LIOUVILLE THEOREMS FOR ANCIENT CALORIC FUNCTIONS VIA OPTIMAL GROWTH CONDITIONS

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Abstract. We prove two Liouville theorems for ancient nonnegative solutions of the heat equation on a complete non-compact Riemannian manifold with Ricci curvature bounded from below by $-K$, $K \geqslant 0$. If, *at any fixed time*, such a solution grows sub-exponentially in space, then it either constant (when $K = 0$) or stationary (if $K > 0$). We also show the optimality of this growth condition through examples.

1. Introduction

Two instances of Liouville's type theorems are the following classical facts:

(i) if u is harmonic on $\mathbb{R}^N$ and is bounded from below, then it is constant;
(ii) if u is harmonic on $\mathbb{R}^N$ and grows sublinearly at infinity, then it is constant.

The first statement follows from the Harnack inequality, while the second one from gradient estimates for harmonic functions. These kind of results received ever increasing attention in the last decades, with generalizations to other partial differential equations, Riemannian manifolds under Ricci curvature lower bounds (see the recent survey [2]), or even to metric measure spaces . In this paper, motivated by [14, 17], we investigate various form of Liouville theorems for the *heat equation* on a *Riemannian manifold*, with the aim to find optimal growth/bound conditions ensuring triviality of its solutions.

Compared to the elliptic case depicted above, Liouville-type theorems in the parabolic setting are more subtle. On the one hand, solving the Cauchy problem in $\mathbb{R}^N$ with initial datum in $C_c^\infty(\mathbb{R}^N)$ shows that no global growth/bound condition can ensure triviality of a solution $u \in C^\infty(\mathbb{R}^N \times]0, +\infty[)$. If, however, we consider *ancient* solutions of the heat equation (i. e., defined on $\mathbb{R}^N \times] - \infty, T[$), things look brighter. Indeed, an immediate byproduct of the parabolic Harnack inequality is the constancy of any bounded solution of the heat equation on $\mathbb{R}^N \times] - \infty, T[$. To explore further the classical case of $\mathbb{R}^N$ we consider the two main examples of ancient solutions, namely

$$u(x,t) := e^{x_N + t}, \qquad v(x,t) = e^{-t}\cos x_N, \qquad x = (x_1, \dots, x_N) \in \mathbb{R}^N \tag{1.1}$$

The first example shows that non-negativity is *not* a Liouville property, meaning that it does not ensure triviality. The second one shows that boundedness *at fixed time* also fails to be a Liouville property for the heat equation on $\mathbb{R}^N$. The best parabolic Liouville theorem in $\mathbb{R}^N$ dates back to Hirschman [7] (see also [19]) and we give now a short proof of it taken from [4].

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Theorem 1.1 (Hirschman)**.** *Let* u *be a non-negative solution of the heat equation on* $\mathbb{R}^N\times\,]-\infty, T_0[$ *such that for a fixed time* $t_0 < T_0$ *it holds*

$$u(x, t_0) \le e^{o(|x|)}, \qquad \text{for } |x| \to +\infty$$

Then, u *is constant.*

Proof. We can assume that $u > 0$. By the Widder representation for ancient positive solutions (see [14]), there exists a non-negative Borel measure μ such that

$$u(x,t) = \int_{\mathbb{R}^N} e^{x\cdot\xi + t|\xi|^2}\, d\mu(\xi). \tag{1.2}$$

By Hölder inequality with respect to the measure $\nu := e^{t_0|\xi|^2}\mu$

$$\begin{aligned} u(sx + (1-s)y, t_0) &= \int_{\mathbb{R}^N} e^{(sx+(1-s)y)\cdot\xi}\, d\nu(\xi) \leqslant \left(\int_{\mathbb{R}^N} e^{x\cdot\xi}\, d\nu(\xi)\right)^s \left(\int_{\mathbb{R}^N} e^{y\cdot\xi}\, d\nu(\xi)\right)^{1-s} \\ &= u^s(x,t_0)\, u^{1-s}(y,t_0), \end{aligned}$$

for all $s \in]0,1[$. Therefore, $x \mapsto \log u(x,t_0)$ is convex, and being sublinear by assumption, it must be constant. Thus $u(x,t_0) \equiv c$ and differentiating under the integral sign (1.2), we obtain

$$0 = P(D_x)u(x,t_0)|_{x=0} = \int_{\mathbb{R}^N} P(\xi)\, d\nu(\xi)$$

for any polinomial P such that $P(0) = 0$. By the Stone-Weierestrass and Riesz representation theorems, this implies that $\operatorname{supp}(\nu) = \{0\}$ and thus $\mu = c\,\delta_0$ for some $c \in \mathbb{R}$. Inserting the latter into (1.2) gives the claim. □

The two examples in (1.1) show that the assumption in the previous Liouville theorem are optimal.

The picture becomes more involved if we substitute $\mathbb{R}^N$ with a general Riemannian manifold. Indeed, if $\mathbb{H}_N$ denotes the real hyperbolic space of dimension N, there are plenty of bounded harmonic functions[1], which are also eternal solutions of $\partial_t - \Delta = 0$. It turns out, however, that this issue can only appear in negative curvature and the following is the more general Liouville type theorem in the Riemannian framework up to now. Recall that a *caloric* function is just a non-negative solution of the heat equation.

Theorem 1.2 (Souplet-Zhang [17])**.** *Let* M *be a complete Riemannian manifold with non-negative Ricci curvature,* $p \in M$ *and* d *be the metric distance. Any ancient caloric* u *such that* $u(x,t) \leqslant e^{o(\mathrm{d}(x,p)+\sqrt{-t})}$ *for* $\mathrm{d}(x,p), -t \to +\infty$*, is constant.*

This result was actually an immediate consequence of a new gradient estimate for positive solutions u of the heat equation in $Q_{R,T} := B_R(p) \times [-T, 0]$, $p \in M$. If $\mathrm{Ric}_M \geqslant -K\,g$ for some $K \geqslant 0$, then the gradient estimate of [17] states that

$$|\nabla \log u|^2 \leqslant C_N\left(K + \frac{1}{R^2} + \frac{1}{T}\right)\log^2(S/u), \qquad S = \sup_{Q_{R,T}} u \tag{1.3}$$

[1] E. g., classical harmonics on $B_1 \subset \mathbb{R}^2$ are also harmonic on $\mathbb{H}_2$ identified with the Poincaré disc.

holds in $Q_{R/2,T/2}$ for any u as above. The previous inequality falls into the wider framework of *parabolic gradient estimates* such as the celebrated Li-Yau's one [13]

$$|\nabla \log u|^2 \leqslant \partial_t \log u + C_N \left(K + \frac{1}{R^2} + \frac{1}{T} \right) \tag{1.4}$$

or the Hamilton inequality [6] (generalized by Kotschvar [10] to the non-compact case)

$$|\nabla \log u|^2 \leqslant C_N \left(K + \frac{1}{T} \right) \log(S/u), \qquad S = \sup_{M \times [-T,0]} u. \tag{1.5}$$

Notice that letting $T \to +\infty$ into this last inequality immediately gives a Liouville theorem for *bounded* ancient caloric functions in the case $K = 0$, but it is only its localized counterpart (1.3) which provides the much weaker sub-exponential growth condition $u \leqslant e^{o(\mathrm{d}(x,p)+\sqrt{-t})}$ as a Liouville property. Still in the case $K = 0$, one can let $R, T \to +\infty$ in (1.4) to obtain that entire ancient caloric functions are non-decreasing in time. Therefore, the supremum S in (1.3) is attained at $t = 0$. If $u(x, 0) \leqslant e^{o(\mathrm{d}(x,p))}$ (which is the optimal growth condition at fixed time), we can then apply (1.3) to $u + 1$ and let $R, T \to +\infty$ to get that u is constant.

Theorem 1.2 has been generalized (with the same growth condition) in various directions, see e.g. [8] and the bibliography therein. We can now state the main result of this note.

Theorem 1.3. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by $-K \leqslant 0$, $p \in M$ and u be an ancient caloric function.*

- *(i) If $K = 0$ and $u(x, t_0) \leqslant e^{o(\mathrm{d}(x,p))}$ for $\mathrm{d}(x, p) \to +\infty$ at some fixed t_0, u is constant.*
- *(ii) If $K > 0$ and $u(x, t) \leqslant e^{o(\mathrm{d}(x,p)-t)}$ for $\mathrm{d}(x, p) - t \to +\infty$, u is stationary (and hence harmonic).*

Let us make some comments on the result. As already pointed out, the case $K = 0$ was already known and provides an *optimal* parabolic Liouville property in the case $\mathrm{Ric}_M \geqslant 0$ more in the spirit of [9]. However, we will prove it without using (1.3), through a Choquet representation for ancient solutions (see Lemma 2.3 below). This method has already been used in [11] to prove the uniqueness of the non-negative Cauchy problem in manifolds with non-negative Ricci curvature bound and sketched in [14, Remark 2.3] to single out a class of non-decreasing in time ancient solutions in the general case $K \geqslant 0$. Remark 2.5 shows, however, that there exist eternal non-negative solutions in $\mathbb{H}_N$ which are exponentially decreasing in time, giving the optimality of the time dependance in *(ii)* above. More generally, the Choquet representation approach allows us to fully deal with the case $K > 0$, which was out of reach of the parabolic gradient estimates (1.3)–(1.5) and is therefore the main novelty of this work. The case $M = \mathbb{H}_N$ discussed above shows that our second Liouville statement is the best one can get through a growth condition.

As a final remark, Liouville properties for ancient caloric functions in the metric measure setting of $\mathrm{RCD}^*(K, N)$ spaces can probably be obtained through the same techniques described here, providing a generalization of the $RCD^*(K, N)$ counterpart of Theorem 1.2. The latter has been proved in the metric measure setting in [8] through a gradient estimate of the form (1.3), but the more general statement of Theorem

1.3 in the RCD* framework can be cooked up via the same ingredients: the granted linearity of the Laplacian is essential in order to apply Choquet theory, the parabolic Harnack inequality holds true since $\mathrm{RCD}^*(K,N)$ verifies doubling and Poincaré, while the relevant Laplacian comparison and comparison principles can be found in [5]. The only additionally needed result is a gradient estimate for eigenfunctions of Yau's type (see Proposition 2.2 below), which follows from the metric version of the parabolic Li-Yau inequality proved in [20].

2. Proof of the main result

By a time translation will always work with caloric functions on $M\times\,]-\infty,1[$. If u is ancient and caloric (i. e., a non-negative ancient solution of the heat equation) then the local Harnack inequality shows that if $u(x_0,t_0)=0$, then u vanishes identically on $M\times\,]-\infty,t_0]$. Since we are supposing that Ric_M is bounded from below, uniqueness of the non-negative Cauchy problem holds (see e. g. [15] and the references therein), and therefore any nontrivial caloric function is strictly positive. Let $\mathcal{C}$ be the cone of caloric functions on $M\times\,]-\infty,1[$. We say that $u\in\mathcal{C}$ is *minimal*, and write $u\in\mathrm{Ext}(\mathcal{C})$, if

$$v\in\mathcal{C}\quad\text{and}\quad v\leqslant u\quad\Rightarrow\quad v=k\,u\quad\text{for some } k\in\mathbb{R}.$$

The following result is basically contained in [11,16], see also [14, Remark 2.3].

Proposition 2.1 (Extremal caloric functions)**.** *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If $v\in\mathrm{Ext}(\mathcal{C})$ then there exists $\lambda\in\mathbb{R}$ and $w\in C^\infty(M)$ solving $\Delta w=\lambda\,w$ such that $v(x,t)=e^{\lambda\,t}\,w(x)$.*

Regarding eigenfunctions, we recall the following a-priori bound of [1], which is a refinement of the classical Yau's gradient estimate in [18].

Proposition 2.2 (Gradient bound for eigenfunctions)**.** *Let (M,g) be a complete N-dimensional Riemannian manifold with $\mathrm{Ric}_M\geqslant-(N-1)\,\kappa\,g$ for $\kappa\geqslant 0$ and w a positive λ-eigenfunction. Then $\lambda\geqslant-(N-1)^2\,\kappa/4$ and*

$$|\nabla\log w|\leqslant\frac{N-1}{2}\left(\sqrt{\kappa+\frac{4\,\lambda}{(N-1)^2}}+\sqrt{\kappa}\right)$$

In particular, by Yau's elliptic Liouville theorem, if $\mathrm{Ric}_M\geqslant 0$, positive nontrivial solutions of $\Delta w=\lambda\,w$ exist only for $\lambda>0$.

Lemma 2.3. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below and $u\in\mathcal{C}$. There exists a Borel probability measure ν on $\mathbb{R}$ and a family of functions $\{w_\lambda\}_{\lambda\in\mathbb{R}}$ such that*

$$u(x,t)=\int_{\mathbb{R}}e^{\lambda\,t}\,w_\lambda(x)\,d\nu\qquad(2.1)$$

where $\lambda\mapsto w_\lambda(x)$ is Borel for any $x\in M$ and w_λ is a positive solution of $\Delta w=\lambda\,w$ for ν a. e. λ.

Proof. The cone $\mathcal{C}$ fails to have a compact base with respect to any useful topology. However, if we equip it with the topology of pointwise convergence, it turns out to be a proper closed a subset of $\mathbb{R}^{M\times\,]-\infty,1[}$ and therefore is weakly complete. We claim that $\mathcal{C}$

is metrizable and hence well-capped in the Choquet sense (see [3, 30.16]). Indeed, let $D \subseteq M\times\,]-\infty,1[$ be denumerable and dense. The local parabolic Harnack inequality implies that the topology of pointwise convergence in D coincides with the pointwise convergence in $\mathcal{C} \subseteq \mathbb{R}^{M\times\,]-\infty,1[}$ (it actually implies locally uniform convergence). This proves metrizability due to D being denumberable and, even more, that $\mathcal{C}$ is second countable and thus separable. As a consequence, $\mathcal{C}$ is a Polish space, and being $\mathrm{Ext}(\mathcal{C})$ a G_δ subset of $\mathcal{C}$[2], it turns out to be Polish as-well.

By Choquet theorem [3, Theorem 30.22], any $u \in \mathcal{C}$ can be represented through a probability measure supported on $\mathrm{Ext}(\mathcal{C})$, i.e. there exists a probability measure μ on $\mathrm{Ext}(\mathcal{C})$ such that for any continuous linear functional Λ

$$\langle \Lambda, u\rangle = \int_{\mathrm{Ext}(\mathcal{C})} \langle \Lambda, v\rangle \, d\mu.$$

Specifying Λ to be the evaluation at $(x,t) \in M\times\,]-\infty,1[$, gives

$$u(x,t) = \int_{\mathrm{Ext}(\mathcal{C})} v(x,t)\, d\mu \qquad \forall (x,t) \in M\times\,]-\infty,1[.$$

Let us fix $p \in M$ and observe that the map $\psi : \mathrm{Ext}(\mathcal{C}) \to \mathbb{R}$ between Polish spaces defined as

$$\psi(v) = v^{-1}(p,0)\, \frac{\partial v}{\partial t}(p,0)$$

is measurable[3] and thus induces a disintegration of the probability measure μ into probability measures $\{\mu_\lambda\}_\lambda$, Borel measurable with respect to λ, such that $\mathrm{supp}(\mu_\lambda) \subseteq \psi^{-1}(\lambda)$. In particular, the Disintegration theorem ensures that there exists a probability measure ν on $\mathbb{R}$ such that

$$u(x,t) = \int_{\mathbb{R}} \int_{\psi^{-1}(\lambda)} v(x,t)\, d\mu_\lambda\, d\nu. \tag{2.2}$$

By Proposition 2.1, any $v \in \mathrm{Ext}(\mathcal{C})$ is of the form $v(x,t) = e^{\lambda t}\, w(x)$ for some $\lambda \in \mathbb{R}$ and $w \geqslant 0$ solving $\Delta w = \lambda\, w$, therefore it holds

$$v(x,t) = e^{\lambda t}\, v(x,0), \qquad \psi(v) = \lambda. \tag{2.3}$$

If $\mathcal{C}_\lambda$ denotes the cone of non-negative solutions to $\Delta w = \lambda\, w$, the latter discussion shows that

$$\psi^{-1}(\lambda) \subseteq \{v : v(x,t) = e^{\lambda t}\, w(x), w \in \mathcal{C}_\lambda\}.$$

The map $\Phi_\lambda : \psi^{-1}(\lambda) \to \mathcal{C}_\lambda$ defined as $\Phi_\lambda(v)(x) = v(x,0)$ is continuous and induces a push-forward measure $(\Phi_\lambda)_*(\mu_\lambda)$ on $\mathcal{C}_\lambda$, which we still denote by μ_λ by a slight abuse of notation. By construction, it satisfies

$$\int_{\psi^{-1}(\lambda)} v(x,0)\, d\mu_\lambda = \int_{\mathcal{C}_\lambda} w(x)\, d\mu_\lambda =: w_\lambda(x)$$

[2] With little effort, it is possible to prove that it is closed.

[3] It is actually continuous by local parabolic regularity, but we wont need it.

and we observe that, being $\lambda \mapsto \mu_\lambda$ Borel, so is $\lambda \mapsto w_\lambda$ and thus $(\lambda, x) \mapsto w_\lambda(x)$. Using the distributional formulation of the equation $\Delta w = \lambda\, w$, namely

$$\int_M w\, \Delta\varphi\, dg = \lambda \int_M w\, \varphi\, dg, \qquad \forall \varphi \in C^\infty_c(M)$$

and Fubini-Tonelli's theorem, it is readily checked that w_λ is a distributional, and thus classical solution of $\Delta w = \lambda\, w$. Finally, by the first relation in (2.3),

$$\int_{\psi^{-1}(\lambda)} v(x,t)\, d\mu_\lambda = e^{\lambda\, t}\, w_\lambda(x)$$

and recalling (2.2) completes the proof of (2.1). Finally, the strong minimum principle ensures that $w_\lambda(x) = 0$ for some $x \in M$ implies $w_\lambda \equiv 0$, so that it suffices to restrict ν to the measurable subset $\{w_\lambda(p) > 0\}$ for a fixed p. □

Lemma 2.4. *Let (M, g) be a complete N-dimensional Riemannian manifold with $\mathrm{Ric}_M \geqslant -(N-1)\, \kappa\, g$, $\kappa \geqslant 0$ and $p \in M$. For $\lambda > 0$ define*

$$\chi_\lambda = \chi_\lambda(\kappa, N) = \frac{N-1}{2} \left(\sqrt{\kappa + \frac{4\,\lambda}{(N-1)^2}} - \sqrt{\kappa} \right). \tag{2.4}$$

There exists $\bar w_\lambda \in \mathrm{Lip}_{\mathrm{loc}}(M)$ such that $\bar w_\lambda(p) = 1$, $\Delta \bar w_\lambda \leqslant \lambda\, \bar w_\lambda$ weakly on M and

$$\begin{cases} \bar w_\lambda(x) \geqslant c(N, \lambda)\, e^{\chi_\lambda\, d(x,p)} & \text{if } \kappa > 0 \\ \bar w_\lambda(x) \geqslant c(N, \varepsilon)\, e^{(\chi_\lambda - \varepsilon)\, d(x,p)} & \text{for any } \varepsilon > 0, \text{ if } \kappa = 0, \end{cases} \tag{2.5}$$

for $x \in M$, where $c(N)$ and $c(N, \varepsilon)$ are suitable positive constants.

Proof. We begin considering the case $\mathrm{Ric}_M \geqslant -(N-1)\, \kappa\, g$ for $\kappa > 0$. Eventually rescaling the metric, we can suppose without loss of generality that $\kappa = 1$. Moreover, the condition $\bar w_\lambda(p) = 1$ can be dropped, as it suffices to eventually multiply by a suitable constant. Let $\mathbb{H}$ denote the real hyperbolic space of dimension N, with corresponding Laplace-Beltrami operator $\Delta_{\mathbb{H}}$ and distance $\mathrm{d}_{\mathbb{H}}$. We identify $\mathbb{H}$ with the open ball $B_1 \subseteq \mathbb{R}^N$ equipped with the Poincaré metric $g = 4\,(1 - |z|^2)^{-2}\mathrm{Id}$, obtaining in particular

$$\mathrm{d}_{\mathbb{H}}(z, 0) = \log\left(\frac{1 + |z|}{1 - |z|} \right), \qquad \forall z \in B_1 \subseteq \mathbb{R}^N. \tag{2.6}$$

The Busemann function b_ν for the geodesic ray γ_ν from 0 with direction ν, $|\nu| = 1$ is explicitly given by

$$b_\nu(z) = \lim_{t \to +\infty} \mathrm{d}_{\mathbb{H}}(\gamma_\nu(t), z) - t = -\log\left(\frac{1 - |z|^2}{|z - \nu|^2} \right),$$

and satisfies

$$|\nabla_{\mathbb{H}} b_\nu| = 1, \qquad \Delta_{\mathbb{H}} b_\nu = N - 1.$$

From the latters, we immediately compute

$$\Delta_{\mathbb{H}}\, e^{\mu\, b_\nu} = \mu\, (\mu + N - 1)\, e^{\mu\, b_\nu}.$$

For $\lambda > 0$, we choose

$$\mu_\lambda := \frac{1}{2}\big(1 - N - \sqrt{(N-1)^2 + 4\,\lambda}\big) < 1 - N$$

and let

$$w_{\lambda,\nu}(z) = e^{\mu_\lambda\, b_\nu(z)} = \left(\frac{1-|z|^2}{|z-\nu|^2}\right)^{-\mu_\lambda}, \tag{2.7}$$

so that, being $\mu_\lambda\,(\mu_\lambda + N - 1) = \lambda$, $w_{\lambda,\nu}$ is a positive λ-eigenfunction. Finally, we let

$$w_\lambda(z) = \int_{\mathbb{S}^{N-1}} w_{\lambda,\nu}(z)\, d\mathcal{H}^{N-1}(\nu), \qquad z \in B_1 \subseteq \mathbb{R}^N,$$

which is again a positive λ-eigenfunction, radial by construction. As such, letting $f(r) = w_\lambda(z)$ with $r = \mathrm{d}_{\mathbb{H}}(z,0)$ and using polar hyperbolic coordinates, f obeys

$$f''(r) + (N-1)\tanh r\, f'(r) = \lambda\, f(r) \qquad f'(0) = 0.$$

Multiplying by $(\cosh r)^{N-1}$ both sides and integrating, we get

$$(\cosh r)^{N-1} f'(r) = \int_0^r \big((\cosh\tau)^{N-1} f'(\tau)\big)'\, d\tau = \lambda \int_0^r (\cosh\tau)^{N-1} f(\tau)\, d\tau \geqslant 0$$

which implies that w_λ is radially increasing. We claim that

$$w_\lambda(z) \geqslant c_N\, e^{\chi_\lambda\, \mathrm{d}_{\mathbb{H}}(z,0)} \tag{2.8}$$

for χ_λ given in (2.4). Using $w_{\lambda,\nu} > 0$ and the expression in (2.7), we get

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} w_{\lambda,\nu}(z)\, d\mathcal{H}^{N-1}(\nu) &\geqslant \int_{\{\nu\in\mathbb{S}^{N-1} : |z-\nu| \leqslant 2\,(1-|z|)\}} \left(\frac{1-|z|^2}{|z-\nu|^2}\right)^{-\mu_\lambda} d\mathcal{H}^{N-1}(\nu) \\ &\geqslant 4^{\mu_\lambda}\left(\frac{1+|z|}{1-|z|}\right)^{-\mu_\lambda} \mathcal{H}^{N-1}\big(\{\nu \in \mathbb{S}^{N-1} : |z-\nu| \leqslant 2\,(1-|z|)\}\big).\end{aligned}$$

Through an elementary geometric argument, we see that it holds

$$\mathcal{H}^{N-1}\big(\{\nu \in \mathbb{S}^{N-1} : |z-\nu| \leqslant 2\,(1-|z|)\}\big) \geqslant c_N\,(1-|z|)^{N-1}$$

for some $c_N > 0$, so that being $\mu_\lambda + N - 1 = -\chi_\lambda$,

$$w_\lambda(z) \geqslant c(N,\lambda)\,\frac{(1+|z|)^{\chi_\lambda+N-1}}{(1-|z|)^{\chi_\lambda}} \geqslant c(N,\lambda)\left(\frac{1+|z|}{1-|z|}\right)^{\chi_\lambda}.$$

Recalling formula (2.6) for the distance $\mathrm{d}_{\mathbb{H}}$ proves (2.8). Finally, let $\bar{w}_\lambda \in \mathrm{Lip}_{loc}(M)$ be defined through

$$\bar{w}_\lambda(x) = w_\lambda(z), \qquad \text{with} \quad \mathrm{d}_{\mathbb{H}}(z,0) = \mathrm{d}(x,p),$$

where d is the usual metric distance in M. Clearly $\bar{w}_\lambda$ is well defined by the radiality of w_λ and (2.8) holds true in M as well by construction. Since $\mathrm{Ric}_M \geqslant -g$, and w_λ is radially increasing, the Laplacian comparison implies that, weakly in M,

$$\Delta\bar{w}_\lambda(x) \leqslant \Delta_{\mathbb{H}} w_\lambda(z) = \lambda\, w_\lambda(z) = \lambda\, \bar{w}_\lambda(x).$$

The case $\kappa = 0$ is easier, as in the model space $\mathbb{R}^N$ we define

$$w_\lambda(z) = \int_{\mathbb{S}^{N-1}} e^{\sqrt{\lambda}\, z\cdot\nu}\, d\mathcal{H}^{N-1}(\nu), \qquad z \in \mathbb{R}^N,$$

which is a radial positive λ-eigenfunction. It is radially increasing by integrating, as before, the corresponding ODE, so that it suffices to prove the pointwise lower bound. To this end, let $\varepsilon > 0$, $z = r\, e_1$, $e_1 = (1, 0, \ldots, 0)$ and compute

$$\int_{\mathbb{S}^{N-1}} e^{\sqrt{\lambda}\, z\cdot\nu}\, d\mathcal{H}^{N-1}(\nu) \geqslant \int_{\{\nu\in\mathbb{S}^{N-1}: e_1\cdot\nu\geqslant(1-\varepsilon)\}} e^{\sqrt{\lambda}\, z\cdot\nu}\, d\mathcal{H}^{N-1}(\nu)$$
$$\geqslant e^{\sqrt{\lambda}\,(1-\varepsilon)\, r}\, \mathcal{H}^{N-1}\big(\{\nu\in\mathbb{S}^{N-1} : e_1\cdot\nu \geqslant (1-\varepsilon)\}\big)$$

which proves the claimed lower bound when $M = \mathbb{R}^N$. Applying the Laplacian comparison as before, we get the claim. □

Remark 2.5. While we focussed on the case $\lambda > 0$, the function $w_{\lambda,\nu}$ given in (2.7) is a positive eigenfunction in $\mathbb{H}_N$ for all μ_λ obeying $\mu_\lambda(\mu_\lambda + N - 1) = \lambda$. Choosing $-(N-1)/2 < \lambda < 0$ and a corresponding μ_λ, we see that the eternal caloric function $u(x,t) := e^{\lambda t} w_{\lambda,\nu}(x)$ on $\mathbb{H}_N \times \mathbb{R}$ is exponentially decreasing in time.

Corollary 2.6. *Let M, p be as above. If $w > 0$ solves $\Delta w = \lambda\, w$ for $\lambda > 0$, then*

$$\liminf_{r\to+\infty} \sup_{\partial B_r(p)} \frac{\log w}{r} \geqslant \chi_\lambda. \tag{2.9}$$

Proof. The statemenet of the corollary in unaffected by multiplying w for positive constants, so we can suppose that $w(p) = 2$. If (2.9) is false, there exists $\varepsilon > 0$ and a sequence $\{r_n\}$ with $r_n \to +\infty$ such that for any sufficiently large n it holds $w \leqslant \exp(\chi_\lambda\,(1-2\,\varepsilon)\, r_n)$ on $\partial B_{r_n}(p)$. Let $\bar{w}_\lambda$ and $c = c(N,\lambda)$ (or $c(N,\varepsilon)$, respectively) be given the previous Lemma. By (2.5), for sufficiently large n it holds

$$\bar{w}_\lambda \geqslant c \exp(\chi_\lambda\,(1-\varepsilon)\, r_n) \geqslant \exp(\chi_\lambda\,(1-2\,\varepsilon)\, r_n) \geqslant w$$

on $\partial B_{r_n}(p)$, so that the weak comparison principle for $-\Delta + \lambda$ in $B_{r_n}(p)$ implies $\bar{w}_\lambda(p) \geqslant w(p)$. As $\bar{w}_\lambda(p) = 1$ and $w(p) = 2$, this is a contradiction. □

Proof. of Theorem 1.3.

Suppose that $\mathrm{Ric}_M \geqslant -(N-1)\,\kappa\, g$ with $\kappa \geqslant 0$, let u be a caloric ancient solution in $M \times]-\infty, 1[$ and consider the representation given in (2.1) of Lemma 2.3. We will prove, separately for $\kappa = 0$ and $\kappa > 0$, that the assumed growth conditions force in both cases $\mathrm{supp}(\nu) = \{0\}$. This in turn implies that u is stationary and harmonic, concluding the proof in the case $\kappa > 0$, while an application Yau's elliptic Liouville theorem will ensure $u \equiv c > 0$ in the case $\kappa = 0$.

Case $\kappa = 0$. Suppose by contradiction that $\mathrm{supp}(\nu) \neq \{0\}$. Recall that $\mathrm{Ric}_M \geqslant 0$ implies that M possesses positive λ-eigenfunctions only for $\lambda \geqslant 0$, therefore we can suppose that there exists $0 < a < b$ such that $\nu([a,b]) > 0$.
As pointed out in the proof of Lemma 2.3, the function $[a,b] \ni \lambda \mapsto w_\lambda \in \mathbb{R}^M$ is Borel when $\mathbb{R}^M$ has the pointwise convergence topology. By Proposition 2.2, the set

$$E := \{\log w : w > 0,\ \Delta w = \lambda\, w \text{ for some } \lambda \in [a,b]\}$$

is equilipschitz, so that pointwise convergence and locally uniform convergence coincide on E, therefore the map

$$[a,b] \ni \lambda \mapsto \varphi(\lambda) := \log w_\lambda \in E$$

is Borel when E is equipped with the locally uniform topology. Consider the metric

$$\mathrm{d}_E(f,g) = \sup_n \left\{ \frac{1}{n} \sup_{B_n(p)} |f-g| \right\} \tag{2.10}$$

which is finite on E due to the above mentioned equilipschitzianity. The topology of (E, d_E) is finer than the locally uniform one, but φ is still Borel to (E, d_E), since

$$\varphi^{-1}(\{\mathrm{d}_E(f,0) \leqslant \delta\}) = \bigcap_n \big\{ \lambda \in \mathbb{R} : \sup_{B_n(p)} |\varphi(\lambda)| \leqslant \delta\, n \big\},$$

Lusin's theorem then provides a compact $K \subseteq [a,b]$ such that

$$\nu(K) \geqslant \nu([a,b])/2, \qquad \varphi\lfloor_K \in C^0(K, (E, \mathrm{d}_E)).$$

Fix a point $\lambda_0 \in K$ such that $\nu\big(I_r(\lambda_0) \cap K\big) > 0$ for all $r > 0$, where $I_r(\lambda_0) = [\lambda_0 - r, \lambda_0 + r]$ and let $\varepsilon > 0$ to be determined. The continuity of φ in λ_0 implies that there exists $r_\varepsilon > 0$ such that for any $n \geqslant 1$,

$$\log w_{\lambda_0} \leqslant \log w_\lambda + \varepsilon\, n, \qquad \text{in } B_n(p),\ \forall \lambda \in I_{r_\varepsilon}(\lambda_0) \cap K.$$

Taking the mean value over $I_{r_\varepsilon}(\lambda_0) \cap K$ with respect to the measure ν and using Jensen inequality we infer that in the ball $B_n(p)$ it holds

$$\log w_{\lambda_0} \leqslant \fint_{I_{r_\varepsilon}(\lambda_0) \cap K} \log w_\lambda \, d\nu + \varepsilon\, n \leqslant \log \left(\fint_{I_{r_\varepsilon}(\lambda_0) \cap K} w_\lambda \, d\nu \right) + \varepsilon\, n$$

and by the positivity of w_λ and the representation (2.1) of u we conclude

$$\log w_{\lambda_0} \leqslant -\log \nu\big(I_{r_\varepsilon}(\lambda_0) \cap K\big) + \log u + \varepsilon\, n$$

in $B_n(p)$. We take the supremum on $\partial B_n(p)$, divide by n and let $n \to +\infty$. By the assumption $\log u(x) \leqslant o(\mathrm{d}(x,p))$, we get

$$\limsup_n \sup_{\partial B_n} \frac{\log w_{\lambda_0}}{n} \leqslant \varepsilon,$$

which gives a contradiction to (2.9) if $\varepsilon < \chi_{\lambda_0}$.

Case $\kappa > 0$. Consider as before the Choquet representation (2.1) of u. The assumption $u(x,t) \leqslant e^{o(\mathrm{d}(x,p)-t)}$ as $\mathrm{d}(x,p) - t \to +\infty$ entails

$$u(p,t) = \int e^{\lambda\, t}\, w_\lambda(p)\, d\nu \leqslant e^{o(-t)}. \tag{2.11}$$

The latter in turn implies that $\nu(]-\infty, 0[) = 0$, for otherwise, being $\lambda \mapsto w_\lambda(p)$ Borel, Lusin's theorem provides a compact $K \subset]-\infty, 0[$ such that

$$\nu(K) > 0, \qquad \lambda \mapsto w_\lambda(p) \in C^0(K, \mathbb{R}).$$

Since $w_\lambda(p) > 0$ for ν a. e. λ, we infer

$$\int e^{\lambda\, t}\, w_\lambda(p)\, d\nu \geqslant e^{t\, \max K} \min_K w_\lambda(p)\, \nu(K),$$

contradicting (2.11) for $t \to -\infty$, due to $\max K < 0$. The rest of the proof follows verbatim as in the previous case, showing that $\nu(]0, +\infty[) = 0$ as well. Therefore $\mathrm{supp}(\nu) = \{0\}$ and thus u is harmonic and stationary.

□

Remark 2.7. In the case $\kappa > 0$ we actually proved the following statement. If there is a point $p \in M$ such that $u(p,t) \leqslant e^{o(-t)}$ as $t \to -\infty$ and t_0 such that $u(x,t_0) \leqslant e^{o(\mathrm{d}(x,p))}$ for $\mathrm{d}(x,p) \to +\infty$, then u is a positive harmonic function.

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