The symbol $(X^*, \|\cdot\|_{X^*})$ denotes the dual space of X, $\langle \cdot, \cdot \rangle$ indicates the duality pairing between X and X^*, while $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X'. We say that $A : X \to X^*$ is of type $(\mathrm{S})_+$ provided

$$x_n \rightharpoonup x \ \text{ in } \ X, \quad \limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \le 0 \quad \Longrightarrow \quad x_n \to x.$$

The function $\Phi : X \to \mathbb{R}$ is called coercive if $\lim_{\|x\|\to+\infty} \Phi(x) = +\infty$ and weakly sequentially lower semicontinuous when

$$x_n \rightharpoonup x \ \text{ in } \ X \quad \Longrightarrow \quad \Phi(x) \le \liminf_{n\to\infty} \Phi(x_n).$$

Suppose $\Phi \in C^1(X)$. We denote by $K(\Phi)$ the critical set of Φ, i.e.,

$$K(\Phi) := \{x \in X : \Phi'(x) = 0\}.$$

The classical Cerami compactness condition for Φ reads as follows:

(C) *Every $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and $(1 + \|x_n\|)\Phi'(x_n) \to 0$ in X^* has a convergent subsequence.*

From now on, Ω indicates a fixed bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. Let $u, v : \Omega \to \mathbb{R}$ be measurable and let $t \in \mathbb{R}$. The symbol $u \le v$ means $u(x) \le v(x)$ for almost every $x \in \Omega$, $t^\pm := \max\{\pm t, 0\}$, $u^\pm(\cdot) := u(\cdot)^\pm$. If u, v belong to a function space, say Y, then we set

$$[u, v] := \{w \in Y : u \le w \le v\}, \quad [u) := \{w \in Y : u \le w\}.$$

The conjugate exponent r' of a number $r \ge 1$ is defined by $r' := r/(r-1)$, while r^* indicates its Sobolev conjugate, namely

$$r^* := \begin{cases} \frac{Nr}{N-r} & \text{when } r < N, \\ +\infty & \text{otherwise.} \end{cases}$$

As usual,

$$\|u\|_r := \left(\int_\Omega |u|^r \, dx\right)^{1/r} \ \forall\, u \in L^r(\Omega), \quad \|u\|_{1,r} := \left(\int_\Omega |\nabla u|^r \, dx\right)^{1/r} \ \forall\, u \in W_0^{1,r}(\Omega),$$

and $W^{-1,r'}(\Omega)$ denotes the dual space of $W_0^{1,r}(\Omega)$. We will also employ the linear space $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u\lfloor_{\partial\Omega} = 0\}$, which is complete with respect to the standard $C^1(\overline{\Omega})$-norm. Its positive cone

$$C_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ in } \overline{\Omega}\}$$

has a nonempty interior given by

$$\mathrm{int}(C_+) = \left\{u \in C_+ : u(x) > 0 \ \forall\, x \in \Omega, \ \frac{\partial u}{\partial n}(x) < 0 \ \forall\, x \in \partial\Omega\right\}.$$

Here $n(x)$ denotes the outward unit normal to $\partial\Omega$ at x.

Let $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ be the nonlinear operator stemming from the negative r-Laplacian, i.e.,

$$\langle A_r(u), v\rangle := \int_\Omega |\nabla u|^{r-2} \nabla u \cdot \nabla v \, dx, \quad u, v \in W_0^{1,r}(\Omega).$$

We know [12, Section 6.2] that A_r is bounded, continuous, strictly monotone, and of type $(S)_+$. The Liusternik-Schnirelmann theory gives an increasing sequence $\{\lambda_{n,r}\}$ of eigenvalues for A_r. The following assertions can be found in [12, Section 6.2].

(p_1) $\lambda_{1,r}$ is positive, isolated, and simple.

(p_2) $\|u\|_r^r \leq \dfrac{1}{\lambda_{1,r}}\|u\|_{1,r}^r$ for all $u \in W_0^{1,r}(\Omega)$.

(p_3) $\lambda_{1,r}$ admits an eigenfunction $\phi_{1,r} \in \mathrm{int}(C_+)$ such that $\|\phi_{1,r}\|_r = 1$.

Proposition 13 of [6] then ensures that

(p_4) If $r \neq \hat{r}$ then $\phi_{1,r}$ and $\phi_{1,\hat{r}}$ are linearly independent.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition

$$|g(x,t)| \leq a(x)\left(1 + |t|^{s-1}\right) \quad \text{in} \quad \Omega \times \mathbb{R},$$

where $a \in L^\infty(\mathbb{R})$, $1 < s \leq p^*$. Set $G(x,\xi) := \int_0^\xi g(x,t)\,dt$ and consider the C^1-functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G(x,u(x))\,dx, \quad u \in W_0^{1,p}(\Omega).$$

Proposition 2.1 ([13], Proposition 2.6)**.** *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 turns out to be a local $W_0^{1,p}(\Omega)$-minimizer of φ.*

Combining this result with the strong comparison principle below, essentially due to Arcoya-Ruiz [3], shows that certain constrained minimizers actually are 'global' critical points. Recall that, given $h_1, h_2 \in L^\infty(\Omega)$,

$$h_1 \prec h_2 \iff \operatorname*{ess\,inf}_K (h_2 - h_1) > 0 \text{ for any nonempty compact set } K \subseteq \Omega.$$

Proposition 2.2. *Let $a \in \mathbb{R}_+$, $h_1, h_2 \in L^\infty(\Omega)$, $u_1 \in C_0^1(\overline{\Omega})$, $u_2 \in \mathrm{int}(C_+)$. Suppose $h_1 \prec h_2$ as well as*

$$-\Delta_p u_i - \Delta_q u_i + a|u_i|^{p-2}u_i = h_i \text{ in } \Omega, \quad i = 1,2.$$

Then, $u_2 - u_1 \in \mathrm{int}(C_+)$.

Throughout the paper, 'for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$', $c_0, c_1, \ldots$ indicate suitable positive constants, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(\cdot, t) = 0$ provided $t \leq 0$, while $F(x,\xi) := \int_0^\xi f(x,t)\,dt$.

The following hypotheses will be posited.

(h_1) There exist $\theta \in [\tau, q]$ and $r \in (p, p^*)$ such that

$$c_1 t^{p-1} + c_2 t^{q-1} \leq f(x,t) \leq c_0\left(t^{\theta-1} + t^{r-1}\right) \quad \forall\,(x,t) \in \Omega \times \mathbb{R}_+,$$

where $c_2 > \lambda_{1,q}$.

(h_2) $\lim_{\xi \to +\infty} \frac{F(x,\xi)}{\xi^p} = +\infty$ uniformly with respect to $x \in \Omega$.

(h_3) $\liminf_{\xi \to +\infty} \frac{f(x,\xi)\xi - pF(x,\xi)}{\xi^\beta} \geq c_3$ uniformly in $x \in \Omega$. Here, $\beta > \tau$ and

$$(r-p)\max\left\{Np^{-1}, 1\right\} < \beta < p^*.$$

(h_4) To every $\rho > 0$ there corresponds $\mu_\rho > 0$ such that $t \mapsto f(x,t) + \mu_\rho t^{p-1}$ is nondecreasing in $[0, \rho]$ for any $x \in \Omega$.

By (h_2)–(h_3) the perturbation $f(x,\cdot)$ is $(p-1)$-superlinear at $+\infty$. In the literature, one usually treats this case via the well-known Ambrosetti-Rabinowitz condition, namely:

(AR) With appropriate $M > 0$, $\sigma > p$ one has both $\operatorname*{ess\,inf}_\Omega F(\cdot, M) > 0$ and

$$0 < \sigma F(x,\xi) \leq f(x,\xi)\xi, \quad (x,\xi) \in \Omega \times [M, +\infty). \tag{2.1}$$

It easily entails $c_3\xi^\sigma \le F(x,\xi)$ in $\Omega\times[M,+\infty)$, which forces (h_2). However, nonlinearities having a growth rate 'slower' than $t^{\sigma-1}$ at $+\infty$ are excluded from (2.1). Thus, assumption (h_3) incorporates in our framework more situations.

Example 2.3. *Let $c_2>\lambda_{1,q}$. The functions $f_1, f_2:\mathbb{R}_+\to\mathbb{R}$ defined by*

$$f_1(t) := \begin{cases} t^{p-1}+c_2t^{\tau-1} & \text{if } 0\le t\le 1,\\ t^{r-1}+c_2t^{q-1} & \text{otherwise,}\end{cases} \qquad f_2(t) := t^{p-1}\log(1+t)+c_2t^{q-1}, \quad t\in\mathbb{R}_+,$$

satisfy (h_1)–(h_4). *Nevertheless, f_1 alone complies with condition (AR).*

3. A bifurcation-type theorem

Write S_λ for the set of positive solutions to (P_λ). Lieberman's nonlinear regularity theory [18, p. 320] and Pucci-Serrin's maximum principle [25, pp. 111,120] yield

$$S_\lambda \subseteq \operatorname{int}(C_+).$$

Put $\mathcal{L} := \{\lambda>0 : S_\lambda\neq\emptyset\}$. Our first goal is to establish some basic properties of $\mathcal{L}$. From now on, $X := W^{1,p}_0(\Omega)$ and $\|\cdot\| := \|\cdot\|_{1,p}$.

Proposition 3.1. *Under* (h_1) *one has $\mathcal{L}\neq\emptyset$.*

Proof. Given $\lambda>0$, consider the C^1-functional $\Psi_\lambda : W^{1,p}_0(\Omega)\to\mathbb{R}$ defined by

$$\Psi_\lambda(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega dx\int_0^{u(x)} g_\lambda(t)\,dt \quad \forall\, u\in W^{1,p}_0(\Omega),$$

where

$$g_\lambda(t) := (t^+)^{\tau-1} + \lambda c_0\left[(t^+)^{\theta-1}+(t^+)^{r-1}\right], \quad t\in\mathbb{R}.$$

Evidently, g_λ fulfills (2.1) once $\sigma\in(p,r)$ and $M>0$ is big enough. So, condition (C) holds true for Ψ_λ. Moreover,

$$u\in\operatorname{int}(C_+) \quad\Longrightarrow\quad \lim_{t\to+\infty}\Psi_\lambda(tu) = -\infty$$

because $r>p$. Observe next that if $s\in[1,p^*]$ then

$$\|u\|_s \le c\|u\|_{p^*} \le C\|u\| \quad \forall\, u\in X,$$

with $C := C(s,\Omega)$. This easily leads to

$$\begin{aligned}\Psi_\lambda(u) &\ge \frac{1}{p}\|u\|^p - c_4\|u\|^\tau - \lambda c_5\left[\|u\|^\theta+\|u\|^r\right]\\ &= \left[\frac{1}{p} - c_4\|u\|^{\tau-p} - \lambda c_5\left(\|u\|^{\theta-p}+\|u\|^{r-p}\right)\right]\|u\|^p, \quad u\in X.\end{aligned} \tag{3.1}$$

Let us set, for any $t>0$,

$$\gamma_\lambda(t) := c_4t^{\tau-p}+\lambda c_5(t^{\theta-p}+t^{r-p}), \quad \hat\gamma_\lambda(t) := (c_4+\lambda c_5)t^{\tau-p}+2\lambda c_5t^{r-p}.$$

From $\tau\le\theta<p<r$ it follows $\lambda c_5t^{\theta-p}\le\lambda c_5\left(t^{\tau-p}+t^{r-p}\right)$, which implies

$$0<\gamma_\lambda(t)\le\hat\gamma_\lambda(t) \quad\text{in}\quad (0,+\infty). \tag{3.2}$$

Since $\lim_{t\to0^+}\hat\gamma_\lambda(t) = \lim_{t\to+\infty}\hat\gamma_\lambda(t) = +\infty$, there exists $t_0>0$ satisfying $\hat\gamma_\lambda'(t_0)=0$. One has

$$t_0 := t_0(\lambda) := \left[\frac{(c_4+\lambda c_5)(p-\tau)}{2\lambda c_5(r-p)}\right]^{\frac{1}{r-\tau}}$$

and, via simple calculations, $\lim_{\lambda\to0^+}\hat\gamma_\lambda(t_0)=0$. On account of (3.1)–(3.2) we can thus find $\lambda_0>0$ such that

$$\Psi_\lambda(u)\ge m_\lambda>0=\Psi_\lambda(0) \quad \text{for all } u\in\partial B(0,t_0),\ \lambda\in(0,\lambda_0).$$

Pick $\lambda\in(0,\lambda_0)$. The mountain pass theorem entails $\Psi'_\lambda(\bar u_\lambda)=0$ and $\Psi_\lambda(\bar u_\lambda)\geq m_\lambda$ with appropriate $\bar u_\lambda\in X$. Hence,

$$\langle A_p(\bar u_\lambda)+A_q(\bar u_\lambda),v\rangle=\int_\Omega\left[(\bar u_\lambda^+)^{\tau-1}+\lambda c_0\left((\bar u_\lambda^+)^{\theta-1}+(\bar u_\lambda^+)^{r-1}\right)\right]v\,dx,\quad v\in X, \tag{3.3}$$

and $\bar u_\lambda\neq 0$. Choosing $v:=-\bar u_\lambda^-$ in (3.3) yields $\|\nabla\bar u_\lambda^-\|_p^p+\|\nabla\bar u_\lambda^-\|_q^q=0$, namely $\bar u_\lambda^-=0$. This forces $\bar u_\lambda\geq 0$ while, by (3.3) again,

$$-\Delta_p\bar u_\lambda-\Delta_q\bar u_\lambda=\bar u_\lambda^{\tau-1}+\lambda c_0\left(\bar u_\lambda^{\theta-1}+\bar u_\lambda^{r-1}\right)\ \text{ in }\ \Omega.$$

Lieberman's nonlinear regularity theory and Pucci-Serrin's maximum principle finally lead to $\bar u_\lambda\in\operatorname{int}(C_+)$. Now define, provided $(x,\xi)\in\Omega\times\mathbb{R}$,

$$\bar f_\lambda(x,\xi):=\begin{cases}(\xi^+)^{\tau-1}+\lambda f(x,\xi^+) & \text{if } \xi\leq\bar u_\lambda(x),\\ \bar u_\lambda(x)^{\tau-1}+\lambda f(x,\bar u_\lambda(x)) & \text{otherwise,}\end{cases}\qquad \bar F_\lambda(x,\xi):=\int_0^\xi\bar f_\lambda(x,t)\,dt.$$

An easy verification ensures that the associated C^1-functional

$$\bar\Phi_\lambda(u):=\frac1p\|\nabla u\|_p^p+\frac1q\|\nabla u\|_q^q-\int_\Omega\bar F_\lambda(x,u(x))\,dx,\quad u\in X,$$

is coercive and weakly sequentially lower semicontinuous. So, it attains its infimum at some point $u_\lambda\in X$. Assumption (h_1) produces

$$\bar\Phi_\lambda(u_\lambda)<0=\bar\Phi_\lambda(0),$$

i.e., $u_\lambda\neq0$, because $\tau<q<p$. As before, from

$$\langle A_p(u_\lambda)+A_q(u_\lambda),v\rangle=\int_\Omega\bar f_\lambda(x,u_\lambda(x))v(x)\,dx\quad\forall\,v\in X \tag{3.4}$$

we infer $u_\lambda\geq0$. Test (3.4) with $v:=(u_\lambda-\bar u_\lambda)^+$, exploit (h_1) again, and recall (3.3) to arrive at

$$\begin{aligned}\langle A_p(u_\lambda)+A_q(u_\lambda),(u_\lambda-\bar u_\lambda)^+\rangle&=\int_\Omega\left[\bar u_\lambda^{\tau-1}+\lambda f(\cdot\,,\bar u_\lambda)\right](u_\lambda-\bar u_\lambda)^+dx\\&\leq\int_\Omega\left[\bar u_\lambda^{\tau-1}+\lambda c_0(\bar u_\lambda^{\theta-1}+\bar u_\lambda^{r-1})\right](u_\lambda-\bar u_\lambda)^+dx\\&=\langle A_p(\bar u_\lambda)+A_q(\bar u_\lambda),(u_\lambda-\bar u_\lambda)^+\rangle,\end{aligned}$$

which entails $u_\lambda\leq\bar u_\lambda$ by monotonicity. Summing up, $u_\lambda\in[0,\bar u_\lambda]\setminus\{0\}$. On account of (3.4), one thus has $u_\lambda\in S_\lambda$ for any $\lambda\in(0,\lambda_0)$. This completes the proof. □

Our next result ensures that $\mathcal{L}$ is an interval.

Proposition 3.2. *Let* (h_1) *be satisfied. If* $\hat\lambda\in\mathcal{L}$ *then* $(0,\hat\lambda)\subseteq\mathcal{L}$.

Proof. Pick $\hat u\in S_{\hat\lambda}$, $\lambda\in(0,\hat\lambda)$, and define, provided $(x,\xi)\in\Omega\times\mathbb{R}$,

$$\hat f_\lambda(x,\xi):=\begin{cases}(\xi^+)^{\tau-1}+\lambda f(x,\xi^+) & \text{if } \xi\leq\hat u(x),\\ \hat u(x)^{\tau-1}+\lambda f(x,\hat u(x)) & \text{otherwise,}\end{cases}\qquad \hat F_\lambda(x,\xi):=\int_0^\xi\hat f_\lambda(x,t)\,dt.$$

The associated energy functional

$$\hat\Phi_\lambda(u):=\frac1p\|\nabla u\|_p^p+\frac1q\|\nabla u\|_q^q-\int_\Omega\hat F_\lambda(x,u(x))\,dx,\quad u\in X,$$

turns out coercive, weakly sequentially lower semicontinuous, besides C^1. Now, arguing exactly as above yields the conclusion. □

A careful reading of this proof allows one to state the next 'monotonicity' property.

Corollary 3.3. *Under hypothesis* (h_1)*, for every* $\hat\lambda\in\mathcal{L}$*,* $u_{\hat\lambda}\in S_{\hat\lambda}$*, and* $\lambda\in(0,\hat\lambda)$ *there exists* $u_\lambda\in S_\lambda$ *such that* $u_\lambda\leq u_{\hat\lambda}$.

Actually, we can prove a more precise assertion.

Proposition 3.4. *Suppose* (h$_1$) *and* (h$_4$) *hold. Then to each* $\hat{\lambda} \in \mathcal{L}$, $u_{\hat{\lambda}} \in S_{\hat{\lambda}}$, $\lambda \in (0, \hat{\lambda})$ *there corresponds* $u_\lambda \in S_\lambda$ *fulfilling* $u_{\hat{\lambda}} - u_\lambda \in \text{int}(C_+)$.

Proof. Write $\rho := \|u_{\hat{\lambda}}\|_\infty$. If μ_ρ is given by (h$_4$) while u_λ comes from Corollary 3.3 then

$$\begin{aligned}
-\Delta_p u_{\hat{\lambda}} - \Delta_q u_{\hat{\lambda}} + \lambda\mu_\rho u_{\hat{\lambda}}^{p-1} &= u_{\hat{\lambda}}^{\tau-1} + \hat{\lambda} f(x, u_{\hat{\lambda}}) + \lambda\mu_\rho u_{\hat{\lambda}}^{p-1} \\
&= u_{\hat{\lambda}}^{\tau-1} + \lambda f(x, u_{\hat{\lambda}}) + \lambda\mu_\rho u_{\hat{\lambda}}^{p-1} + (\hat{\lambda} - \lambda) f(x, u_{\hat{\lambda}}) \\
&\geq u_\lambda^{\tau-1} + \lambda f(x, u_\lambda) + \lambda\mu_\rho u_\lambda^{p-1} = -\Delta_p u_\lambda - \Delta_q u_\lambda + \lambda\mu_\rho u_\lambda^{p-1}
\end{aligned} \tag{3.5}$$

because $u_\lambda \leq u_{\hat{\lambda}}$ and $f(x,t) \geq 0$ once $t \geq 0$. The function $h(x) := (\hat{\lambda} - \lambda) f(x, u_{\hat{\lambda}}(x))$ lies in $L^\infty(\Omega)$. Indeed, on account of (h$_1$), we have

$$0 \leq h(x) \leq c_0(\hat{\lambda} - \lambda)\left[\|u\|_\infty^{\theta-1} + \|u\|_\infty^{r-1}\right] \quad \forall\, x \in \Omega.$$

Pick any compact set $K \subseteq \Omega$. Recalling that $u_{\hat{\lambda}} \in \text{int}(C_+)$ and using (h$_1$) again gives

$$h(x) \geq (\hat{\lambda} - \lambda)\left[c_1 u_{\hat{\lambda}}(x)^{p-1} + c_2 u_{\hat{\lambda}}(x)^{q-1}\right] \geq \left(c_1 \inf_K u_{\hat{\lambda}}^{p-1} + c_2 \inf_K u_{\hat{\lambda}}^{q-1}\right) > 0, \quad x \in \Omega,$$

whence $0 \prec h$. Now, (3.5) combined with Proposition 2.2 entails $u_{\hat{\lambda}} - u_\lambda \in \text{int}(C_+)$. □

The interval $\mathcal{L}$ turns out to be bounded.

Proposition 3.5. *Let* (h$_1$) *and* (h$_4$) *be satisfied. If* $\lambda^* := \sup \mathcal{L}$ *then* $\lambda^* < \infty$.

Proof. Fix $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda$. Note that we can suppose $\lambda > 1$, otherwise $\mathcal{L}$ would be bounded, which of course entails $\lambda^* < \infty$. Define

$$g_\lambda(x,\xi) := \begin{cases} \lambda\left[c_1(\xi^+)^{p-1} + c_2(\xi^+)^{q-1}\right] & \text{if } \xi \leq u_\lambda(x), \\ \lambda\left[c_1 u_\lambda(x)^{p-1} + c_2 u_\lambda(x)^{q-1}\right] & \text{otherwise,} \end{cases} \quad G_\lambda(x,\xi) := \int_0^\xi g_\lambda(x,t)dt$$

for every $(x,\xi) \in \Omega \times \mathbb{R}$, as well as

$$\Psi_\lambda(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_\lambda(x, u(x))\, dx, \quad u \in X.$$

The same arguments employed before yield here a global minimum point, say $\bar{u}_\lambda$, to Ψ_λ. So, in particular,

$$\langle A_p(\bar{u}_\lambda) + A_q(\bar{u}_\lambda), v\rangle = \int_\Omega g_\lambda(x, \bar{u}_\lambda(x)) v(x)\, dx \quad \forall\, v \in X. \tag{3.6}$$

Choosing $v := -\bar{u}_\lambda^-$ first and then $v := (\bar{u}_\lambda - u_\lambda)^+$ we obtain $\bar{u}_\lambda \in [0, u_\lambda]$; cf. the proof of Proposition 3.1. Since, by (p$_3$) in Section 2, $u_\lambda, \phi_{1,q} \in \text{int}(C_+)$, through [22, Proposition 1] one has $t\phi_{1,q} \leq u_\lambda$, with $t > 0$ small enough. Thus, on account of (p$_3$) again,

$$\begin{aligned}
\Psi_\lambda(t\phi_{1,q}) &= \frac{1}{p}\|\nabla(t\phi_{1,q})\|_p^p + \frac{1}{q}\|\nabla(t\phi_{1,q})\|_q^q - \int_\Omega G_\lambda(x, t\phi_{1,q}(x))\, dx \\
&= \frac{t^p}{p}\|\nabla\phi_{1,q}\|_p^p + \frac{t^q}{q}\|\nabla\phi_{1,q}\|_q^q - \int_\Omega \lambda\left(c_1 \frac{t^p}{p}\phi_{1,q}^p + c_2 \frac{t^q}{q}\phi_{1,q}^q\right) dx \\
&= \frac{t^p}{p}\|\nabla\phi_{1,q}\|_p^p + \frac{t^q}{q}\lambda_{1,q} - \lambda c_1 \frac{t^p}{p}\|\phi_{1,q}\|_p^p - \lambda c_2 \frac{t^q}{q} \\
&\leq \frac{t^p}{p}\|\nabla\phi_{1,q}\|_p^p + \frac{t^q}{q}(\lambda_{1,q} - \lambda c_2) \\
&< \frac{t^p}{p}\|\nabla\phi_{1,q}\|_p^p + \frac{t^q}{q}\lambda_{1,q}(1 - \lambda) = c_6 t^p - c_7 t^q.
\end{aligned}$$

Now, recall that $q < p$ and decrease t when necessary to achieve

$$\Psi_\lambda(\bar{u}_\lambda) = \min_X \Psi_\lambda \leq \Psi_\lambda(t\phi_{1,q}) < 0 = \Psi_\lambda(0),$$

i.e., $\bar{u}_\lambda \neq 0$. Summing up, $\bar{u}_\lambda \in [0, u_\lambda] \setminus \{0\}$, whence, by (3.6), it turns out a positive solution of the equation

$$-\Delta_p u - \Delta_q u = \lambda c_1 |u|^{p-2} u + \lambda c_2 |u|^{q-2} u \quad \text{in} \quad \Omega.$$

Due to [5, Theorem 2.4], this prevents λ from being arbitrary large, as desired. □

Le us finally prove that $\mathcal{L} = (0, \lambda^*]$. From now on, $\Phi_\lambda : X \to \mathbb{R}$ will denote the C^1-energy functional associated with problem (P_λ). Evidently,

$$\Phi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{\tau}\|u^+\|_\tau^\tau - \lambda \int_\Omega F(x, u^+(x))\, dx \quad \forall\, u \in X. \tag{3.7}$$

Proposition 3.6. *Under* (h_1), (h_3), *and* (h_4) *one has* $\lambda^* \in \mathcal{L}$.

Proof. Pick any $\{\lambda_n\} \subseteq (0, \lambda^*)$ fulfilling $\lambda_n \uparrow \lambda^*$. Via Corollary 3.3, construct a sequence $\{u_n\} \subseteq X$ such that $u_n \in S_{\lambda_n}$, $u_n \le u_{n+1}$. Then

$$\langle A_p(u_n) + A_q(u_n), v\rangle = \int_\Omega u_n^{\tau-1} v\, dx + \lambda_n \int_\Omega f(\cdot\,, u_n) v\, dx, \qquad v \in X. \tag{3.8}$$

We can also assume $\Phi_\lambda(u_n) < 0$ (see the proof of Proposition 3.1), which means

$$\|\nabla u_n\|_p^p + \frac{p}{q}\|\nabla u_n\|_q^q - \frac{p}{\tau}\|u_n\|_\tau^\tau - \lambda_n \int_\Omega pF(x, u_n(x))\, dx < 0. \tag{3.9}$$

Testing (3.8) with $v := u_n$ gives

$$\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q = \|u_n\|_\tau^\tau + \lambda_n \int_\Omega f(\cdot\,, u_n) u_n\, dx. \tag{3.10}$$

Since $q < p$ while $\lambda_1 \le \lambda_n$, from (3.9)–(3.10) it follows

$$\int_\Omega [f(\cdot\,, u_n)u_n - pF(\cdot\,, u_n)]\, dx \le \frac{1}{\lambda_1}\left(\frac{p}{\tau} - 1\right)\|u_n\|_\tau^\tau \quad \forall\, n \in \mathbb{N}. \tag{3.11}$$

Observe next that, thanks to (h_1) and (h_3), one has

$$f(x,\xi)\xi - pF(x,\xi) \ge c_8 \xi^\beta - c_9 \quad \text{in} \quad \Omega \times \mathbb{R}_+\,.$$

Consequently, (3.11) becomes

$$c_8 \|u_n\|_\beta^\beta \le \frac{1}{\lambda_1}\left(\frac{p}{\tau} - 1\right)\|u_n\|_\tau^\tau + c_{10} \le c_{11}\|u_n\|_\beta^\tau + c_{10}, \qquad n \in \mathbb{N},$$

because $\tau < \beta$. This clearly forces

$$\|u_n\|_\beta \le c_{12} \quad \forall\, n \in \mathbb{N}. \tag{3.12}$$

If $r \le \beta$ then $\{u_n\}$ turns out also bounded in $L^r(\Omega)$. Using (3.10) besides (h_1) entails

$$\begin{aligned} \|u_n\|^p &\le \|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q \le \|u_n\|_\tau^\tau + \lambda^* \int_\Omega f(\cdot\,, u_n)u_n\, dx \\ &\le |\Omega|^{1-\tau/r}\|u_n\|_r^\tau + \lambda^* c_0 \int_\Omega (u_n^\theta + u_n^r)\, dx \\ &\le |\Omega|^{1-\tau/r}\|u_n\|_r^\tau + \lambda^* c_0 \int_\Omega [(1 + u_n^r) + u_n^r]\, dx, \end{aligned} \tag{3.13}$$

whence $\{u_n\} \subseteq X$ is bounded. Suppose now $\beta < r < p^*$. Two cases may occur.
1) $p < N$. Let $t \in (0,1)$ satisfy

$$\frac{1}{r} = \frac{1-t}{\beta} + \frac{t}{p^*}. \tag{3.14}$$

The interpolation inequality [12, p. 905] yields $\|u_n\|_r \le \|u_n\|_\beta^{1-t}\|u_n\|_{p^*}^t$. Via (3.12) we thus obtain

$$\|u_n\|_r^r \le c_{13}\|u_n\|_{p^*}^{tr}, \quad n \in \mathbb{N}. \tag{3.15}$$

Reasoning exactly as before and exploiting (3.15) produce

$$\|u_n\|^p \le \|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q \le c_{14}\left(1+\|u_n\|_{p^*}^{tr}\right) \le c_{15}\left(1+\|u_n\|^{tr}\right). \tag{3.16}$$

Finally, note that $tr < p$. Indeed, $(r-p)\frac{N}{p} < \beta$ due to (h_3), while

$$tr < p \iff \frac{r-\beta}{p^*-\beta} < \frac{p}{p^*} \iff (r-p)\frac{N}{p} < \beta;$$

cf. (3.14). Now, the boundedness of $\{u_n\} \subseteq X$ directly stems from (3.16).
2) $p \ge N$, which implies $p^* = +\infty$. We will repeat the previous argument with p^* replaced by any $\sigma > r$. Accordingly, if $t \in (0,1)$ fulfills $\frac{1}{r} = \frac{1-t}{\beta} + \frac{t}{\sigma}$ then $tr = \frac{\sigma(r-\beta)}{\sigma-\beta}$. Since, thanks to (h_3) again,

$$\lim_{\sigma\to+\infty} \frac{\sigma(r-\beta)}{\sigma-\beta} = r-\beta < p,$$

one arrives at $tr < p$ for σ large enough. This entails $\{u_n\} \subseteq X$ bounded once more.
Hence, in either case, we may assume

$$u_n \rightharpoonup u^* \text{ in } X \quad \text{and} \quad u_n \to u^* \text{ in } L^r(\Omega), \tag{3.17}$$

where a subsequence is considered when necessary. Testing (3.8) with $v := u_n - u^*$ thus yields, as $n \to +\infty$,

$$\lim_{n\to+\infty} \langle A_p(u_n) + A_q(u_n), u_n - u^*\rangle = 0,$$

whence, by monotonicity of A_q,

$$\limsup_{n\to+\infty} \left[\langle A_p(u_n), u_n - u^*\rangle + \langle A_q(u), u_n - u^*\rangle\right] \le 0.$$

On account of (3.17) it follows

$$\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u^*\rangle \le 0.$$

Recalling that A_p enjoys the $(S)_+$-property, we infer $u_n \to u^*$ in X, besides $0 \le u_n \le u^*$ for all $n \in \mathbb{N}$.
Finally, let $n \to +\infty$ in (3.8) to get

$$\langle A_p(u^*) + A_q(u^*), v\rangle = \int_\Omega (u^*)^{\tau-1} v\,dx + \lambda^* \int_\Omega f(\cdot, u^*) v\,dx \quad \forall v \in X,$$

i.e., $u^* \in S_{\lambda^*}$ and, a fortiori, $\lambda^* \in \mathcal{L}$. □

Some meaningful (bifurcation) properties of the set S_λ will now be established.

Proposition 3.7. *Suppose* (h_1)–(h_4) *hold true. Then, for every* $\lambda \in (0, \lambda^*)$*, problem* (P_λ) *admits two solutions* $u_0, \hat{u} \in \operatorname{int}(C_+)$ *such that* $u_0 \le \hat{u}$*. Moreover,* u_0 *is a local minimizer of the associated energy functional* Φ_λ*.*

Proof. Fix $\lambda \in (0, \lambda^*)$ and choose $\eta \in (\lambda, \lambda^*)$. By Proposition 3.2, there exists $u_\eta \in S_\eta$ while Proposition 3.4 provides $u_0 \in S_\lambda$ satisfying

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}([0, u_\eta]). \tag{3.18}$$

The same reasoning adopted in the proof of Proposition 3.2 ensures here that u_0 is a global minimum point to the functional

$$\Phi_{\lambda,\eta}(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega F_{\lambda,\eta}(x, u(x))\,dx, \quad u \in X,$$

where $F_{\lambda,\eta}(x,\xi) := \int_0^\xi f_{\lambda,\eta}(x,t)\,dt$, with

$$f_{\lambda,\eta}(x,\xi) := \begin{cases} (\xi^+)^{\tau-1} + \lambda f(x,\xi^+) & \text{if } \xi \le u_\eta(x), \\ u_\eta(x)^{\tau-1} + \lambda f(x, u_\eta(x)) & \text{otherwise.} \end{cases}$$

By (3.18), u_0 turns out a local $C^1_0(\overline{\Omega})$-minimizer of Φ_λ, because $\Phi_\lambda\lfloor_{[0,u_\eta]}=\Phi_{\lambda,\eta}\lfloor_{[0,u_\eta]}$. Via Proposition 2.1 we then see that this remains valid with $C^1_0(\overline{\Omega})$ replaced by X. Set

$$f_0(x,\xi):=\begin{cases} u_0(x)^{\tau-1}+\lambda f(x,u_0(x)) & \text{if } \xi\le u_0(x),\\ \xi^{\tau-1}+\lambda f(x,\xi) & \text{otherwise,}\end{cases}\quad F_0(x,\xi):=\int_0^\xi f_0(x,t)\,dt, \tag{3.19}$$

$(x,\xi)\in\Omega\times\mathbb{R}$, as well as

$$\Phi_0(u):=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega F_0(x,u(x))\,dx\quad\forall\, u\in X. \tag{3.20}$$

From (3.19) and the nonlinear regularity theory it follows $u_0\in K(\Phi_0)\subseteq[u_0)\cap\mathrm{int}(C_+)$. We may thus assume

$$K(\Phi_0)\cap[u_0,u_\eta]=\{u_0\}, \tag{3.21}$$

or else a second solution of (P_λ) bigger than u_0 would exist. Bearing in mind the proof of Proposition 3.6 and making small changes to accommodate the truncation at $u_0(x)$ shows that Φ_0 satisfies condition (C). Let us next truncate $f_0(x,\cdot)$ at $u_\eta(x)$ to construct a new Carathéodory function $\tilde f$, with primitive $\tilde F$ and associated functional $\tilde\Phi$, defined like in (3.20) but replacing F_0 by $\tilde F$. Evidently,

$$K(\tilde\Phi)=K(\Phi_0)\cap[u_0,u_\eta],$$

whence $K(\tilde\Phi)=\{u_0\}$ because of (3.21). Since $\tilde\Phi$ is coercive and weakly sequentially lower semicontinuous, it possesses a global minimum point that must coincide with u_0. An easy verification gives $\Phi_0\lfloor_{[0,u_\eta]}=\tilde\Phi\lfloor_{[0,u_\eta]}$. So, thanks to (3.18), u_0 turns out a local $C^1_0(\overline{\Omega})$-minimizer of Φ_0. This still holds when X replaces $C^1_0(\overline{\Omega})$; cf. Proposition 2.1. We may suppose $K(\Phi_0)$ finite, otherwise infinitely many solutions of (P_λ) bigger than u_0 do exist. Adapting the argument exploited in [1, Proposition 29] provides $\rho\in(0,1)$ such that

$$\Phi_0(u_0)<m_0:=\inf\{\Phi_0(u):\|u-u_0\|=\rho\}. \tag{3.22}$$

Finally, if $u\in\mathrm{int}(C_+)$ then simple calculations based on (h_2) entail $\Phi_0(tu)\to-\infty$ as $t\to+\infty$. Therefore, the mountain pass theorem can be applied, and there is $\hat u\in X$ fulfilling

$$\hat u\in K(\Phi_0),\quad \Phi_0(\hat u)\ge m_0. \tag{3.23}$$

Via (3.22)–(3.23) one has $u_0\neq\hat u$ while the inclusion $K(\Phi_0)\subseteq[u_0)\cap\mathrm{int}(C_+)$ forces $u_0\le\hat u$, which ends the proof. □

Proposition 3.8. *Under* (h_1)–(h_4)*, the solution set* S_λ *admits a smallest element* $\bar u_\lambda$ *for every* $\lambda\in\mathcal{L}$.

Proof. A standard procedure ensures that S_λ turns out downward directed; see, e.g., [10, Section 4]. Lemma 3.10 at p. 178 of [17] yields

$$\operatorname{ess\,inf} S_\lambda=\inf\{u_n:n\in\mathbb{N}\} \tag{3.24}$$

for some decreasing sequence $\{u_n\}\subseteq S_\lambda$. Consequently, $0\le u_n\le u_1$ and

$$\langle A_p(u_n)+A_q(u_n),v\rangle=\int_\Omega\left[u_n^{\tau-1}+\lambda f(\cdot\,,u_n)\right]v\,dx\quad\forall\, v\in X. \tag{3.25}$$

Due to (h_1), testing (3.25) with $v:=u_n$ we thus obtain

$$\begin{aligned}\|u_n\|^p\le\|\nabla u_n\|_p^p+\|\nabla u_n\|_q^q&=\int_\Omega\left[u_n^\tau+\lambda f(\cdot\,,u_n)u_n\right]dx\\ &\le\int_\Omega\left[u_n^\tau+\lambda c_0\left(u_n^\theta+u_n^r\right)\right]dx\le\int_\Omega\left[u_1^\tau+\lambda c_0\left(u_1^\theta+u_1^r\right)\right]dx,\quad n\in\mathbb{N},\end{aligned}$$

namely $\{u_n\}\subseteq X$ is bounded. Like before (cf. the proof of Proposition 3.6), this gives $u_n\to\bar u_\lambda$ in X, where a subsequence is considered if necessary. So, from (3.25) it easily follows

$$\langle A_p(\bar u_\lambda)+A_q(\bar u_\lambda),v\rangle=\int_\Omega\left[\bar u_\lambda^{\tau-1}+\lambda f(\cdot\,,\bar u_\lambda)\right]v\,dx\quad\forall\, v\in X.$$

Showing that $\bar{u}_\lambda \neq 0$ will entail $\bar{u}_\lambda \in S_\lambda$, whence the conclusion by (3.24). To the aim, consider the problem

$$-\Delta_p u - \Delta_q u = u^{\tau-1} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{3.26}$$

Its energy functional

$$\Phi_0(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{\tau}\|u^+\|_\tau^\tau, \quad u \in X,$$

turns out coercive and weakly sequentially lower semicontinuous. Hence, there exists $\tilde{u} \in X$ satisfying $\Phi_0(\tilde{u}) = \inf_X \Phi_0$. One has $u_0 \neq 0$, because $\Phi_0(\tilde{u}) < 0 = \Phi_0(0)$ (the argument is like in the proof of Proposition 3.5). Further, $\Phi_0'(\tilde{u}) = 0$, i.e.,

$$\langle A_p(\tilde{u}) + A_q(\tilde{u}), v\rangle = \int_\Omega (\tilde{u}^+)^{\tau-1} v\, dx \quad \forall\, v \in X.$$

Choosing $v := -\tilde{u}^-$ we see that u is a positive solution to (3.26). Actually, $\tilde{u} \in \text{int}(C_+)$ and, through a standard procedure [15, Lemma 3.1], $\tilde{u}$ turns out unique.

Claim: $\tilde{u} \le u$ for all $u \in S_\lambda$.

Indeed, for any fixed $u \in S_\lambda$, define

$$\Psi(w) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega dx \int_0^{w(x)} g(x,t)\, dt, \quad w \in X,$$

where

$$g(x,t) := \begin{cases} (t^+)^{\tau-1} & \text{if } t \le u(x), \\ u(x)^{\tau-1} & \text{otherwise} \end{cases} \quad \forall\, (x,t) \in \Omega \times \mathbb{R}.$$

The following assertions can be easily verified.

- $\Psi(u^*) = \inf_X \Psi$, with appropriate $u^* \in X$.
- $\Psi(u^*) < 0 = \Psi(0)$, whence $u^* \neq 0$.
- $u^* \in K(\Psi) \subseteq [0, u] \cap C_+$.

Therefore, u^* is a positive solution of (3.26). By uniqueness, this implies $u^* = \tilde{u}$. Thus, a fortiori, $\tilde{u} \le u$.

The claim brings $\tilde{u} \le u_n$, $n \in \mathbb{N}$, which in turn provides $0 < \tilde{u} \le \bar{u}_\lambda$, as desired. □

Let us finally come to some meaningful properties of the map

$$k : \lambda \in \mathcal{L} \mapsto \bar{u}_\lambda \in C_0^1(\overline{\Omega}).$$

Proposition 3.9. *Suppose* (h_1)–(h_4) *hold true. Then the function* k *is both*

(i_1) *strictly increasing, namely* $\bar{u}_{\lambda_2} - \bar{u}_{\lambda_1} \in \text{int}(C_+)$ *if* $\lambda_1 < \lambda_2$, *and*
(i_2) *left-continuous.*

Proof. Pick $\lambda_1, \lambda_2 \in \mathcal{L}$ such that $\lambda_1 < \lambda_2$. Since $\bar{u}_{\lambda_2} \in S_{\lambda_2}$, Proposition 3.4 yields $u_{\lambda_1} \in S_{\lambda_1}$ fulfilling $\bar{u}_{\lambda_2} - u_{\lambda_1} \in \text{int}(C_+)$, while Proposition 3.8 entails $\bar{u}_{\lambda_1} \le u_{\lambda_1}$. Hence, $\bar{u}_{\lambda_2} - \bar{u}_{\lambda_1} \in \text{int}(C_+)$. This shows ($i_1$).

If $\lambda_n \to \lambda^-$ in $\mathcal{L}$ then, by (i_1), the sequence $\{\bar{u}_{\lambda_n}\}$ turns out increasing. Its boundedness in X immediately stems from (h_1); see the previous proof. Now, repeat the argument below (3.17) to arrive at

$$\bar{u}_{\lambda_n} \to \tilde{u}_\lambda \text{ in } X, \tag{3.27}$$

whence $\tilde{u}_\lambda \in S_\lambda \subseteq \text{int}(C_+)$. We finally claim that $\tilde{u}_\lambda = \bar{u}_\lambda$. Assume on the contrary

$$\bar{u}_\lambda(x_0) < \tilde{u}_\lambda(x_0) \text{ for some } x_0 \in \Omega. \tag{3.28}$$

Lieberman's nonlinear regularity theory gives $\{\bar{u}_n\} \subseteq C_0^{1,\alpha}(\overline{\Omega})$ as well as

$$\|\bar{u}_{\lambda_n}\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_{16} \quad \forall\, n \in \mathbb{N}.$$

Since the embedding $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ is compact, (3.27) becomes

$$\bar{u}_{\lambda_n} \to \tilde{u}_\lambda \text{ in } C_0^1(\overline{\Omega}).$$

Because of (3.28), this implies $\bar{u}_\lambda(x_0) < \bar{u}_{\lambda_n}(x_0)$ for any n large enough, against (i_1). Consequently, $\tilde{u}_\lambda = \bar{u}_\lambda$, and (i_2) follows from (3.27). □

Gathering Propositions 3.1–3.9 together we obtain the following

Theorem 3.10. *Let* (h_1)–(h_4) *be satisfied. Then, there exists* $\lambda^* > 0$ *such that problem* (P_λ) *admits*

(j_1) *at least two solutions* $u_0, \hat{u} \in \mathrm{int}(C_+)$, *with* $u_0 \le \hat{u}$, *for every* $\lambda \in (0, \lambda^*)$,
(j_2) *at least one solution* $u^* \in \mathrm{int}(C_+)$ *when* $\lambda = \lambda^*$,
(j_3) *no positive solutions for all* $\lambda > \lambda^*$,
(j_4) *a smallest positive solution* $\bar{u}_\lambda \in \mathrm{int}(C_+)$ *provided* $\lambda \in (0, \lambda^*]$.

Moreover, the map $\lambda \in (0, \lambda^*] \mapsto \bar{u}_\lambda \in C_0^1(\overline{\Omega})$ *is strictly increasing and left-continuous.*

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