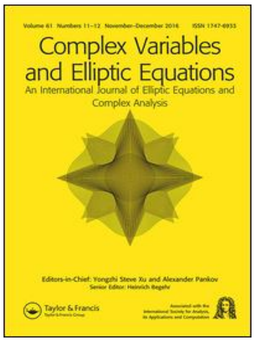

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Existence and multiplicity of positive solutions for the fractional Laplacian under subcritical or critical growth

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ABSTRACT
We study a Dirichlet type problem for an equation involving the fractional Laplacian and a reaction term subject to either subcritical or critical growth conditions, depending on a positive parameter. Applying a critical point result of Bonanno, we prove existence of one or two positive solutions as soon as the parameter lies under an (explicitly determined) value. As an application, we find two positive solutions for a fractional Ambrosetti–Brezis–Cerami problem.

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1. Introduction

This paper is devoted to the Dirichlet problem for a pseudo-differential equation of fractional order:

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1}$$

Here $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ is a bounded domain with $C^{1,1}$ boundary, and the leading operator is the fractional Laplacian defined for all $u \in \mathcal{S}(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = 2\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy. \tag{2}$$

The autonomous reaction $f \in C(\mathbb{R})$ is assumed to be non-negative and dominated at infinity by a power of u, namely, for all $t \in \mathbb{R}$

$$0 \le f(t) \le a_0(1 + |t|^{q-1}) \ (a_0 > 0,\ q \le 2_s^*), \tag{3}$$

where $2_s^* = 2N/(N - 2s)$ denotes the critical exponent for the fractional Sobolev space $H^s(\mathbb{R}^N)$ (see [1]). Finally, $\lambda > 0$ is a parameter.

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Problem (1) admits a variational formulation by means of the energy functional

$$J_\lambda(u) = \frac{[u]_s^2}{2} - \lambda \int_\Omega F(u)\,\mathrm{d}x,$$

where $[\,\cdot\,]_s$ denotes the Gagliardo seminorm and F is a primitive of f, i.e. weak solutions of (1) coincide with critical points of J_λ in a convenient subspace of $H^s(\mathbb{R}^N)$ (see Section 2 below for details). We note that, for $\lambda = 1$, problem (1) embraces the Dirichlet problem with pure power nonlinearities:

$$\begin{cases} (-\Delta)^s\, u = \mu u^{p-1} + u^{q-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{4}$$

with $1 < p < q \le 2_s^*$ and $\mu > 0$.

For a general introduction to the fractional Laplacian, we refer to [1–4]. The study of (1) (or closely related problems) via variational methods started from the work of Servadei and Valdinoci [5,6]. Here we distinguish between the *subcritical* ($q < 2_s^*$ in (3)) and *critical* ($q = 2_s^*$) cases. In the subcritical case, we mention for instance the contributions of [7–14] and the monograph [15].

In the critical case, the main difficulty lies in the fact that J_λ does not satisfy the (usual in variational methods) Palais-Smale compactness condition. In particular, problem (4) with $p = 2$, $q = 2_s^*$ represents a fractional counterpart of the famous Brezis–Nirenberg problem [16]. Again, the first result in this direction is due to Servadei and Valdinoci [17] (see also [18–20]). Later, Barrios et al. [21] studied (4) with $1 < p < q = 2_s^*$, which for $s = 1$ reduces to the problem with concave–convex nonlinearities studied by Ambrosetti, Brezis and Cerami in [22]. In particular, they proved that in the concave case $1 < p < 2$, for $\mu > 0$ small enough, such problem has at least two positive solutions $u_\mu < w_\mu$, employing both topological (sub-supersolutions) and variational methods.

Our approach to problem (1) is purely variational, mainly based on a critical point theorem of Bonanno [23] and some of its consequences, presented in [24–26]. The main feature of such method is a strategy to find a local minimizer of a J_λ-type functional, which only requires a *local* Palais-Smale condition. Our results are

(a) In the subcritical case ($q < 2_s^*$), we apply an abstract result of [24] and explicitly compute a real number $\lambda^* > 0$ s.t. problem (1) admits at least two positive solutions u_λ, v_λ for all $\lambda \in (0, \lambda^*)$.

(b) In the critical case ($q = 2_s^*$), we first study a generalization of problem (4), explicitly determining a real number $\mu^* > 0$ s.t. there exist at least one positive solution u_μ for all $\mu \in (0, \mu^*)$. Then, we focus on (4) with $1 < p < 2 < q = 2_s^*$ and, applying the mountain pass theorem, we produce a second positive solution $w_\mu > u_\mu$ for all $\mu \in (0, \mu^*)$ (here we mainly follow [26]).

To our knowledge, this is the first application of the ideas of [23] in the field of fractional Laplacian equations. A noteworthy difference with respect to the classical elliptic case is: in this approach it is essential to explicitly compute $J_\lambda(\bar{u})$ at some Sobolev-type function

$\bar{u} : \Omega \to \mathbb{R}$, which is usually chosen in such a way to have a piecewise constant $|\nabla \bar{u}|$. In the fractional framework, functions may have no gradient at all, and the computation of the Gagliardo seminorm is often prohibitive, so $\bar{u}$ will be chosen as (a multiple of) the solution of a fractional torsion equation in a ball (see (7)).

We also remark that our main result in part (*b*) is formally equivalent to the main result of [21], but with two substantial differences: the first solution u_μ is found as a local minimizer of J_λ (instead of being detected via sub-supersolutions, and *a posteriori* proved to be a minimizer), and moreover the interval $(0, \mu^*)$ is *explicitly* determined (although possibly not optimal).

The paper has the following structure: in Section 2, we collect the necessary preliminaries; in Section 3, we develop part (*a*) of our study; in Sections 4 and 5, we focus on part (*b*).

Notation: Throughout the paper, for any $A \subset \mathbb{R}^N$, we shall set $A^c = \mathbb{R}^N \setminus A$. By $|A|$ we will denote either the N-dimensional Lebesgue measure or the $(N-1)$-dimensional Hausdorff measure of A, which will be clear from the context. For any two measurable functions u, v, $u = v$ in A will stand for $u(x) = v(x)$ for a.e. $x \in A$ (and similar expressions). We will often write $t^\nu = |t|^{\nu-1}t$ for $t \in \mathbb{R}$, $\nu > 1$. For any $t \in \mathbb{R}$, we set $t^\pm = \max\{\pm t, 0\}$. By $B_r(x)$, we denote the open ball centered at $x \in \mathbb{R}^N$ of radius $r > 0$. For all $\nu \in [1, \infty]$, $\|\cdot\|_\nu$ denotes the standard norm of $L^\nu(\Omega)$ (or $L^\nu(\mathbb{R}^N)$, which will be clear from the context). Every function u defined in Ω will be identified with its 0-extension to $\mathbb{R}^N$. Moreover, C will denote a positive constant (whose value may change line by line).

2. Preliminaries

We begin by recalling some basic notions about fractional Sobolev spaces (for details we refer to [1]). We define the Gagliardo seminorm by setting for all measurable $u : \mathbb{R}^N \to \mathbb{R}$

$$[u]_s = \left[\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right]^{\frac{1}{2}}.$$

Accordingly, we define the space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \, [u]_s < \infty \right\}.$$

The embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous, and the fractional Talenti constant is given by the following lemma (see [27, Theorem 1.1] and [1, Proposition 3.6]):

Lemma 2.1: *We have*

$$T(N, s) = \max_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{2_s^*}}{[u]_s} = \frac{s^{\frac{1}{2}} \Gamma(\frac{N-2s}{2})^{\frac{1}{2}} \Gamma(N)^{\frac{s}{N}}}{2^{\frac{1}{2}} \pi^{\frac{N+2s}{4}} \Gamma(1-s)^{\frac{1}{2}} \Gamma(\frac{N}{2})^{\frac{s}{N}}} > 0,$$

the maximum being attained at the functions

$$u(x) = \frac{a}{(b + |x - x_0|^2)^{\frac{N-2s}{2}}} \quad (a, b > 0, \; x_0 \in \mathbb{R}^N).$$

Now we establish a variational formulation for (1), following [6] (see also [28]). Set

$$H_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \Omega^c \right\},$$

a Hilbert space under the inner product

$$\langle u, v \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y$$

and the corresponding norm $\|u\| = [u]_s$ (see [6, Lemma 7]). The dual space of $H_0^s(\Omega)$ is denoted $H^{-s}(\Omega)$. By Lemma 2.1 and Hölder's inequality, for any $\nu \in [1, 2_s^*]$ the embedding $H_0^s(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous and for all $u \in H_0^s(\Omega)$ we have

$$\|u\|_\nu \leq T(N, s) |\Omega|^{\frac{2_s^* - \nu}{2_s^* \nu}} \|u\|. \tag{5}$$

Further, the embedding is compact iff $\nu < 2_s^*$ (see [6, Lemma 8]).

In order to deal with problem (1) variationally, we assume the hypotheses on the reaction f:

$\mathbf{H}_0$ $f \in C(\mathbb{R})$, $F(t) = \int_0^t f(\tau) \, \mathrm{d}\tau$, and

(i) $f(t) \geq 0$ for all $t \in \mathbb{R}$;
(ii) $f(t) \leq a_0(1 + |t|^{2_s^* - 1})$ for all $t \in \mathbb{R}$ ($a_0 > 0$).

We set for all $u \in H_0^s(\Omega)$, $\lambda > 0$

$$\Phi(u) = \frac{\|u\|^2}{2}, \quad \Psi(u) = \int_\Omega F(u) \, \mathrm{d}x, \quad J_\lambda(u) = \Phi(u) - \lambda \Psi(u)$$

(Ψ is well defined by virtue of hypothesis $\mathbf{H}_0$ (i) (ii)). Then $\Phi, \Psi, J_\lambda \in C^1(H_0^s(\Omega))$ with

$$\langle J_\lambda'(u), \varphi \rangle = \langle u, \varphi \rangle - \lambda \int_\Omega f(u) \varphi \, \mathrm{d}x$$

for all $u, \varphi \in H_0^s(\Omega)$. We say that u is a (weak) solution of problem (1) if $J_\lambda'(u) = 0$ in $H^{-s}(\Omega)$, that is, for all $\varphi \in H_0^s(\Omega)$ we have

$$\langle u, \varphi \rangle = \lambda \int_\Omega f(u) \varphi \, \mathrm{d}x. \tag{6}$$

The regularity theory for fractional Dirichlet problems was essentially developed in [29] (see also [21,28]). While smooth in Ω, solutions are in general singular on $\partial\Omega$, so the best global regularity we can expect is weighted Hölder continuity, in the following sense. Set for all $x \in \overline{\Omega}$

$$\mathrm{d}_\Omega(x) = \mathrm{dist}(x, \Omega^c),$$

then define the spaces

$$C_s^0(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} \in C^0(\overline{\Omega}) \right\}, \quad \|u\|_{0,s} = \left\| \frac{u}{\mathrm{d}_\Omega^s} \right\|_\infty,$$

and for any $\alpha \in (0, 1)$

$$C_s^\alpha(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} \in C^\alpha(\overline{\Omega}) \right\},$$

$$\|u\|_{\alpha,s} = \|u\|_{0,s} + \sup_{x \neq y} \frac{|u(x)/\mathrm{d}_\Omega^s(x) - u(y)/\mathrm{d}_\Omega^s(y)|}{|x-y|^\alpha}.$$

The positive order cone of $C_s^0(\overline{\Omega})$ has a nonempty interior given by

$$\mathrm{int}(C_s^0(\overline{\Omega})_+) = \left\{ u \in C_s^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} > 0 \text{ in } \overline{\Omega} \right\}.$$

For the reader's convenience, we recall from [28, Theorems 2.3, 3.2 and Lemma 2.7] the main properties of weak solutions:

Proposition 2.2: *Let* $\mathbf{H}_0$ *hold,* $u \in H_0^s(\Omega)$ *be a weak solution of* (1). *Then:*

(i) *(a priori bound)* $u \in L^\infty(\Omega)$*;*
(ii) *(regularity)* $u \in C_s^\alpha(\overline{\Omega})$ *with* $\alpha \in (0, s]$ *depending only on* s *and* Ω*;*
(iii) *(Hopf's lemma) if* $u \neq 0$*, then* $u \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

By Proposition 2.2 (iii), we see that, whenever $u \in H_0^s(\Omega) \setminus \{0\}$ satisfies (6), then in particular $u > 0$ in Ω. Moreover, assuming further that f is locally Lipschitz in $\mathbb{R}$, from [29, Corollary 1.6], we deduce that $u \in C^\beta(\Omega)$ for any $\beta \in [1, 1+2s)$, which along with Proposition 2.2 (ii) implies that for all $x \in \mathbb{R}^N$ the mapping

$$y \mapsto \frac{u(x) - u(y)}{|x-y|^{N+2s}}$$

lies in $L^1(\mathbb{R}^N)$. Then, testing (6) with any $\varphi \in C_c^\infty(\Omega)$ and applying (2), we have

$$\int_\Omega (-\Delta)^s u\varphi \,\mathrm{d}x = \langle u, \varphi \rangle = \int_\Omega f(u)\varphi \,\mathrm{d}x,$$

i.e. u solves (1) pointwisely.

We also recall the following result, relating the local minimizers of the energy functional J_λ in $H_0^s(\Omega)$ and in $C_s^0(\overline{\Omega})$, respectively (see [28, Theorem 1.1], [21, Proposition 2.5], and [30, Theorem 1.1] for a nonlinear extension), namely an analog for the fractional case of the main result of [31]:

Proposition 2.3: *Let* $\mathbf{H}_0$ *hold,* $u \in H_0^s(\Omega)$. *Then, the following are equivalent:*

(i) *there exists* $\rho > 0$ *s.t.* $J_\lambda(u+v) \geq J_\lambda(u)$ *for all* $v \in H_0^s(\Omega) \cap C_s^0(\overline{\Omega})$, $\|v\|_{0,s} \leq \rho$*;*
(ii) *there exists* $\sigma > 0$ *s.t.* $J_\lambda(u+v) \geq J_\lambda(u)$ *for all* $v \in H_0^s(\Omega)$, $\|v\| \leq \sigma$.

As pointed out in the Introduction, we will make use of the fractional torsion equation on a ball:

$$\begin{aligned} &(-\Delta)^s u_R = 1 \quad \text{in } B_R(x_0) \\ &u_R = 0 \quad \text{in } B_R(x_0)^c, \end{aligned} \tag{7}$$

where $x_0 \in \mathbb{R}^N$, $R > 0$. The solution of (7) (defined as in (6)) is unique, given by

$$u_R(x) = A(N,s)(R^2 - |x - x_0|^2)_+^s, \; A(N,s) = \frac{s\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}\Gamma(1+s)\Gamma(1-s)}$$

(see [2, p. 33] or [29, Equation (1.4)]). This simple example is popular in fractional regularity theory, as it shows that solutions of Dirichlet problems may be singular at the boundary. For future use, we compute some norms of u_R:

Lemma 2.4: *For all $x_0 \in \mathbb{R}^N$, $R > 0$ we have*

(i) $\|u_R\|_\nu = A(N,s)[\frac{\pi^{\frac{N}{2}}\Gamma(1+\nu s)R^{N+2\nu s}}{\Gamma(\frac{N+2\nu s+2}{2})}]^{\frac{1}{\nu}}$ *for all* $\nu \geq 1$;

(ii) $[u_R]_s = [\frac{s\Gamma(\frac{N}{2})R^{N+2s}}{2\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})}]^{\frac{1}{2}}$.

Proof: First we recall the well-known formulas

$$|\partial B_1(0)| = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}, \quad \int_0^1 (1-\rho^2)^\alpha \rho^{N-1}\, d\rho = \frac{\Gamma(\frac{N}{2})\Gamma(1+\alpha)}{2\Gamma(\frac{N+2\alpha+2}{2})} \quad (\alpha > 0),$$

then for all $\nu \geq 1$, we compute

$$\begin{aligned}\int_{B_R(x_0)} u_R^\nu(x)\, dx &= A(N,s)^\nu \int_{B_R(x_0)} (R^2 - |x-x_0|^2)^{\nu s}\, dx \\ &= A(N,s)^\nu R^{N+2\nu s}|\partial B_1(0)| \int_0^1 (1-\rho^2)^{\nu s}\rho^{N-1}\, d\rho \\ &= A(N,s)^\nu \frac{\pi^{\frac{N}{2}}\Gamma(1+\nu s)R^{N+2\nu s}}{\Gamma(\frac{N+2\nu s+2}{2})},\end{aligned}$$

which implies (i). Furthermore, testing (7) with $u_R \in H_0^s(B_R(x_0))$ and applying (i) with $\nu = 1$, we have

$$\begin{aligned}[u_R]_s^2 &= \int_{B_R(x_0)} u_R\, dx \\ &= A(N,s)\frac{\pi^{\frac{N}{2}}\Gamma(1+s)R^{N+2s}}{\Gamma(\frac{N+2s+2}{2})} \\ &= \frac{s\Gamma(\frac{N}{2})R^{N+2s}}{2\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})},\end{aligned}$$

which gives (ii). ■

Remark 2.5: We note that some results here are affected by the definition (2), which is the same adopted in [21]. Other works on the subject, for instance [1,2,27], define the fractional

Laplacian as

$$(-\Delta)^s u(x) = C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,\ C(N,s) = \frac{2^{2s}s\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)} > 0,$$

where the multiplicative constant is required to equivalently define $(-\Delta)^s$ by means of the Fourier transform. In this paper, explicit constants are one of the main issues, so we decide to follow the standard of [21] in order to easily compare similar results.

3. Two positive solutions under subcritical growth

In this section, following [24] as a model, we study (1) under the hypotheses:

$\mathbf{H}_1 f \in C(\mathbb{R})$, $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$ satisfy

(i) $f(t) \geq 0$ for all $t \in \mathbb{R}$;
(ii) $f(t) \leq a_p|t|^{p-1} + a_q|t|^{q-1}$ for all $t \in \mathbb{R}$ ($1 \leq p < 2 < q < 2_s^*$, $a_p, a_q > 0$);
(iii) $\lim_{t\to 0^+} \frac{F(t)}{t^2} = \infty$
(iv) $0 < \rho F(t) \leq f(t)t$ for all $t \geq M$ ($\rho > 2, M > 0$).

Hypotheses $\mathbf{H}_1$ conjure for f a subcritical, superlinear growth at infinity, as well as a sublinear growth near the origin, while (iv) is an Ambrosetti–Rabinowitz condition.

First, we recall the classical Palais-Smale condition at level $c \in \mathbb{R}$, for a functional $J \in C^1(X)$ on a Banach space X:

$(PS)_c$ Every sequence (u_n) in X, s.t. $J(u_n) \to c$ and $J'(u_n) \to 0$ in X^*, has a convergent subsequence.

We say that J satisfies (PS), if J satisfies $(PS)_c$ for any $c \in \mathbb{R}$.

We will apply the following abstract result, slightly rephrased from [24, Theorem 2.1]:

Theorem A: *Let X be a Banach space, $\Phi, \Psi \in C^1(X)$, $J_\lambda = \Phi - \lambda\Psi$ $(\lambda > 0)$, $r \in \mathbb{R}$, $\bar{u} \in X$ satisfy*

(A_1) $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$;
(A_2) $0 < \Phi(\bar{u}) < r$;
(A_3) $\sup_{\Phi(u)\leq r} \frac{\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$;
(A_4) $\inf_{u\in X} J_\lambda(u) = -\infty$ *for all* $\lambda \in I_r = (\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, [\sup_{\Phi(u)\leq r} \frac{\Psi(u)}{r}]^{-1})$.

Then, for all $\lambda \in I_r$ for which J_λ satisfies (PS), there exist $u_\lambda, v_\lambda \in X$ s.t.

$$J'_\lambda(u_\lambda) = J'_\lambda(v_\lambda) = 0, \quad J_\lambda(u_\lambda) < 0 < J_\lambda(v_\lambda).$$

Let $T(N,s) > 0$ be defined by Lemma 2.1, set

$$\lambda^* = \frac{1}{2T(N,s)^2|\Omega|^{\frac{2_s^*-2}{2_s^*}}} \left(\frac{a_p}{p}\right)^{\frac{2-q}{q-p}} \left(\frac{a_q}{q}\right)^{\frac{p-2}{q-p}} \left(\frac{2-p}{q-2}\right)^{\frac{2-p}{q-p}} \frac{q-2}{q-p} > 0. \tag{8}$$

We have the multiplicity result:

Theorem 3.1: *Let* $\mathbf{H}_1$ *hold,* $\lambda^* > 0$ *be defined by (8). Then, for all* $\lambda \in (0, \lambda^*)$*, (1) has at least two solutions* $u_\lambda, v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Proof: Without loss of generality, we may assume $f(t) = 0$ for all $t \le 0$. We are going to apply Theorem A. Set $X = H_0^s(\Omega)$ and define Φ, Ψ, J_λ as in Section 2, then clearly $\Phi, \Psi \in C^1(H_0^s(\Omega))$ and

$$\inf_{u \in H_0^s(\Omega)} \Phi(u) = \Phi(0) = \Psi(0) = 0,$$

hence hypothesis (A_1) holds. Set

$$r = \frac{|\Omega|^{\frac{2}{2_s^*}}}{2T(N,s)^2} \left[\frac{a_p q(2-p)}{a_q p(q-2)} \right]^{\frac{2}{q-p}} > 0. \tag{9}$$

For all $u \in H_0^s(\Omega)$, $\Phi(u) \le r$, we have $\|u\| \le (2r)^{\frac{1}{2}}$. So, by hypotheses $\mathbf{H}_1$ (i) (ii), along with (5), (8) and (9), we obtain

$$\begin{aligned}
\frac{\Psi(u)}{r} &\le \frac{a_p}{pr}\|u\|_p^p + \frac{a_q}{qr}\|u\|_q^q \\
&\le \frac{a_p}{pr} T(N,s)^p |\Omega|^{\frac{2_s^*-p}{2_s^*}} (2r)^{\frac{p}{2}} + \frac{a_q}{qr} T(N,s)^q |\Omega|^{\frac{2_s^*-q}{2_s^*}} (2r)^{\frac{q}{2}} \\
&= 2T(N,s)^2 |\Omega|^{\frac{2_s^*-2}{2_s^*}} \left(\frac{a_p}{p}\right)^{\frac{q-2}{q-p}} \left(\frac{a_q}{q}\right)^{\frac{2-p}{q-p}} \left(\frac{2-p}{q-2}\right)^{\frac{p-2}{q-p}} \\
&\quad + 2T(N,s)^2 |\Omega|^{\frac{2_s^*-2}{2_s^*}} \left(\frac{a_p}{p}\right)^{\frac{q-2}{q-p}} \left(\frac{a_q}{q}\right)^{\frac{2-p}{q-p}} \left(\frac{2-p}{q-2}\right)^{\frac{q-2}{q-p}} \\
&= 2T(N,s)^2 |\Omega|^{\frac{2_s^*-2}{2_s^*}} \left(\frac{a_p}{p}\right)^{\frac{q-2}{q-p}} \left(\frac{a_q}{q}\right)^{\frac{2-p}{q-p}} \left(\frac{2-p}{q-2}\right)^{\frac{p-2}{q-p}} \frac{q-p}{q-2} = \frac{1}{\lambda^*}.
\end{aligned}$$

Summarizing,

$$\sup_{\Phi(u) \le r} \frac{\Psi(u)}{r} \le \frac{1}{\lambda^*}. \tag{10}$$

Now fix $\lambda \in (0, \lambda^*)$. Since $\partial\Omega$ is $C^{1,1}$, we can find $x_0 \in \mathbb{R}^N$, $R > 0$ largest s.t. $B_R(x_0) \subseteq \Omega$. Let $K > 0$ be s.t.

$$K \frac{s\Gamma(\frac{N}{2})\Gamma(1+2s)\Gamma(\frac{N+2s+2}{2})R^{2s}}{\pi^{\frac{N}{2}}\Gamma(1+s)^2\Gamma(1-s)\Gamma(\frac{N+4s+2}{2})} > \frac{1}{\lambda}. \tag{11}$$

By $\mathbf{H}_1$ (iii), we can find $\varepsilon > 0$ s.t. for all $t \in [0, \varepsilon]$

$$F(t) \ge Kt^2. \tag{12}$$

Finally, fix

$$0 < \delta < \min\left\{ \left[\frac{4\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})r}{s\Gamma(\frac{N}{2})R^{N+2s}} \right]^{1/2}, \frac{2\pi^{\frac{N}{2}}\Gamma(1+s)\Gamma(1-s)\varepsilon}{s\Gamma(\frac{N}{2})R^{2s}} \right\}. \tag{13}$$

Now let u_R be the solution of (7) in $B_R(x_0)$, and set $\bar{u} = \delta u_R \in H^s_0(\Omega)$. Then we have by Lemma 2.4(ii) and (13)

$$\Phi(\bar{u}) = \frac{s\Gamma(\frac{N}{2})R^{N+2s}\delta^2}{4\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})} < r,$$

which implies (A_2). Besides, by (13) we have for all $x \in \Omega$

$$0 \le \bar{u}(x) \le \frac{s\Gamma(\frac{N}{2})R^{2s}\delta}{2\pi^{\frac{N}{2}}\Gamma(1+s)\Gamma(1-s)} < \varepsilon,$$

hence by (12) and Lemma 2.4(i)

$$\Psi(\bar{u}) \ge \int_\Omega K\bar{u}^2\,\mathrm{d}x = K\delta^2\|u_R\|_2^2 = K\frac{s^2\Gamma(\frac{N}{2})^2\Gamma(1+2s)R^{N+4s}}{4\pi^{\frac{N}{2}}\Gamma(1+s)^2\Gamma(1-s)^2\Gamma(\frac{N+4s+2}{2})}\delta^2.$$

The relations above and (11) imply

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge K\frac{s\Gamma(\frac{N}{2})\Gamma(1+2s)\Gamma(\frac{N+2s+2}{2})R^{2s}}{\pi^{\frac{N}{2}}\Gamma(1+s)^2\Gamma(1-s)\Gamma(\frac{N+4s+2}{2})} > \frac{1}{\lambda}.$$

Recalling that $\lambda < \lambda^*$, by (10) we have

$$\sup_{\Phi(u)\le r}\frac{\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})},$$

which yields at once (A_3) and $\lambda \in I_r$. By $\mathbf{H}_1$ (iv) we can find $C > 0$ s.t. for all $t \ge M$

$$F(t) \ge Ct^\rho. \tag{14}$$

Now pick $w \in C^\infty_c(\Omega) \setminus \{0\}$. By (14), and recalling that $F(t) \ge 0$ for all $t \in \mathbb{R}$, we have for all $\tau > 0$

$$\begin{aligned} J_\lambda(\tau w) &\le \frac{\|w\|^2}{2}\tau^2 - \lambda\int_{\{w\le M/\tau\}} F(\tau w)\,\mathrm{d}x - \lambda\int_{\{w>M/\tau\}} C(\tau w)^\rho\,\mathrm{d}x \\ &\le \frac{\|w\|^2}{2}\tau^2 - \lambda\int_\Omega C(\tau w)^\rho\,\mathrm{d}x + \lambda\int_{\{w\le M/\tau\}} CM^\rho\,\mathrm{d}x \\ &\le \frac{\|w\|^2}{2}\tau^2 - \lambda C\|w\|_\infty^\rho|\Omega|\tau^\rho + \lambda CM^\rho|\Omega| \end{aligned}$$

and the latter tends to $-\infty$ as $\tau \to \infty$ (since $\rho > 2$). So we see that (A_4) holds as well.

Finally, we prove that J_λ satisfies (PS). Let (u_n) be a sequence in $H^s_0(\Omega)$ s.t. $|J_\lambda(u_n)| \le C$, $J'_\lambda(u_n) \to 0$ in $H^{-s}(\Omega)$. Then, for all $n \in \mathbb{N}$ we have

$$\frac{\|u_n\|^2}{2} - \lambda \int_\Omega F(u_n)\,dx \le C \tag{15}$$

and for all $\varphi \in H^s_0(\Omega)$

$$\left|\langle u_n, \varphi\rangle - \lambda \int_\Omega f(u_n)\varphi\,dx\right| \le \|J'_\lambda(u_n)\|\,\|\varphi\| \tag{16}$$

Multiplying (15) by $\rho > 2$ (as in $\mathbf{H}_1$ (iv)), testing (16) with u_n, and subtracting,

$$\begin{aligned}\frac{\rho-2}{2}\|u_n\|^2 &\le \lambda \int_\Omega \big(\rho F(u_n) - f(u_n)u_n\big)\,dx + \|J'_\lambda(u_n)\|\,\|u_n\| + C \\ &\le \lambda \int_{\{0\le u_n\le M\}} C\left(|u_n|^p + |u_n|^q\right)\,dx + \|J'_\lambda(u_n)\|\,\|u_n\| + C \\ &\le \lambda C(M^p + M^q)|\Omega| + \|J'_\lambda(u_n)\|\,\|u_n\| + C.\end{aligned}$$

So (u_n) is bounded in $H^s_0(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup u$ in $H^s_0(\Omega)$, $u_n \to u$ in $L^p(\Omega), L^q(\Omega)$, and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. Testing (16) this time with $u_n - u \in H^s_0(\Omega)$, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\|u_n - u\|^2 &\le \langle u, u_n - u\rangle + \lambda \int_\Omega \left(a_p|u_n|^{p-1} + a_q|u_n|^{q-1}\right)|u_n - u|\,dx \\ &\quad + \|J'_\lambda(u_n)\|\,\|u_n - u\| \\ &\le \langle u, u_n - u\rangle + \lambda\left(a_p\|u_n\|_p^{p-1}\|u_n - u\|_p + a_q\|u_n\|_q^{q-1}\|u_n - u\|_q\right) \\ &\quad + \|J'_\lambda(u_n)\|\,\|u_n - u\|,\end{aligned}$$

(where we used $\mathbf{H}_1$ (ii) and Hölder's inequality), and the latter tends to 0 as $n \to \infty$. So, $u_n \to u$ in $H^s_0(\Omega)$. (Note that we actually proved that J_λ is unbounded from below and satisfies (PS) for *all* $\lambda > 0$.)

By Theorem A, there exist $u_\lambda, v_\lambda \in H^s_0(\Omega)$ s.t.

$$J'_\lambda(u_\lambda) = J'_\lambda(v_\lambda) = 0, \quad J_\lambda(u_\lambda) < 0 < J_\lambda(v_\lambda).$$

Therefore, $u_\lambda, v_\lambda \not\equiv 0$ solve (1). By $\mathbf{H}_1$ (i) and Proposition 2.2, finally, we have $u_\lambda, v_\lambda \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$. ■

We focus now on problem (4), with $1 < p < 2 < q < 2^*_s$ (subcritical case) and $\mu > 0$. Set

$$\mu^* = \left[2T(N,s)^2|\Omega|^{\frac{2^*_s-2}{2^*_s}}\right]^{\frac{p-q}{q-2}} p\, q^{\frac{2-p}{q-2}} \left(\frac{2-p}{q-2}\right)^{\frac{2-p}{q-2}} \left(\frac{q-2}{q-p}\right)^{\frac{q-p}{q-2}} > 0. \tag{17}$$

We have the multiplicity result:

Corollary 3.2: *Let* $1 < p < 2 < q < 2_s^*$, $\mu^* > 0$ *be defined by* (17). *Then, for all* $\mu \in (0, \mu^*)$ (4) *has at least two solutions* $u_\mu, v_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Proof: Set for all $t \in \mathbb{R}$, $\mu \in (0, \mu^*)$

$$f(t) = \mu(t^+)^{p-1} + (t^+)^{q-1}.$$

Then f satisfies $\mathbf{H}_1$ with $a_p = \mu$, $a_q = 1$, and any $\rho \in (2, q)$. In view of (17), here (8) rephrases as

$$\lambda^* = \frac{1}{2T(N,s)^2|\Omega|^{\frac{2_s^*-2}{2_s^*}}} p^{\frac{q-2}{q-p}} q^{\frac{2-p}{q-p}} \left(\frac{2-p}{q-2}\right)^{\frac{2-p}{q-p}} \frac{q-2}{q-p} \mu^{\frac{2-q}{q-p}} > 1.$$

Hence we can apply Theorem 3.1 with $\lambda = 1$ and find $u_\mu, v_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ solutions to (4). ■

We present an example for Corollary 3.2:

Example 3.3: Set $s = \frac{1}{2}$, $p = \frac{3}{2}$, $q = 3$, $N = 2$ and

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\right\}.$$

Then we have $2_{1/2}^* = 4 > 3$, $|\Omega| = 6\pi$, while Lemma 2.1 gives

$$T\left(2, \frac{1}{2}\right) = \frac{(\frac{1}{2})^{\frac{1}{2}}\Gamma(\frac{1}{2})^{\frac{1}{2}}\Gamma(2)^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{3}{4}}\Gamma(\frac{1}{2})^{\frac{1}{2}}\Gamma(1)^{\frac{1}{4}}} = \frac{1}{2\pi^{\frac{3}{4}}}.$$

Therefore, (17) becomes

$$\mu^* = \left[2\left(\frac{1}{2\pi^{\frac{3}{4}}}\right)^2 (6\pi)^{\frac{1}{2}}\right]^{-\frac{3}{2}} \frac{3}{2} 3^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{2}{3}\right)^{\frac{3}{2}} = \frac{2^{\frac{3}{4}}\pi^{\frac{3}{2}}}{3^{\frac{3}{4}}}.$$

By Corollary 3.2, for all $\mu \in (0, \mu^*)$ (4) has at least two positive solutions.

4. One positive solution under critical growth

In this section, we study the slight generalization of problem (4):

$$\begin{cases} (-\Delta)^s u = \mu g(u) + u^{2_s^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{18}$$

with $\mu > 0$ and assuming the hypotheses on g:

$\mathbf{H}_2$ $g \in C(\mathbb{R})$, $G(t) = \int_0^t g(\tau)\, d\tau$ satisfy

(i) $g(t) \geq 0$ for all $t \in \mathbb{R}$;
(ii) $g(t) \leq a_p|t|^{p-1}$ for all $t \in \mathbb{R}$ ($p \in (1, 2_s^*)$, $a_p > 0$);
(iii) $\lim_{t\to 0^+} \frac{G(t)}{t^2} = \infty$.

Note that, due to hypothesis $\mathbf{H}_1$ (iii), problem (18) reduces to (4) with $g(t) = t^{p-1}$ only for $p \in (1, 2)$ (concave case). Although, the results of this section also embrace the case $p \in [2, 2_s^*)$ (linear/convex case).

Due to the presence of the critical term $u^{2_s^*-1}$ in (18), we cannot apply Theorem A, as the associated energy functional does not satisfy (PS) in general. So we introduce the local Palais-Smale condition for functionals of the type $J_\lambda = \Phi - \lambda\Psi$, with $\Phi, \Psi \in C^1(X)$, $\lambda > 0$, defined on a Banach space X, and $r > 0$:

$(PS)^r$ Every sequence (u_n) in X, s.t. $(J_\lambda(u_n))$ is bounded in $\mathbb{R}$, $J'(u_n) \to 0$ in X^*, and $\Phi(u_n) \leq r$ for all $n \in \mathbb{N}$, has a convergent subsequence.

In this case, our main tool is the local minimum result, slightly rephrased from [26, Theorem 3.3]:

Theorem B: *Let X be a Banach space, $\Phi, \Psi \in C^1(X)$, $J_\lambda = \Phi - \lambda\Psi$ ($\lambda > 0$), $r \in \mathbb{R}$; $\bar{u} \in X$ satisfy*

(B_1) $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$;
(B_2) $0 < \Phi(\bar{u}) < r$;
(B_3) $\sup_{\Phi(u)\leq r} \dfrac{\Psi(u)}{r} < \dfrac{\Psi(\bar{u})}{\Phi(\bar{u})}$.

Let

$$I_r = \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \left[\sup_{\Phi(u)\leq r} \frac{\Psi(u)}{r} \right]^{-1} \right).$$

Then, for all $\lambda \in I_r$ for which J_λ satisfies $(PS)^r$, there exists $u_\lambda \in X$ s.t.

$$0 < \Phi(u_\lambda) < r, \quad J_\lambda(u_\lambda) = \min_{0<\Phi(u)<r} J_\lambda(u).$$

Set for all $\mu > 0$, $t \in \mathbb{R}$

$$f(t) = \mu g(t) + (t^+)^{2^*-1}, \; F(t) = \int_0^t f(\tau)\,d\tau,$$

then define $\Phi, \Psi \in C^1(H_0^s(\Omega))$ as in Section 2. Further, for all $\lambda > 0$ set $J_\lambda = \Phi - \lambda\Psi$. Set for all $r, \mu > 0$

$$\lambda_r^* = \min\left\{ \left[\frac{2^{\frac{2_s^*}{2}} T(N,s)^{2_s^*} r^{\frac{2_s^*-2}{2}}}{2_s^*} + \mu \frac{2^{\frac{p}{2}} a_p T(N,s)^p |\Omega|^{\frac{2_s^*-p}{2_s^*}} r^{\frac{p-2}{2}}}{p} \right]^{-1}, \frac{1}{T(N,s)^{2_s^*}} \left[\frac{s}{2Nr} \right]^{\frac{2s}{N-2s}} \right\}. \tag{19}$$

We prove now that J_λ satisfies $(PS)^r$ for all $r > 0$ and all $\lambda > 0$ small enough:

Lemma 4.1: *Let $r, \mu > 0$, $\lambda_r^* > 0$ be defined by* (19). *Then J_λ satisfies $(PS)^r$ for all $\lambda \in (0, \lambda_r^*)$.*

Proof: Let (u_n) be a sequence in $H_0^s(\Omega)$ s.t. $(J_\lambda(u_n))$ is bounded, $J'_\lambda(u_n) \to 0$ in $H^{-s}(\Omega)$, and $\Phi(u_n) \le r$ for all $n \in \mathbb{N}$. Then (u_n) is bounded in $H_0^s(\Omega)$, hence in $L^{2_s^*}(\Omega)$ (Lemma 2.1). Passing to a subsequence we have $u_n \rightharpoonup u$ in $H_0^s(\Omega)$, $L^{2_s^*}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$, $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, and $J_\lambda(u_n) \to c$. First we see that

$$J'_\lambda(u) = 0. \tag{20}$$

Indeed, since $(u_n^{2_s^*-1})$ is bounded in $L^{(2_s^*)'}(\Omega)$, up to a further subsequence we have $u_n^{2_s^*-1} \rightharpoonup u^{2_s^*-1}$ in $L^{(2_s^*)'}(\Omega)$, while by $\mathbf{H}_2$ (i) (ii) we have $g(u_n) \to g(u)$ in $L^{p'}(\Omega)$. So, for all $\varphi \in H_0^s(\Omega)$ we have

$$\begin{aligned}\langle J'_\lambda(u_n), \varphi\rangle &= \langle u_n, \varphi\rangle - \lambda \int_\Omega u_n^{2_s^*-1}\varphi \, dx - \lambda\mu \int_\Omega g(u_n)\varphi \, dx \\ &\to \langle u, \varphi\rangle - \lambda \int_\Omega u^{2_s^*-1}\varphi \, dx - \lambda\mu \int_\Omega g(u)\varphi \, dx = \langle J'_\lambda(u), \varphi\rangle,\end{aligned}$$

which along with $J'_\lambda(u_n) \to 0$ gives (20). Besides,

$$J_\lambda(u) > -r. \tag{21}$$

Indeed, since $u_n \rightharpoonup u$ in $H_0^s(\Omega)$ and Φ is convex, we have $\Phi(u) \le r$, i.e. $\|u\| \le (2r)^{\frac{1}{2}}$. So using Lemma 2.1, (5) with $\nu = p$, (19), and $\lambda < \lambda_r^*$, we have

$$\begin{aligned} J_\lambda(u) &\ge -\lambda\Psi(u) \\ &\ge -\frac{\lambda}{2_s^*}\|u\|_{2_s^*}^{2_s^*} - \frac{\lambda\mu a_p}{p}\|u\|_p^p \\ &\ge -\frac{\lambda}{2_s^*}T(N,s)^{2_s^*}(2r)^{\frac{2_s^*}{2}} - \frac{\lambda\mu a_p}{p}T(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}(2r)^{\frac{p}{2}} \\ &\ge -\lambda r\left[\frac{2^{\frac{2_s^*}{2}}T(N,s)^{2_s^*}r^{\frac{2_s^*-2}{2}}}{2_s^*} + \mu\frac{2^{\frac{p}{2}}a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}r^{\frac{p-2}{2}}}{p}\right] \\ &\ge -\frac{\lambda r}{\lambda_r^*}, \end{aligned}$$

and the latter gives (21) since $\lambda > \lambda_r^*$. Now set $v_n = u_n - u$. We have

$$\lim_n \left[\Phi(v_n) - \frac{\lambda}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*}\right] = c - J_\lambda(u). \tag{22}$$

Indeed, since $v_n \rightharpoonup 0$ in $H_0^s(\Omega)$, we have

$$\|v_n\|^2 = \|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2 = \|u_n\|^2 - \|u\|^2 + o(1)$$

(as $n \to \infty$). Since $v_n \rightharpoonup 0$ in $L^{2_s^*}(\Omega)$, by the Brezis-Lieb Lemma [32, Theorem 1] we have

$$\|v_n\|_{2_s^*}^{2_s^*} = \|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*} + o(1).$$

Since $u_n \to u$ in $L^p(\Omega)$, we have $G(u_n) \to G(u)$ in $L^1(\Omega)$. So,

$$\begin{aligned}\Phi(v_n) - \frac{\lambda}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*} &= [\Phi(u_n) - \Phi(u)] - \frac{\lambda}{2_s^*}\left[\|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*}\right] \\ &\quad - \lambda\mu \int_\Omega [G(u_n) - G(u)]\,\mathrm{d}x + o(1) \\ &= J_\lambda(u_n) - J_\lambda(u) + o(1) \to c - J_\lambda(u).\end{aligned}$$

On the other hand,

$$\lim_n \left[\|v_n\|^2 - \lambda\|v_n\|_{2_s^*}^{2_s^*}\right] = 0. \tag{23}$$

Indeed, arguing as above and recalling that $g(u_n)u_n \to g(u)u$ in $L^1(\Omega)$, we have

$$\begin{aligned}\|v_n\|^2 - \lambda\|v_n\|_{2_s^*}^{2_s^*} &= [\|u_n\|^2 - \|u\|^2] - \lambda\left[\|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*}\right] \\ &\quad - \lambda\mu \int_\Omega [g(u_n)u_n - g(u)u]\,\mathrm{d}x + o(1) \\ &= \langle J'_\lambda(u_n), u_n\rangle - \langle J'_\lambda(u), u\rangle + o(1),\end{aligned}$$

and the latter tends to 0 as $n \to \infty$, by $J'_\lambda(u_n) \to 0$, boundedness of (u_n), and (20). Recalling that (v_n) is bounded in $H_0^s(\Omega)$, up to a subsequence we have $\|v_n\| \to \beta \geq 0$. We prove that

$$\beta = 0, \tag{24}$$

arguing by contradiction. Assume $\beta > 0$. Then, by (23) we have

$$\beta^2 = \lim_n \lambda\|v_n\|_{2_s^*}^{2_s^*} \leq \lambda T(N,s)^{2_s^*}\beta^{2_s^*},$$

hence

$$\beta \geq \left[\frac{1}{\lambda T(N,s)^{2_s^*}}\right]^{\frac{1}{2_s^*-2}}.$$

By (21) and (22), we also have

$$\left(\frac{1}{2} - \frac{1}{2_s^*}\right)\beta^2 = c - J_\lambda(u) < 2r,$$

hence

$$\beta < \left[\frac{2Nr}{s}\right]^{\frac{1}{2}}.$$

Comparing the last inequalities and recalling (19), we get

$$\lambda > \frac{1}{T(N,s)^{2_s^*}}\left[\frac{s}{2Nr}\right]^{\frac{2s}{N-2s}} \geq \lambda_r^*,$$

a contradiction. So (24) is proved, which means $u_n \to u$ in $H_0^s(\Omega)$. Thus, J_λ satisfies $(PS)^r$. ■

Set

$$\mu^* = \min\left\{\left[\frac{2_s^*}{2^{\frac{2_s^*+2}{2}}T(N,s)^{2_s^*}}\right]^{\frac{2}{2_s^*-2}}, \frac{s}{3NT(N,s)^{\frac{N}{s}}}\right\}^{\frac{2-p}{2}} \frac{p}{2^{\frac{p+2}{2}}a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}} > 0. \tag{25}$$

We have the existence result for problem (18):

Theorem 4.2: *Let* $\mathbf{H}_2$ *hold,* $\mu^* > 0$ *be defined by* (25). *Then, for all* $\mu \in (0,\mu^*)$, (18) *has at least one solution* $u_\mu \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.

Proof: Fix $\mu \in (0,\mu^*)$ and set

$$r = \min\left\{\left[\frac{2_s^*}{2^{\frac{2_s^*+2}{2}}T(N,s)^{2_s^*}}\right]^{\frac{2}{2_s^*-2}}, \frac{s}{3NT(N,s)^{\frac{N}{s}}}\right\} > 0. \tag{26}$$

By (25), (26) we have

$$\frac{2^{\frac{2_s^*}{2}}T(N,s)^{2_s^*}r^{\frac{2_s^*-2}{2}}}{2_s^*} + \mu\frac{2^{\frac{p}{2}}a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}r^{\frac{p-2}{2}}}{p} \leq \frac{1}{2} + \frac{\mu}{2\mu^*} < 1,$$

as well as

$$\frac{1}{T(N,s)^{2_s^*}}\left[\frac{s}{2Nr}\right]^{\frac{2s}{N-2s}} \geq \frac{1}{T(N,s)^{2_s^*}}\left[\frac{s}{2N}\frac{3NT(N,s)^{\frac{N}{s}}}{s}\right]^{\frac{2s}{N-2s}} = \left(\frac{3}{2}\right)^{\frac{2s}{N-2s}} > 1,$$

hence by (19) we have $\lambda_r^* > 1$.

We intend to apply Theorem B. First, we see that hypothesis (B_1) holds. Then, for all $u \in H_0^s(\Omega)$, $\Phi(u) \leq r$ we have by $\mathbf{H}_2$ (i) (ii), Lemma 2.1, and (5)

$$\begin{aligned}
\frac{\Psi(u)}{r} &\leq \frac{\|u\|_{2_s^*}^{2_s^*}}{2_s^*r} + \mu\frac{a_p\|u\|_p^p}{pr} \\
&\leq \frac{T(N,s)^{2_s^*}(2r)^{\frac{2_s^*}{2}}}{2_s^*r} + \mu\frac{a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}(2r)^{\frac{p}{2}}}{pr} \\
&\leq \frac{1}{\lambda_r^*}.
\end{aligned}$$

On the other hand, by $\mathbf{H}_2$ (iii) we have

$$\lim_{t\to 0^+} \frac{F(t)}{t^2} = \infty.$$

So, arguing as in the proof of Theorem 3.1, we can find $\bar{u} \in H_0^s(\Omega)$ s.t.

$$0 < \Phi(\bar{u}) < r, \quad \frac{\Psi(\bar{u})}{\Phi(\bar{u})} > \frac{1}{\lambda_r^*},$$

which ensures (B_2) and (B_3). Finally, since $\lambda_r^* > 1$, by Lemma 4.1 the functional J_1 satisfies $(PS)^r$.

Since $1 \in I_r$, from Theorem B, we deduce the existence of a (relabeled) function $u_\mu \in H_0^s(\Omega)$ s.t.

$$0 < \Phi(u_\mu) < r, \; J_1(u_\mu) = \min_{0<\Phi(u_\mu)<r} J_1(u).$$

In particular, we have $J_1'(u_\mu) = 0$ in $H^{-s}(\Omega)$. Thus, by Proposition 2.2, $u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ is a solution of (18). ■

Remark 4.3: The proof of Theorem 4.2 gives additional information: u_μ is a local minimizer of J_1 in $H_0^s(\Omega)$, satisfies the bound $\|u_\mu\| < (2r)^{\frac{1}{2}}$, and the mapping $\mu \mapsto J_1(u_\mu)$ is decreasing in $(0, \mu^*)$.

5. Two positive solutions under critical growth

Finally, we turn to problem (4) with $q = 2_s^*$, namely, the Ambrosetti–Brezis–Cerami problem for the fractional Laplacian:

$$\begin{cases} (-\Delta)^s\, u = \mu u^{p-1} + u^{2_s^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{27}$$

with $p \in (1, 2)$, $\mu > 0$. This is a special case of (18) with $g(t) = (t^+)^{p-1}$, which satisfies $\mathbf{H}_2$ with $a_p = 1$. We know from [21, Theorem 1.1] that (27) has at least two positive solutions for all $\mu > 0$ small enough. Our last result yields an explicitly estimate of 'how small' μ should be, given by (25) which in the present case rephrases as

$$\mu^* = \min\left\{\left[\frac{2_s^*}{2^{\frac{2_s^*+2}{2}}T(N,s)^{2_s^*}}\right]^{\frac{2}{2_s^*-2}}, \frac{s}{3NT(N,s)^{\frac{N}{s}}}\right\}^{\frac{2-p}{2}} \frac{p}{2^{\frac{p+2}{2}}T(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}} > 0. \tag{28}$$

Indeed, we have the multiplicity result:

Theorem 5.1: *Let $p \in (1, 2)$, $\mu^* > 0$ be defined by* (28). *Then, for all $\mu \in (0, \mu^*)$,* (27) *has at least two solutions $u_\mu, w_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, $u_\mu < w_\mu$ in Ω.*

Proof: Fix $\mu \in (0, \mu^*)$, define $f \in C(\mathbb{R})$, $\Phi, \Psi \in C^1(H_0^s(\Omega))$ as in Section 4, and set for brevity $J = J_1 = \Phi - \Psi$. From Theorem 4.2 and Remark 4.3, we know that there exists $u_\mu \in H_0^s(\Omega) \cap \mathrm{int}(C_s^0(\overline{\Omega})_+)$ which solves (27) and is a local minimizer of J. Set for all $(x, t) \in \Omega \times \mathbb{R}$

$$\tilde{f}(x,t) = f(u_\mu(x) + t^+) - f(u_\mu(x)),$$
$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,\tau)\,\mathrm{d}\tau = F(u_\mu(x) + t^+) - F(u_\mu(x)) - f(u_\mu(x))t^+.$$

For all $v \in H_0^s(\Omega)$ set

$$\tilde{\Psi}(v) = \int_\Omega \tilde{F}(x,v)\,\mathrm{d}x, \quad \tilde{J}(v) = \Phi(v) - \tilde{\Psi}(v).$$

As in Section 2, it is easily seen that $\tilde{J} \in C^1(H_0^s(\Omega))$ and all its critical points solve the (nonautonomous) auxiliary problem

$$\begin{cases} (-\Delta)^s\, v = \tilde{f}(x,v) & \text{in } \Omega \\ v = 0 & \text{in } \Omega^c. \end{cases} \tag{29}$$

The functionals $\tilde{J}$ and J are related to each other by the inequality for all $v \in H_0^s(\Omega)$:

$$\tilde{J}(v) \geq J(u_\mu + v^+) - J(u_\mu) + \frac{\|v^-\|^2}{2}. \tag{30}$$

Indeed, we have $v^\pm \in H_0^s(\Omega)$ and, setting

$$\Omega_+ = \{x \in \Omega : v(x) > 0\}, \quad \Omega_- = \Omega \setminus \Omega_+,$$

from $v = v^+ - v^-$ we have

$$\begin{aligned}\|v\|^2 &= \|v^+\|^2 + \|v^-\|^2 - 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(v^+(x) - v^+(y))(v^-(x) - v^-(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\geq \|v^+\|^2 + \|v^-\|^2,\end{aligned}$$

as the integrand vanishes everywhere but in $\Omega_+ \times \Omega_-$ and in $\Omega_- \times \Omega_+$, where is is negative. So we have

$$\begin{aligned}\tilde{J}(v) &= \frac{\|v\|^2}{2} - \int_\Omega \tilde{F}(x,v)\,\mathrm{d}x \\ &\geq \frac{\|v^+\|^2}{2} + \frac{\|v^-\|^2}{2} - \int_\Omega \left[F(u_\mu + v^+) - F(u_\mu) - f(u_\mu)v^+\right]\mathrm{d}x \\ &= \frac{\|u_\mu + v^+\|^2}{2} - \frac{\|u_\mu\|^2}{2} - \langle u_\mu, v^+\rangle + \frac{\|v^-\|^2}{2} \\ &\quad - \int_\Omega \left[F(u_\mu + v^+) - F(u_\mu) - f(u_\mu)v^+\right]\mathrm{d}x\end{aligned}$$

$$= J(u_\mu + v^+) - J(u_\mu) + \frac{\|v^-\|^2}{2}$$

(where we used that u_μ solves (27)).

We claim that 0 is a local minimizer of $\tilde{J}$. Indeed, by Proposition 2.3, there exists $\rho > 0$ s.t. for all $v \in H^s_0(\Omega) \cap C^0_s(\overline{\Omega})$, $\|v\|_{0,s} \le \rho$ we have $J(u_\mu + v) \ge J(u_\mu)$. Then, for any such v we have as well $\|v^+\|_{0,s} \le \rho$, which along with (30) implies

$$\tilde{J}(v) \ge J(u_\mu + v^+) - J(u_\mu) + \frac{\|v^-\|^2}{2} \ge 0 = \tilde{J}(0).$$

So, 0 is a local minimizer of $\tilde{J}$ in $C^0_s(\overline{\Omega})$ and hence, by Proposition 2.3 again, it is such also in $H^s_0(\Omega)$. In particular, $\tilde{J}'(0) = 0$ in $H^{-s}(\Omega)$, i.e. 0 solves (29).

From now on we closely follow [21]. Arguing by contradiction, assume that 0 is the *only* critical point of $\tilde{J}$ in $H^s_0(\Omega)$. Under such assumption, by [21, Lemma 2.10] $\tilde{J}$ satisfies $(PS)_c$ at any level $c < c^*$, where

$$c^* = \frac{s}{NT(N,s)^{\frac{N}{s}}}. \tag{31}$$

Fix $x_0 \in \Omega$, and for all $\varepsilon > 0$, define $v_\varepsilon \in H^s(\mathbb{R}^N)$ by setting for all $x \in \mathbb{R}^N$

$$v_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x - x_0|^2)^{\frac{N-2s}{2}}}.$$

By Lemma 2.1, we have

$$\|v_\varepsilon\|_{2^*_s} = T(N,s)[v_\varepsilon]_s. \tag{32}$$

Now fix $r > 0$ s.t. $\overline{B}_r(x_0) \subset \Omega$, $\eta \in C^\infty(\mathbb{R}^N)$ s.t. $\eta = 1$ in $B_{\frac{r}{2}}(x_0)$, $\eta = 0$ in $B^c_1(x_0)$, and $0 \le \eta \le 1$ in $\mathbb{R}^N$, then define $w_\varepsilon \in H^s_0(\Omega)$ by setting for all $x \in \mathbb{R}^N$

$$w_\varepsilon(x) = \frac{\eta(x)v_\varepsilon(x)}{\|\eta v_\varepsilon\|_{2^*_s}}.$$

Clearly $\|w_\varepsilon\|_{2^*_s} = 1$. Besides, we will prove that for all $\varepsilon > 0$ small enough

$$\max_{\tau \ge 0} \tilde{J}(\tau w_\varepsilon) < c^*. \tag{33}$$

Assume $N > 4s$. Then, by [5, Propositions 21, 22] we find for all $\varepsilon > 0$ small enough

$$\|w_\varepsilon\|^2 \le \frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}$$
$$\|w_\varepsilon\|_2^2 \ge C\varepsilon^{2s} - C\varepsilon^{N-2s}$$

($C > 0$ denotes several constants, independent of ε). By convexity we have for all $x \in \Omega$, $t \ge 0$

$$\tilde{F}(x,t) \ge \frac{t^{2^*_s}}{2^*_s} + \frac{C}{2} u_\mu(x)^{2^*_s - 2} t^2.$$

Using (32) and the relations above, we see that for all $\varepsilon > 0$ small enough and all $\tau \ge 0$

$$\tilde{J}(\tau w_\varepsilon) \le \frac{\tau^2}{2}\|w_\varepsilon\|^2 - \frac{\tau^{2^*_s}}{2^*_s}\|w_\varepsilon\|_{2^*_s}^{2^*_s} - \frac{C\tau^2}{2}\int_\Omega u_\mu^{2^*_s - 2} w_\varepsilon^2 \, dx$$

$$\leq \frac{\tau^2}{2}\left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s} - C'\varepsilon^{2s}\right] - \frac{\tau^{2_s^*}}{2_s^*} =: h_\varepsilon(\tau) \tag{34}$$

($C, C' > 0$ independent of ε). Now we focus on the mapping $h_\varepsilon \in C^1(\mathbb{R}_+)$. First we note that

$$\lim_{\tau\to\infty} h_\varepsilon(\tau) = -\infty,$$

so there exists $\tau_\varepsilon \geq 0$ s.t.

$$h_\varepsilon(\tau_\varepsilon) = \max_{\tau\geq 0} h_\varepsilon(\tau).$$

If $\tau_\varepsilon = 0$, from (34) we immediately deduce (33). So, let $\tau_\varepsilon > 0$. Differentiating h_ε, we get

$$\tau_\varepsilon = \left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s} - C'\varepsilon^{2s}\right]^{\frac{1}{2_s^*-2}},$$

which tends to $T(N,s)^{-\frac{2}{2_s^*-2}} > 0$ as $\varepsilon \to 0^+$. So, taking $\varepsilon > 0$ small enough, we have $\tau_\varepsilon \geq \tau_0 > 0$. Set

$$\tilde{\tau}_\varepsilon = \left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\right]^{\frac{1}{2_s^*-2}},$$

and note that the mapping

$$\tau \mapsto \frac{\tau^2}{2}\left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\right] - \frac{\tau^{2_s^*}}{2_s^*}$$

is increasing in $[0, \tilde{\tau}_\varepsilon]$. So we have

$$\begin{aligned}
h_\varepsilon(\tau_\varepsilon) &= \frac{\tau_\varepsilon^2}{2}\left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\right] - \frac{\tau_\varepsilon^{2_s^*}}{2_s^*} - \frac{C'\varepsilon^{2s}\tau_\varepsilon^2}{2} \\
&\leq \frac{\tilde{\tau}_\varepsilon^2}{2}\left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\right] - \frac{\tilde{\tau}_\varepsilon^{2_s^*}}{2_s^*} - C''\varepsilon^{2s} \\
&= \frac{s}{N}\left[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\right]^{\frac{N}{2s}} - C''\varepsilon^{2s}.
\end{aligned}$$

Since $N-2s > 2s$, for all $\varepsilon > 0$ small enough we have by (31)

$$h_\varepsilon(\tau_\varepsilon) < \frac{s}{NT(N,s)^{\frac{N}{s}}} = c^*.$$

Then, by (34) we obtain (33). The cases $2s < N \leq 4s$ are treated in similar ways, see [21, Lemma 2.11]. As a byproduct of (34) we have that $\tilde{J}(\tau w_\varepsilon) \to -\infty$ as $\tau \to \infty$, so we can find $\bar{\tau} > 0$ s.t.

$$\tilde{J}(\bar{\tau} w_\varepsilon) < 0.$$

Since $\tilde{J}$ has a local minimum at 0 and no other critical point, we can find $\sigma \in (0, \|\bar{\tau} w_\varepsilon\|)$ s.t. $\tilde{J}(v) > 0$ for all $v \in H_0^s(\Omega)$, $\|v\| = \sigma$. That is, $\tilde{J}$ exhibits a mountain pass geometry around 0. Set

$$\Gamma = \left\{\gamma \in C([0,1], H^s_0(\Omega)) : \gamma(0) = 0, \gamma(1) = \bar{\tau} w_\varepsilon\right\}, \quad c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \tilde{J}(\gamma(t)).$$

Clearly, $\gamma(t) = t\bar{\tau}w_\varepsilon$ define a path of the family Γ, so by (33) we have

$$c \le \max_{t\in[0,1]} \tilde{J}(t\bar{\tau}w_\varepsilon) < c^*.$$

Thus, $\tilde{J}$ satisfies $(PS)_c$. By the Mountain Pass Theorem, there exists $v \in H^s_0(\Omega) \setminus \{0\}$ s.t. $\tilde{J}'(v) = 0$ in $H^{-s}(\Omega)$, a contradiction. So we have proved the existence of $v_\mu \in H^s_0(\Omega) \setminus \{0\}$ s.t. $\tilde{J}'(v_\mu) = 0$ in $H^{-s}(\Omega)$. Such v_μ solves (29), and by monotonicity of f we have for a.e. $x \in \Omega$

$$\tilde{f}(x, v_\mu(x)) = f(u_\mu(x) + v^+_\mu(x)) - f(u_\mu(x)) \ge 0,$$

so by the fractional Hopf lemma (see for instance [28, Lemma 2.7], as Proposition 2.2 here does not apply) we have $v_\mu \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$. Now set

$$w_\mu = u_\mu + v_\mu \in \mathrm{int}(C^0_s(\overline{\Omega})_+).$$

Clearly $w_\mu > u_\mu$ in Ω, and for all $\varphi \in H^s_0(\Omega)$ we have

$$\begin{aligned}
\langle J'(w_\mu), \varphi\rangle &= \langle u_\mu + v_\mu, \varphi\rangle - \int_\Omega f(u_\mu + v_\mu)\varphi \,\mathrm{d}x \\
&= \left[\langle u_\mu, \varphi\rangle - \int_\Omega f(u_\mu)\varphi \,\mathrm{d}x\right] + \left[\langle v_\mu, \varphi\rangle - \int_\Omega \tilde{f}(x, v_\mu)\varphi \,\mathrm{d}x\right] \\
&= \langle J'(u_\mu), \varphi\rangle + \langle \tilde{J}'(v_\mu), \varphi\rangle = 0,
\end{aligned}$$

so w_μ solves (27), which concludes the proof. ∎

Finally, we present an example:

Example 5.2: Let $s = \frac{1}{2}, N = 2, p = \frac{3}{2}$ and $\Omega \subset \mathbb{R}^2$ be as in Example 3.3, but set this time $q = 2^*_{1/2} = 4$. Recall that in such case

$$T\left(2, \frac{1}{2}\right) = \frac{1}{2\pi^{\frac{3}{4}}}.$$

So, (28) yields

$$\mu^* = \frac{3^{\frac{1}{8}}\pi^{\frac{5}{4}}}{2^{\frac{11}{8}}}.$$

By Theorem 5.1, for all $\mu \in (0, \mu^*)$ problem (27) has at least two positive solutions $u_\mu < w_\mu$.

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