

Finite time collapse in chemotaxis systems with logistic-type superlinear source

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We consider the following quasilinear Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla(u \nabla v) + g(u), & (x, t) \in \Omega \times [0, T_{max}), \\ 0 = \Delta v - v + u, & (x, t) \in \Omega \times [0, T_{max}), \end{cases}$$

on a ball $\Omega \equiv B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, $R > 0$, under homogeneous Neumann boundary conditions and nonnegative initial data. The source term $g(u)$ is superlinear and of logistic type, that is, $g(u) = \lambda u - \mu u^k$, $k > 1$, $\mu > 0$, $\lambda > 0$, and T_{max} is the blow-up time. The solution (u, v) may or may not blow-up in finite time. Under suitable conditions on data, we prove that the function u , which blows up in $L^\infty(\Omega)$ -norm, blows up also in $L^p(\Omega)$ -norm for some $p > 1$. Moreover, a lower bound of the lifespan (or blow-up time when it is finite) T_{max} is derived. In addition, if $\Omega \subset \mathbb{R}^3$ a lower bound of T_{max} is explicitly computable.

KEYWORDS

chemotaxis, finite time blow-up, logistic source

MSC CLASSIFICATION

33B44; 92C17

1 | INTRODUCTION

In many biological phenomena, the chemotaxis, the biased movement of cells (or organisms) in response to chemical gradients, plays an important role in coordinating cell migration (see previous studies^{1–3}). The movement is referred to as chemoattractant if the cells move toward the increasing signal concentration ($\chi > 0$), whereas it is called chemorepulsion whenever the cells move away from the increasing signal concentration ($\chi < 0$) with χ in (2).

In 1970, Keller and Segel³ derived a celebrated model to describe this event. The model has been extensively studied since 1970s, and a number of variations have been proposed and examined, and the properties of their solutions investigated, as the existence of global bounded solutions and the question whether the chemotaxis model allows for a chemotactic collapse, that is, if the system possesses solutions that blow up in finite or infinite time.^{1,4,5}

The topic of blow-up solutions has been addressed by several authors also for more general operators and from different points of view (see, for instance, Marras and Porru⁶ for some results concerning the elliptic case and previous studies^{7–9} for parabolic systems under various boundary conditions).

Our aim is to study the parabolic-elliptic problem:

$$\begin{cases} u_t = \Delta u - \nabla(u \nabla v) + g(u), & (x, t) \in \Omega \times [0, T_{max}), \\ 0 = \Delta v - v + u, & (x, t) \in \Omega \times [0, T_{max}), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times [0, T_{max}), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

with $\Omega \equiv B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, $R > 0$, $g(u) = \lambda u - \mu u^k$, $k > 1$, $\mu > 0$, $\lambda > 0$, the nonnegative initial datum $u_0 \in C^0(\bar{\Omega})$ and where $\frac{\partial}{\partial \nu}$ stands for the normal derivative on $\partial\Omega$.

System (1) is a particular case of the following initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \chi \nabla(u \nabla v) + g(u), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (2)$$

with $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded domain with smooth boundary, $\tau > 0$, $\chi \in \mathbb{R}$, and $g(u)$ is a source term.

We recall that

- If $\tau = 1$, $\chi > 0$, and $g(u) = 0$, (2) is the classical Keller–Segel system introduced by Keller and Segel.³
- If $\tau = 0$, $\chi > 0$, and $g(u) = 0$, we have a simpler model that reflects that the signal substance diffuses much faster than cells move (parabolic-elliptic Keller–Segel system) and the question of blow-up and global existence of solution was studied for instance in previous studies.^{10–12}
- If $\tau = 0$, $\chi = 1$, and $g(u) = 0$ and if $\Omega = B_R(0) \subset \mathbb{R}^n$, or $\Omega = \mathbb{R}^n$, $n \geq 3$, $R > 0$, Souplet and Winkler¹³ consider radially symmetric solutions of the following parabolic-elliptic Keller–Segel–Patlak system

$$\begin{cases} u_t = \Delta u - \nabla(u \nabla v) & x \in \Omega, t > 0, \\ 0 = \Delta v + u - M, & x \in \Omega, t > 0, \end{cases} \quad (3)$$

with Neumann boundary conditions, $u(x, 0) = u_0(x)$ in Ω and

$$M := \begin{cases} \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx & \text{if } \Omega = B_R, \\ 0 & \text{if } \Omega = \mathbb{R}^n. \end{cases}$$

The authors study the blow-up asymptotics of radially decreasing solutions of (3) and show that the final profile satisfies $C_1 |x|^{-2} \leq u(x, T) \leq C_2 |x|^{-2}$ with convergence in $L^1(\Omega)$ as $t \rightarrow T$, the time existence of the solution.

- If $\tau = 0$, $\chi = 1$, and $g(u) = \lambda u - \mu u^k$, if $k = 2$, $\forall \lambda$ either $n \leq 2$, $\mu > 0$ or $n \geq 3$, $\mu \leq \frac{n-2}{n}$, then no blow-up occurs; if $k > 2$, $\forall \lambda$ the same conclusion holds.¹⁴ If $k > 1$, $\mu > 0$, and $\lambda \in \mathbb{R}$, that is, g is a source term of logistic superlinear degradation type, recently, Winkler¹⁵ proves that in low-dimensional spatial settings (compared with higher dimensional case in Winkler¹⁵) under a dimensional-dependent range of k , when $\Omega \equiv B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, the solution of (2) blows up in finite time in $L^\infty(\Omega)$ -norm.
- If $\tau = 1$, $\chi > 0$, and $g(u) = 0$, for the following more general system:

$$\begin{cases} u_t = \nabla \cdot [(u + \alpha)^{m_1 - 1} \nabla u - \chi u (u + \alpha)^{m_2 - 2} \nabla v], & \text{in } \Omega \times (0, T), \\ v_t = \Delta v - v + u, & \text{in } \Omega \times (0, T), \end{cases} \quad (4)$$

under Neumann boundary conditions and initial conditions, where Ω is a general bounded domain in \mathbb{R}^n with smooth boundary, $\alpha > 0$, $\chi > 0$, $m_1, m_2 \in \mathbb{R}$, and $T > 0$, Nishino and Yokota¹⁶ derived a lower bound of blow-up time.

- If $\tau = 0$, $\chi > 0$, $g(u) = 0$, and $M := \frac{1}{|\Omega|} \int_{\Omega} u_0(x) dx$, Marras et al.¹⁷ investigate the blow-up solutions of the following:

$$\begin{cases} u_t = \nabla \cdot [(u + \alpha)^{m_1 - 1} \nabla u - \chi u (u + \alpha)^{m_2 - 2} \nabla v], \\ 0 = \Delta v - M + u, \\ u_\nu = v_\nu = 0, \\ u(x, 0) = u_0(x), \\ \int_{\Omega} v(x, t) dx = 0, \end{cases} \quad (5)$$

with $(x, t) \in \Omega \times (0, T_{max})$, Ω is a smooth and bounded domain of \mathbb{R}^n , with $n \geq 1$, T_{max} is the blow-up time, $\alpha > 0$, and m_1, m_2 are real numbers. Under some links between the above parameters m_1, m_2 and the extra condition $\int_{\Omega} v(x, t) dx =$

0, they prove that if $p_0 > \frac{n}{2}(m_2 - m_1)$, any blowing up classical solution in $L^\infty(\Omega)$ -norm blows up also in $L^{p_0}(\Omega)$ -norm, and a lower bound of the blow-up time T_{max} is derived.

- If $\tau = 0$ and $\chi > 0$ and $g(u) \leq au - \mu u^2$ a source term of logistic type ($a \geq 0, \mu > 0$), another interesting model was achieved by Cao and Zheng¹⁸; the following quasilinear parabolic-elliptic Keller–Segel system is considered:

$$\begin{cases} u_t = \nabla(\phi(u)\nabla u) - \chi\nabla(u\nabla v) + g(u), & x \in \Omega \quad t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, \quad t > 0, \end{cases} \quad (6)$$

with Neumann boundary conditions and $u(x, 0) = u_0(x)$ in $\Omega \subset \mathbb{R}^n$, $n \geq 1$, a bounded convex domain with smooth boundary, $\phi(s) > 0$ for $s > 0$, $\phi(s) \geq ks^p$, $k > 0$, $p \in \mathbb{R}$. There are three nonlinear mechanisms included in this model: the nonlinear diffusion $\nabla(\phi(u)\nabla u)$, the aggregation $\chi\nabla(u\nabla v)$, and the logistic absorption $g(u)$; they observe that the nonlinear diffusion with the logistic absorption dominate the aggregation, so that the unique classical solution is global in time and bounded, regardless of the initial data, if $\mu > \chi \left(1 - \frac{2}{n(1-p)_+}\right)$, which enlarge the parameter range $\mu > \chi \frac{n-2}{n}$ present when in the system $g(u) = 0$.

For other results, see the references in the papers cited above.

Our purpose is to find a lower bound T of the blow-up time T_{max} , so that there exists a safe interval of existence of the solution (u, v) to system (1), $[0, T]$ with $T < T_{max}$. First, we prove that $u(x, t)$, which blows up in $L^\infty(\Omega)$ -norm (see Winkler¹⁵), blows up also in $L^p(\Omega)$ -norm, $p > \frac{n}{2}$, by improving a result of Freitag.¹⁹

Winkler¹⁵ proves that, assuming some restrictions on k and u_0 , the solution of (1) blows up in finite time, in $L^\infty(\Omega)$ -norm, with $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, $R > 0$.

This result is contained in the following theorem.

Theorem 1 (Winkler¹⁵). *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 3$ and $R > 0$, and let $\lambda \in \mathbb{R}$, $\mu > 0$, and $k > 1$ be such that*

$$k < \begin{cases} \frac{7}{6} & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases} \quad (7)$$

Then, for all $L > 0$, $m > 0$, and $m_0 \in (0, m)$, one can find $r_0 = r_0(R, \lambda, \mu, k, L, m, m_0) \in (0, R)$ with the property that whenever $u_0 \in C^0(\bar{\Omega})$ such that

$$u_0(x) \leq L|x|^{-n(n-1)} \quad \text{for all } x \in \Omega \quad (8)$$

as well as

$$\int_{\Omega} u_0(x) dx \leq m \quad \text{but} \quad \int_{B_{r_0}} u_0 \geq m_0, \quad (9)$$

there exists $T_{max} \in (0, \infty)$ and a classical solution (u, v) of (1) with

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) & \text{and} \\ v \in C^{2,0}(\bar{\Omega} \times (0, T_{max})), \end{cases} \quad (10)$$

which blows up at $t = T_{max}$ in the sense that

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (11)$$

Now, we can state our first main result, which provides that the classical solution of (1), blows up in L^p -norm at finite time.

Theorem 2. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$. Then, a classical solution (u, v) to system (1) for $t \in (0, T_{max})$, provided by Theorem 1, is such that for all $p > \frac{n}{2}$*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty. \quad (12)$$

Define for all $p > 1$ the energy function

$$\Psi(t) = \frac{1}{p} \|u\|_{L^p(\Omega)}^p \quad \text{with} \quad \Psi_0 = \Psi(0) = \frac{1}{p} \|u_0\|_{L^p(\Omega)}^p. \quad (13)$$

Theorem 3. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$. Then, for all $p > \frac{n}{2}$ and positive constants B_1, B_2, B_3 depending on λ, p, n such that the blow-up time T_{max} of the classical solution (u, v) to system (1), provided by Theorem 2, satisfies the following estimate:*

$$T_{max} \geq \int_{\Psi_0}^{\infty} \frac{d\eta}{B_1\eta + B_2\eta^{\gamma_1} + B_3\eta^{\gamma_2}}, \quad (14)$$

$$\text{with } \gamma_1 = \frac{p+1}{p}, \quad \gamma_2 = \frac{2(p+1)-n}{2p-n}.$$

In the next theorem, assuming $\Omega \subset \mathbb{R}^3$, a safe interval of existence of the solution $[0, T]$, $T < T_{max}$ is obtained since we can derive an explicit lower bound for T_{max} .

To this end, we introduce the function

$$\Phi(t) = \|u\|_{L^2(\Omega)}^2 \quad \text{with} \quad \Phi_0 = \Phi(0) := \|u_0\|_{L^2(\Omega)}^2. \quad (15)$$

We observe that, under the hypotheses of Theorem 1, if the solution (u, v) of (1) blows up in $L^\infty(\Omega)$ -norm, from Theorem 2 (with $p = 2$), it blows up also in $L^2(\Omega)$ -norm at $t = T_{max}$.

We remark that the choice of the domain $\Omega \subset \mathbb{R}^3$ is due to the use of a Sobolev-type inequality valid only in \mathbb{R}^3 .

Theorem 4. *Let $\Omega = B_R(0) \subset \mathbb{R}^3$, $R > 0$ and (u, v) be a classical solution of (1) for $t \in (0, T_{max})$, provided by Theorem 1. Then, Φ , defined in (15), satisfies the following first order differential inequality:*

$$\Phi'(t) \leq A\Phi^3(t), \quad (16)$$

with A a positive constant depending on $\|u_0\|_{L^2(\Omega)}$, k , λ , μ , $|\Omega|$.

From Theorem 4, as a consequence, we have

Corollary 1. *Under the assumptions of Theorem 4, let (u, v) be a solution of (1) and $\Phi(t)$ and Φ_0 defined in (15). Then, there exists a safe interval of existence of (u, v) say $[0, T]$ with*

$$T = \frac{1}{2A\Phi_0^2} \leq T_{max}. \quad (17)$$

We remark that $\frac{1}{2A\Phi_0^2}$ is explicitly computable.

This paper is organized as follows.

In Section 2, we collect some results to be used in the proofs of the main theorems. In Section 3, we prove that $u(x, t)$, which blows up in $L^\infty(\Omega)$ -norm, blows up also in $L^p(\Omega)$ -norm with $p > \frac{n}{2}$ (Theorem 2). Moreover, by using a Gagliardo–Nirenberg inequality, we prove Theorem 6. Section 4 is dedicated to the case $\Omega \subset \mathbb{R}^3$ and contains the proofs of Theorem 4 and a corollary where a safe interval of existence of (u, v) say $[0, T]$ is derived with T an explicit lower bound of the blow-up T_{max} .

2 | PRELIMINARIES

In this section, we state some known results to be used in the proofs of the main theorems.

Throughout the paper, we will assume the conditions contained in the Theorem 1.

We need the following Gagliardo–Nirenberg inequality.

Lemma 1. *Let Ω be a bounded and smooth domain of \mathbb{R}^n , $n \geq 1$. Let $r \geq 1$, $0 < q \leq p \leq \infty$, $s > 0$. Then, there exists a constant $c_{GN} > 0$ such that*

$$\|w\|_{L^p(\Omega)} \leq c_{GN} \left(\|\nabla w\|_{L^r(\Omega)}^a \|w\|_{L^q(\Omega)}^{1-a} + \|w\|_{L^s(\Omega)} \right) \quad (18)$$

for all $w \in L^q(\Omega)$ with $\nabla w \in L^r(\Omega)$, and $a := \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{n} - \frac{1}{r}} \in [0, 1)$.

Proof. See Nirenberg.²⁰, p125 □

Lemma 2. Let $n \in \mathbb{N}$, $n \geq 3$. Assume

$$p_0 > \frac{n}{2}. \quad (19)$$

Then, for all $p > p_0$, it holds that

$$0 < \theta_1 < 1, \quad \theta_1 = \frac{\frac{p}{2p_0} - \frac{p}{2(p+1)}}{\frac{p}{2p_0} - \frac{1}{2} + \frac{1}{n}} \quad (20)$$

$$0 < \beta_1 < 1, \quad \beta_1 = \frac{p+1}{p} \theta_1, \quad (21)$$

$$0 < \theta_2 < 1, \quad \theta_2 = \frac{n}{2(p+1)}, \quad (22)$$

$$0 < \beta_2 < 1, \quad \beta_2 = \frac{p+1}{p} \theta_2 = \frac{n}{2p}. \quad (23)$$

Proof. From $p > p_0 > \frac{n}{2}$, we have $p > 1 - \frac{2}{n}$ and $\frac{p}{p+1} > 1 - \frac{2}{n}$ and (20) follows.

The result (21) follows from hypothesis (19); in fact, we have $\frac{1}{2p_0} < \frac{1}{n}$ from which we obtain $\frac{p+1}{2p_0} - \frac{1}{2} < \frac{p}{2p_0} - \frac{1}{2} + \frac{1}{n}$ and (21) follows. Easily, we obtain also (22) and (23). □

Lemma 3. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$ be a bounded and smooth domain, $u \in C^0(\Omega)$ a positive function, and p, k , two positive real numbers such that $p + k - 1 > p > 0$. Then, we have

$$\int_{\Omega} u^{p+k-1} dx \geq |\Omega|^{\frac{1-k}{p}} \left(\int_{\Omega} u^p dx \right)^{\frac{p+k-1}{p}}. \quad (24)$$

Proof. The inequality follows from Hölder's inequality. □

Lemma 4. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$ be a bounded and smooth domain, and $\lambda \in \mathbb{R}$, $\mu > 0$, $k > 1$. Then, for a solution (u, v) of (1), we have

$$\int_{\Omega} u dx \leq \bar{m}, \quad \text{for all } t \in (0, T_{max}), \quad (25)$$

with

$$\bar{m} = \max \left\{ \int_{\Omega} u_0 dx, \left(\frac{\lambda}{\mu} |\Omega|^{k-1} \right)^{\frac{1}{k-1}} \right\}. \quad (26)$$

Proof. From the first equation in (1), we obtain

$$\frac{d}{dt} \int_{\Omega} u dx = \lambda \int_{\Omega} u dx - \mu \int_{\Omega} u^k dx \leq \lambda \int_{\Omega} u dx - \mu |\Omega|^{1-k} \left(\int_{\Omega} u dx \right)^k, \quad (27)$$

where in the last term, we used Hölder's inequality: $\int_{\Omega} u^k \leq |\Omega|^{\frac{k-1}{k}} \left(\int_{\Omega} u \right)^k$. From (27), we infer that $y = \int_{\Omega} u dx$ satisfies

$$\begin{cases} y'(t) \leq \lambda y(t) - \bar{\mu} y^k(t), & \bar{\mu} = \mu |\Omega|^{1-k}, \text{ for all } t \in [0, T_{max}) \\ y(0) = y_0. \end{cases} \quad (28)$$

Upon an ODE comparison argument, this entails that

$$y(t) \leq \bar{m}, \quad \text{for all } t \in (0, T_{\max}).$$

This clearly proves the lemma. \square

Lemma 5. *Let Ω be a bounded domain in \mathbb{R}^3 assumed to be star shaped and convex in two orthogonal directions. For any nonnegative $w \in C^1(\Omega)$, the following inequality holds:*

$$\int_{\Omega} w^3 dx \leq \sqrt{2} \left[a_1^{\frac{3}{2}} \left(\int_{\Omega} w^2 dx \right)^{\frac{3}{2}} + \frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3} \left(\int_{\Omega} w^2 \right)^3 + \frac{3a_2^{\frac{3}{2}}\epsilon_1}{4} \int_{\Omega} |\nabla w|^2 dx \right], \quad (29)$$

with $\epsilon_1 > 0$ a suitable constant, and

$$a_1 = \frac{3}{2\rho_0}, \quad a_2 = \frac{d}{\rho_0} + 1, \quad \rho_0 = \min_{\partial\Omega} x_i \nu_i > 0, \quad d^2 = \max_{\Omega} x_i x_i.$$

Proof. The proof easily follows from the inequality (see lemma A2 in Payne et al.²¹)

$$\int_{\Omega} w^3 dx \leq \left\{ \frac{3}{2\rho_0} \int_{\Omega} w^2 dx + \left(\frac{d}{\rho_0} + 1 \right) \left(\int_{\Omega} w^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla w|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{3}{2}}. \quad (30)$$

In fact, in (30), firstly, we apply the following arithmetic inequality:

$$(a + b)^{\frac{3}{2}} \leq \sqrt{2} \left(a^{\frac{3}{2}} + b^{\frac{3}{2}} \right), \quad a, b > 0, \quad (31)$$

to have

$$\int_{\Omega} w^3 dx \leq \sqrt{2} \left\{ \left(\frac{3}{2\rho_0} \int_{\Omega} w^2 dx \right)^{\frac{3}{2}} + \left(\frac{d}{\rho_0} + 1 \right)^{\frac{3}{2}} \left(\int_{\Omega} w^2 dx \right)^{\frac{3}{4}} \left(\int_{\Omega} |\nabla w|^2 dx \right)^{\frac{3}{4}} \right\},$$

and then, from an application of Young's inequality, we get (29). \square

3 | BLOW-UP IN L^P -NORM

Throughout this section, we are under the hypotheses of Theorem 1.

The goal of this section is to extend the result of Freitag (theorem 2.2 in Freitag¹⁹) to solution (u, v) of problem (1). In order to prove Theorem 3, first, we state the following lemmas.

Lemma 6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, a bounded and smooth domain and (u, v) be a solution of (1). If for some $p_0 > \frac{n}{2}$, there exists a constant C such that*

$$\|u\|_{L^{p_0}(\Omega)} \leq C, \quad \text{for all } t \in (0, T_{\max}), \quad (32)$$

then, for some $\hat{C} > 0$ and $p > p_0$

$$\|u\|_{L^p(\Omega)} \leq \hat{C}, \quad \text{for all } t \in (0, T_{\max}). \quad (33)$$

Proof. Let $\Psi(t)$ be defined in (13) with $p > p_0$. Differentiating $\Psi(t)$, we have

$$\begin{aligned} \Psi'(t) &= \int_{\Omega} u^{p-1} u_t dx = \int_{\Omega} u^{p-1} \Delta u dx - \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla v) dx \\ &\quad + \lambda \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+k-1} dx = J_1 + J_2 + J_3 + J_4. \end{aligned} \quad (34)$$

Now, we consider separately the four terms of (34).

$$\begin{aligned} J_1 &= \int_{\Omega} \nabla (u^{p-1} \nabla u) dx - (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 dx = \\ &\quad - \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx \end{aligned} \quad (35)$$

$$J_2 = - \int_{\Omega} u^{p-1} \nabla u \nabla v dx - \int_{\Omega} u^p \Delta v dx. \quad (36)$$

We can estimate the first term in (36) as follows:

$$\begin{aligned} \int_{\Omega} u^{p-1} \nabla u \nabla v dx &= \int_{\Omega} \nabla (u^{p-1} u \nabla v) dx - \int_{\Omega} u \nabla (u^{p-1} \nabla v) dx \\ &= -(p-1) \int_{\Omega} u^{p-1} \nabla u \nabla v dx - \int_{\Omega} u^p \Delta v dx, \end{aligned}$$

from which we obtain

$$\int_{\Omega} u^{p-1} \nabla u \nabla v dx = -\frac{1}{p} \int_{\Omega} u^p \Delta v dx. \quad (37)$$

Replacing (37) into (36), we arrive at

$$\begin{aligned} J_2 &= - \left(1 - \frac{1}{p}\right) \int_{\Omega} u^p \Delta v dx = - \left(1 - \frac{1}{p}\right) \int_{\Omega} u^p v dx \\ &\quad + \left(1 - \frac{1}{p}\right) \int_{\Omega} u^{p+1} dx \leq \left(1 - \frac{1}{p}\right) \int_{\Omega} u^{p+1} dx. \end{aligned} \quad (38)$$

In the last term of (38), we now use the Gagliardo–Nirenberg inequality (18) with $w = u^{\frac{p}{2}}$, $r = 2$, $\mathbf{p} = 2\frac{p+1}{p}$, $\mathbf{q} = \frac{2p_0}{p}$, $\mathbf{s} = \frac{2}{p}$. We have

$$\int_{\Omega} u^{p+1} dx = \|u^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{2\frac{p+1}{p}} \leq c_{GN} \left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2\frac{p+1}{p}\theta_1} \|u^{\frac{p}{2}}\|_{L^{\frac{2p_0}{p}}(\Omega)}^{2\frac{p+1}{p}(1-\theta_1)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2\frac{p+1}{p}} \right),$$

with $\theta_1 \in (0, 1)$ and $\frac{p+1}{p}\theta_1 \in (0, 1)$ defined in Lemma 2, having also made use of

$$(a + b)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha), \quad \text{for any } a, b \geq 0, \alpha > 0. \quad (39)$$

Because Lemma 2 and (32) hold, we apply Young's inequality in the previous inequality arriving to

$$\begin{aligned}
\int_{\Omega} u^{p+1} dx &\leq c_{GN} \left(\int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx \right)^{\frac{p+1}{p} \theta_1} \left(\int_{\Omega} u^{p_0} dx \right)^{(p+1)(1-\theta_1)} \\
&\quad + c_{GN} \left(\int_{\Omega} u dx \right)^{p+1} \leq c_{GN} \epsilon_1 \beta_1 \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + \\
&\quad c_{GN} \epsilon_1^{\frac{\beta_1}{1-\beta_1}} (1-\beta_1) C^{\frac{(p+1)(1-\theta_1)}{1-\beta_1}} + c_{GN} \bar{m}^{p+1} \\
&= c_1 \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + c_2,
\end{aligned} \tag{40}$$

valid for any $\epsilon_1 > 0$, $c_1 = c_1(\epsilon_1) = c_{GN} \epsilon_1 \beta_1$, $\beta_1 = \frac{p+1}{p} \theta_1 \in (0, 1)$, $c_2 = c_2(\epsilon_1) = c_{GN} \epsilon_1^{\frac{\beta_1}{1-\beta_1}} (1-\beta_1) C^{\frac{(p+1)(1-\theta_1)}{1-\beta_1}} + c_{GN} \bar{m}^{p+1}$, with \bar{m} and C defined, respectively, in (26) and (32).

Replacing (40) into (38) leads to

$$J_2 \leq c_1 \left(1 - \frac{1}{p} \right) \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + c_2 \left(1 - \frac{1}{p} \right). \tag{41}$$

In the third term of (34), we use, in order, Hölder's and Young's inequalities to obtain

$$\begin{aligned}
J_3 &\leq \lambda \left(\int_{\Omega} u^{p+k-1} dx \right)^{\frac{p}{p+k-1}} |\Omega|^{\frac{k-1}{p+k-1}} \\
&\leq \lambda \epsilon_2 \frac{p}{p+k-1} \int_{\Omega} u^{p+k-1} dx + \lambda \frac{k-1}{p+k-1} \epsilon_2^{-\frac{p}{k-1}} |\Omega| \\
&= c_3 \int_{\Omega} u^{p+k-1} dx + c_4,
\end{aligned} \tag{42}$$

with $\epsilon_2 > 0$, $c_3 = c_3(\epsilon_2) = \lambda \epsilon_2 \frac{p}{p+k-1}$, $c_4 = c_4(\epsilon_2) = \lambda \frac{k-1}{p+k-1} \epsilon_2^{-\frac{p}{k-1}} |\Omega|$.

Taking into account that

$$J_4 = -\mu \int_{\Omega} u^{p+k-1} dx, \tag{43}$$

we now substitute (35) and (41)–(43) in (34) to have

$$\begin{aligned}
\Psi' &\leq -\frac{(p-1)}{p} \left(\frac{4}{p} - c_1 \right) \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx \\
&\quad - (\mu - c_3) \int_{\Omega} u^{p+k-1} dx + c_5,
\end{aligned} \tag{44}$$

with $c_5 = c_2 \left(1 - \frac{1}{p} \right) + c_4$.

In (44), we choose ϵ_1 , such that $\frac{4}{p} - c_1 \geq 0$ and ϵ_2 , such that $\mu - c_3 \geq 0$. Neglecting the negative term $-\frac{(p-1)}{p} \left(\frac{4}{p} - c_1\right) \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx$ and using (24) in the second term of (44), we obtain

$$\Psi' \leq -c_6 \Psi^\gamma + c_5,$$

with $c_6 = (\mu - c_3) |\Omega|^{\frac{1-k}{p}} \frac{p+k-1}{p}$ and $\gamma = \frac{p+k-1}{p}$.

Thanks to this result, we arrive at this initial problem

$$\begin{cases} \Psi'(t) \leq c_5 - c_6 \Psi^\gamma(t) & t \in (0, T_{max}), \\ \Psi(0) = \frac{1}{p} \int_{\Omega} u_0^p, \end{cases}$$

so an application of a comparison principle leads to

$$\Psi(t) \leq \max \left\{ \Psi(0), \left(\frac{c_5}{c_6} \right)^{\frac{1}{\gamma}} \right\} =: \hat{C} \text{ for all } t \in (0, T_{max}). \quad (45)$$

Moreover, from this bound, elliptic regularity results applied to the second equation of system (1), that is, $-\Delta v + v = u$, imply $v \in L^\infty((0, T_{max}); W^{2,p}(\Omega))$ and, hence, $\nabla v \in L^\infty((0, T_{max}); W^{1,p}(\Omega))$ and from Sobolev embedding theorems, we have $v \in L^\infty((0, T_{max}); C^{[2-n/p]}(\bar{\Omega}))$ and $\nabla v \in L^\infty((0, T_{max}); L^q(\Omega))$ for all $n < q < p^* =: \frac{np}{n-p}$. \square

Lemma 7. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded and smooth domain and (u, v) be a classical solution to system (1). If for some $\frac{n}{2} < p < n$, there exists $\hat{C} > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq \hat{C} \text{ for all } t \in (0, T_{max}), \quad (46)$$

then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \hat{C} \text{ for all } t \in (0, T_{max}). \quad (47)$$

Proof. For any $x \in \Omega$ and $t \in (0, T_{max})$, we set $t_0 = \max\{t_0, t-1\}$, and we consider the representation formula for u :

$$\begin{aligned} u(\cdot, t) &\leq e^{(t-t_0)\Delta} u(\cdot, t_0) - \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v)(\cdot, s) ds \\ &\quad + \int_{t_0}^t e^{(t-s)\Delta} [\lambda u(\cdot, s) - \mu u^k(\cdot, s)] ds \\ &=: u_1(\cdot, t) + u_2(\cdot, t) + u_3(\cdot, t), \end{aligned} \quad (48)$$

and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} + \|u_3(\cdot, t)\|_{L^\infty(\Omega)}, \quad (49)$$

Following the steps of lemma 4.1 in Vigliani and Woolley,²² we obtain

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \max \{ \|u_0\|_{L^\infty(\Omega)}, 2\bar{m}C_S \} := c_7, \quad (50)$$

where C_S is a positive constant and \bar{m} is defined in (26), and

$$\|u_2(\cdot, t)\|_{L^\infty(\Omega)} \leq c_8, \quad (51)$$

with c_8 a positive constant which plays the analogous role of the constant \hat{C}_S of lemma 4.1 in Viglialoro and Woolley.²²

Now, we prove that there exists a constant $c_9 > 0$ such that $\|u_3\|_{L^\infty(\Omega)} \leq c_9$. To this end, we firstly observe that

$$h(u) = \lambda u - \mu u^k \leq h(u_*) := c_9,$$

with $u_* = \left(\frac{\lambda}{\mu k}\right)^{\frac{1}{k-1}}$.

We have

$$\begin{aligned} \|u_3(\cdot, t)\|_{L^\infty(\Omega)} &\leq \int_{t_0}^t \|e^{(t-s)\Delta} [\lambda u(\cdot, s) - \mu u^k(\cdot, s)]\|_{L^\infty(\Omega)} ds \\ &\leq \int_{t_0}^t \|c_9 e^{(t-s)\Delta}\|_{L^\infty(\Omega)} ds = c_9(t - t_0) \leq c_9. \end{aligned} \quad (52)$$

From (50) to (52), we arrive at (47) with $\hat{C} = c_7 + c_8 + c_9$. □

Proof of Theorem 2. □

Proof. From Theorem 1, the unique local classical solution of (1) blows up at $t = T_{max}$ in the sense $\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ (11). By contradiction, we prove that it blows up also in L^p -norm. In fact, if exist $p_0 > \frac{n}{2}$ and $C > 0$ such that

$$\|u\|_{L^{p_0}(\Omega)} \leq C,$$

then from Lemma 6, there exists a constant $\hat{C} > 0$ such that

$$\|u\|_{L^p(\Omega)} < \hat{C} \quad \text{for all } t \in (0, T_{max}),$$

and

$$\begin{cases} u \in L^\infty((0, T_{max}); L^p(\Omega)) & \text{for } p > \frac{n}{2}, \\ u \nabla v \in L^\infty((0, T_{max}); L^{q_1}(\Omega)) & \text{for all } q_1 > n + 2. \end{cases} \quad (53)$$

From Lemma 7, there exists $\hat{C} > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \text{for all } t \in (0, T_{max}),$$

which is in contradiction to the hypothesis (11), so that, if u blows up in L^∞ -norm and $p > p_0 > \frac{n}{2}$, then u blows up also in L^p -norm. □

Proof of Theorem 3. □

Proof. We start from (34), and we use (35) and (38) to write

$$\begin{aligned} \Psi'(t) &\leq -\frac{4(p-1)}{p^2} \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + \left(1 - \frac{1}{p}\right) \int_{\Omega} u^{p+1} dx + \lambda \int_{\Omega} u^p dx \\ &\quad - \mu \int_{\Omega} u^{p+k-1} dx. \end{aligned} \quad (54)$$

In the second term of (54), we apply the Gagliardo–Nirenberg inequality (18) with $p = 2\frac{p+1}{p}$, $r = q = s = 2$,

$$\begin{aligned} \int_{\Omega} u^{p+1} &= \left\| u^{\frac{p}{2}} \right\|_{L^{\frac{2p+1}{p}}(\Omega)}^{2\frac{p+1}{p}} \leq \bar{c}_{GN} \left\| \nabla u^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{2\frac{p+1}{p}\theta_2} \left\| u^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{2\frac{p+1}{p}(1-\theta_2)} \\ &\quad + \bar{c}_{GN} \left\| u^{\frac{p}{2}} \right\|_{L^2}^{2\frac{p+1}{p}}, \end{aligned} \quad (55)$$

where θ_2 is defined in (22), having also make use of (39).

Using the expression of θ_2 , we rewrite (55), and then applying Young's inequality, we have

$$\begin{aligned} \int_{\Omega} u^{p+1} &\leq \bar{c}_{GN} \left(\int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx \right)^{\frac{n}{2p}} \left(\int_{\Omega} u^p dx \right)^{\frac{2(p+1)-n}{2p}} \\ &\quad + \bar{c}_{GN} \left(\int_{\Omega} u^p dx \right)^{\frac{p+1}{p}} \leq a_1 \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx \\ &\quad + a_2 \left(\int_{\Omega} u^p dx \right)^{\frac{2(p+1)-n}{2p-n}} + \bar{c}_{GN} \left(\int_{\Omega} u^p dx \right)^{\frac{p+1}{p}}, \end{aligned} \quad (56)$$

with $a_1 = a_1(\varepsilon_1) = \frac{n}{2p} \varepsilon_1 \bar{c}_{GN}$, $a_2 = a_2(\varepsilon_1) = \frac{2p-n}{2p} \varepsilon_1^{-\frac{n}{(2-n)p}} \bar{c}_{GN}$, $\varepsilon_1 > 0$.

By replacing (56) and (24) into (54), we arrive at

$$\begin{aligned} \Psi'(t) &\leq - \left(\frac{p-1}{p} \right) \left(\frac{4}{p} - a_1 \right) \int_{\Omega} |\nabla u^{\frac{p}{2}}|^2 dx + \\ &\quad \left(1 - \frac{1}{p} \right) a_2 \left(\int_{\Omega} u^p dx \right)^{\frac{2(p+1)-n}{2p-n}} + \left(1 - \frac{1}{p} \right) \bar{c}_{GN} \left(\int_{\Omega} u^p dx \right)^{\frac{p+1}{p}} \\ &\quad + \lambda \int_{\Omega} u^p dx - \mu |\Omega|^{\frac{1-k}{p}} \left(\int_{\Omega} u^p dx \right)^{\frac{p+k-1}{p}}. \end{aligned} \quad (57)$$

Choosing ε_1 in (57) such that $\left(\frac{p-1}{p} \right) \left(\frac{4}{p} - a_1 \right) \geq 0$, we can neglect the first term and the fifth (negative) term in (57). Using the definition of $\Psi(t) = \frac{1}{p} \int_{\Omega} u^p dx$, we obtain the following first-order differential inequality on Ψ :

$$\Psi'(t) \leq B_1 \Psi + B_2 \Psi^{\frac{p+1}{p}} + B_3 \Psi^{\frac{2(p+1)-n}{2p-n}}, \quad (58)$$

with $B_1 = \lambda p$, $B_2 = \left(1 - \frac{1}{p} \right) \bar{c}_{GN} p^{\frac{p+1}{p}}$, $B_3 = \left(1 - \frac{1}{p} \right) a_2 p^{\frac{2(p+1)-n}{2p-n}}$.

Integrating (58) from 0 to T_{max} , we obtain (14). \square

4 | AN EXPLICIT LOWER BOUND OF T_{max} in $\Omega \subset \mathbb{R}^3$

In this section, we consider the L^2 -norm of u defined in (15) as $\Phi(t) = \|u\|_{L^2(\Omega)}^2$, $t \in [0, T_{max})$ with $\Phi_0 = \Phi(0) := \|u_0\|_{L^2(\Omega)}^2$.

Under the hypotheses of Theorem 1, we assume the spatial convex domain $\Omega \subset \mathbb{R}^3$. Let $[0, T]$, $T < T_{max}$ be the time interval of existence of the solution of (1): we have $\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$. From Theorem 2, selecting $p = 2$ (which fits with the choice $n = 3$ in the condition $p > \frac{n}{2}$), necessarily the classical solution (u, v) of (1) blows up in L^2 -norm at $t = T_{max}$. In this situation, we prove that $\Phi(t)$ satisfies a differential inequality of the first order stated in Theorem 4, and as a consequence, we determine a lower bound of the lifespan T_{max} by proving Corollary 1.

Proof of Theorem 4. \square

Proof. By differentiating (15) and using the equation in (1), we have

$$\begin{aligned} \Phi'(t) &= 2 \int_{\Omega} uu_t dx = 2 \int_{\Omega} u \Delta u dx - 2 \int_{\Omega} u \nabla \cdot (u \nabla v) dx \\ &\quad + 2\lambda \int_{\Omega} u^2 dx - 2\mu \int_{\Omega} u^{k+1} dx = I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (59)$$

We now estimate the terms in (59) in order to arrive to a first-order differential inequality in terms of powers of Φ .

$$I_1 = 2 \int_{\Omega} u \Delta u dx = 2 \int_{\Omega} \nabla \cdot (u \nabla u) dx - 2 \int_{\Omega} |\nabla u|^2 dx = -2 \int_{\Omega} |\nabla u|^2 dx. \quad (60)$$

Using the divergence theorem and the second equation in (1), we can write

$$\begin{aligned} I_2 &= -2 \int_{\Omega} u \nabla \cdot (u \nabla v) dx = - \int_{\Omega} \nabla \cdot (u^2 \Delta v) dx - \int_{\Omega} u^2 \Delta v dx \\ &= - \int_{\Omega} u^2 \Delta v dx = - \int_{\Omega} u^2 v dx + \int_{\Omega} u^3 dx. \end{aligned} \quad (61)$$

To bound the last term in (61) in terms of Φ and $\int_{\Omega} |\nabla u|^2 dx$, firstly, we make use of (29) (with $w = u$) in Lemma 5. Neglecting the negative term $-\int_{\Omega} u^2 v dx$, we obtain

$$I_2 \leq \sqrt{2} a_1^{\frac{3}{2}} \left(\int_{\Omega} u^2 dx \right)^{\frac{3}{2}} + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4 \epsilon_1^3} \left(\int_{\Omega} u^2 \right)^3 + \sqrt{2} \frac{3 a_2^{\frac{3}{2}} \epsilon_1}{4} \int_{\Omega} |\nabla u|^2 dx. \quad (62)$$

Using Hölder's inequality, we bound the last term in (59):

$$I_4 = -2\mu \int_{\Omega} u^{k+1} dx \leq -2\mu |\Omega|^{\frac{1-k}{2}} \left(\int_{\Omega} u^2 dx \right)^{\frac{k+1}{2}}. \quad (63)$$

By replacing (60), (62), and (63) in (59), we get

$$\begin{aligned} \Phi'(t) &\leq 2\lambda \int_{\Omega} u^2 dx - 2\mu |\Omega|^{\frac{1-k}{2}} \left(\int_{\Omega} u^2 dx \right)^{\frac{k+1}{2}} + \sqrt{2} a_1^{\frac{3}{2}} \left(\int_{\Omega} u^2 dx \right)^{\frac{3}{2}} \\ &\quad + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4 \epsilon_1^3} \left(\int_{\Omega} u^2 \right)^3 + \left(\sqrt{2} \frac{3 a_2^{\frac{3}{2}} \epsilon_1}{4} - 2 \right) \int_{\Omega} |\nabla u|^2 dx. \end{aligned}$$

Choosing $\epsilon_1 = \frac{4\sqrt{2}}{3} a_2^{-\frac{3}{2}}$, we have

$$\Phi'(t) \leq 2\lambda \Phi - 2\mu |\Omega|^{\frac{1-k}{2}} \Phi^{\frac{k+1}{2}} + \sqrt{2} a_1^{\frac{3}{2}} \Phi^{\frac{3}{2}} + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4 \epsilon_1^3} \Phi^3. \quad (64)$$

Because $1 < k < 7/6$, and $p > \frac{n}{2}$, $u(x, t)$ blows up in L^2 -norm at finite time T_{max} , then $\Phi(t)$ can be nondecreasing, so that $\Phi(t) \geq \Phi_0$ with $t \in [0, T_{max})$, or non increasing (possibly with some kind of oscillations), in which case there exists a time $t_1 \in [0, T_{max})$ where $\Phi(t_1) = \Phi_0$. As a consequence, $\Phi(t) \geq \Phi_0$ for all $t \in [t_1, T_{max})$. It implies that

$$\frac{\Phi}{\Phi_0} \leq \left(\frac{\Phi}{\Phi_0} \right)^{\frac{k+1}{2}}, \quad t \in [t_1, T_{max}),$$

from which

$$-\Phi^{\frac{k+1}{2}} \leq -\Phi \Phi_0^{\frac{k-1}{2}}, \quad t \in [t_1, T_{max}). \quad (65)$$

Moreover,

$$\Phi^{\frac{3}{2}} \leq \Phi^3 \Phi_0^{-\frac{3}{2}}, \quad t \in [t_1, T_{max}). \quad (66)$$

We replace (65) and (66) into (64) to have

$$\begin{aligned} \Phi'(t) &\leq 2 \left(\lambda - \mu |\Omega|^{\frac{1-k}{2}} \Phi_0^{\frac{k-1}{2}} \right) \Phi + \left(\sqrt{2} a_1^{\frac{3}{2}} \Phi_0^{-\frac{3}{2}} + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4e_1^3} \right) \Phi^3 \\ &= A_1 \Phi + A_2 \Phi^3, \quad t \in [t_1, T_{max}), \end{aligned}$$

with $A_1 = 2 \left(\lambda - \mu |\Omega|^{\frac{1-k}{2}} \Phi_0^{\frac{k-1}{2}} \right)$, $A_2 = \sqrt{2} a_1^{\frac{3}{2}} \Phi_0^{-\frac{3}{2}} + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4e_1^3}$.

At last, we can write

$$\bar{\Phi}'(t) \leq A \bar{\Phi}^3, \quad t \in [t_1, T_{max}), \quad (67)$$

where the positive constant A depends on $\|u_0\|_{L^2(\Omega)}$, k , μ , $|\Omega|$, so defined

$$A = \begin{cases} A_1 \Phi_0^{-2} + A_2, & \text{if } \lambda > \mu |\Omega|^{\frac{1-k}{2}} \Phi_0^{\frac{k-1}{2}}, \\ A_2, & \text{if } \lambda \leq \mu |\Omega|^{\frac{1-k}{2}} \Phi_0^{\frac{k-1}{2}}, \end{cases} \quad (68)$$

and (16) is proved. □

Proof of Corollary 1. □

Proof. Integrating (67) from t_1 to T_{max} , we lead to

$$\frac{1}{2\Phi_0^2} = \int_{\Phi_0}^{\infty} \frac{d\eta}{\eta^3} \leq \int_{t_1}^{T_{max}} A \, d\tau \leq \int_0^{T_{max}} A \, d\tau = A T_{max}, \quad (69)$$

from which we obtain (17): it means that the solution of (1) exists bounded in the interval $[0, T]$, with $T = \frac{1}{2A\Phi_0^2}$, a lower bound of the lifespan T_{max} . □

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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