INFLUENCE OF NONLINEAR PRODUCTIONS ON THE GLOBAL SOLVABILITY OF AN ATTRACTION-REPULSION CHEMOTAXIS SYSTEM

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ABSTRACT. This paper is dedicated to the attraction-repulsion chemotaxis-system

$$(\diamond)\qquad \begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) & \text{in } \Omega\times(0,T_{max}),\\ 0 = \Delta v + f(u) - \beta v & \text{in } \Omega\times(0,T_{max}),\\ 0 = \Delta w + \gamma u^r - \delta w & \text{in } \Omega\times(0,T_{max}), \end{cases}$$

defined in Ω, a smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 2$. Moreover, $\gamma,\beta,\delta,\chi,\xi > 0$, $r \geq 1$ and f is a suitably regular function generalizing, for $u \geq 0$ and $\alpha, s > 0$, the prototype $f(u) = \alpha u^s$. We focus our analysis on the value $T_{max} \in (0,\infty]$, since it establishes the temporal interval of existence of solutions (u,v,w) to problem ($\diamond$). To be precise, once the problem is endowed with zero-flux boundary conditions we prove the following results, all excluding chemotactic collapse scenarios under precise correlations between the attraction and repulsive effects prescribing the model:

- for every $\alpha,\beta,\gamma,\delta,\chi > 0$, and $r > s \geq 1$, there exists $\xi^* > 0$ such that if $\xi > \xi^*$, any sufficiently regular initial datum $u_0(x) \geq 0$ emanates a unique classical solution (u,v,w) to problem ($\diamond$) which is global, i.e. $T_{max} = \infty$, and such that u, v and w are uniformly bounded;
- for every $\alpha,\beta,\gamma,\delta,\chi > 0$, and $s > r \geq 1$, there exists $\xi_* > 0$ such that if $\xi \geq \xi_*$, any sufficiently regular initial datum $u_0(x) \geq 0$ enjoying precise smallness assumptions emanates a unique classical solution (u,v,w) to problem ($\diamond$) which is global, i.e. $T_{max} = \infty$, and such that u, v and w are uniformly bounded;
- for every $\alpha,\beta,\gamma,\delta,\chi,\xi > 0$, and $0 < s < 1$ and $r = 1$, any sufficiently regular initial datum $u_0(x) \geq 0$ emanates a unique classical solution (u,v,w) to problem ($\diamond$) which is global, i.e. $T_{max} = \infty$, and such that u, v and w are uniformly bounded.

Further, in a remark of the manuscript, we also address an open question posed in [Vig19].

1. INTRODUCTION AND PRESENTATION OF THE MAIN RESULTS

The chemotaxis is the movement of certain cells, situated in an environment, along the concentration gradient of a chemical signal/stimulus produced by a substance which is therein inhomogeneously distributed. The first models of chemotaxis phenomena were introduced by Keller and Segel in [KS70, KS71a, KS71b];

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they are formulated in terms of partial differential equations and have been largely developed during the last decades in the fields of both theoretical and applied mathematics.

Among the countless variants of these landmark models, it is worthwhile to consider the general situation where the motion of a certain cell density $u = u(x,t)$, distributed inside an insulated domain (zero-flux on the border), is dictated by the natural diffusion (below indicated with A) of the cells which attract and repulse each other through two coupled chemosensitivity effects: ($B < 0$) chemoattractant and ($C > 0$) chemorepellent. Additionally, the chemical signal $w = w(x,t)$, responsible for moving away one cell from the other, spreads and is produced by a law indicated by E, whilst the other one, $v = v(x,t)$ which has a coalescence impact on the cells, grows with F. A more involved scenario is obtained when, in addition to what was now specified, an external source (D) influences the kinetics of the cells by providing and dissipating density; the corresponding mathematical formulation reads

$$\begin{cases} u_t = \nabla \cdot (A(u,v,w)\nabla u + B(u,v,w)\nabla v + C(u,v,w)\nabla w) + D(u,v,w) \\ \tau v_t = \Delta v + E(u,v,w) \\ \tau w_t = \Delta w + F(u,v,w) \end{cases} \tag{1}$$

We analyze this system in $\Omega \times (0, T_{max})$, where Ω is a bounded and smooth domain in $\mathbb{R}^n$, $n \geq 2$, and where $T_{max} \in (0,\infty]$ establishes the temporal interval up to which the evolution of u, v and w advances. Additionally, A, B, C, D, E, F are sufficiently regular functions of their arguments and $\tau \in \{0,1\}$ is a parameter; for $\tau = 0$ the cell diffusion is rather slower than the diffusion of the chemicals. Moreover, we equip (1) with Neumann boundary conditions, imposing equal to zero the outward normal derivative, denoted with $(\cdot)_\nu$, of $u, v,$ and w on $\partial\Omega$, and for $\tau = 1$ (i.e. $\tau = 0$) complementing the system with nonnegative and sufficiently regular initial distributions for $u_0(x)$, $v_0(x)$ and $w_0(x)$ (i.e. initial distribution for $u_0(x)$).

Experiments and observations indicate that the aforementioned cellular movement may present certain instabilities, as the so-called *chemotactic collapse*, the mechanism resulting in aggregation processes for the cell distribution, eventually blowing up/exploding at finite time. This is well known in the classical Keller–Segel system, obtained from (1) by letting $A(u,v,w) \equiv 1$, $B(u,v,w) = -\chi v$, $\chi > 0$, $C(u,v,w) = D(u,v,w) \equiv 0$ and $E(u,v,w) = -v + u$, and eliminating the third unknown w; see, for instance, [HW01] and [Win13] for the paraolic-parabolic case (i.e. $\tau = 1$) and [JL92] and for [Nag01] for the parabolic-elliptic one ($\tau = 0$). On the other hand, the introduction of external and/or chemo-diffusive smoothing effects may prevent blow-up scenarios, even in the presence of a superlinear growth rate for the chemoattractant v: in [GST16] it is proved that for $A(u,v,w) \equiv 1$, $B(u,v,w) = -\chi u^m$, $C(u,v,w) \equiv 0$, $D(u,v,w) = \mu u(1-u^\alpha)$, $\tau = 0$, $E(u,v,w) = u^\gamma - v$, with $m, \gamma, \alpha \geq 1$ and $\chi, \mu > 0$, and under the assumptions $\alpha > m + \gamma - 1$ or $\alpha = m + \gamma - 1$ and $\mu > \frac{n\alpha - 2}{2(m-1)+n\alpha\chi}$, with $n \in \mathbb{N}$, any nonnegative and sufficiently regular initial datum provides to (1) a unique classical solution, which is uniformly bounded. Moreover the same holds for the critical case $\alpha = m + \gamma - 1$ and $\mu > \frac{n\alpha - 2}{2(m-1)+n\alpha\chi}$, as established in [HT17]. (We also mention [LT16, Win18] for the critical analysis on both blowing-up and global solutions to Keller–Segel systems with super and sublinear stimulus production.) Further, for B, C, D and E as above but for $A(u,v,w) \simeq u^{q-1}$, $q \geq 1$, in [WZ18] it is shown that there exists $\mu^* = \mu^*(m,q,\chi,\alpha,n) \geq 0$ such that when $q > m + \gamma - \frac{2}{n}$ or

$\alpha > m+\gamma-1$ or $\alpha = m+\gamma-1$ and $\mu > \mu^*$ system (1) possesses a global bounded classical solution for any sufficiently smooth initial data.

Conversely, the sole insertion in the classical Keller–Segel model of a repulsive effect coming from another chemical substance does not suffice to avoid δ-formations for the cells' density. More precisely, confining our attention to the linear diffusion case $A(u,v,w) \equiv 1$, and fixing $B(u,v,w) = -\chi\nabla u$ and $C(u,v,w) = \xi\nabla u$ $(\chi,\xi > 0)$, $D(u,v,w) \equiv 0$ and productions $E(u,v,w) = \alpha u - \beta v$, $F(u,v,w) = \gamma u - \delta w$ $(\alpha,\beta,\gamma,\delta > 0)$, for $\tau = 0$, we have that the value $\xi\gamma - \chi\alpha$, measuring in some sense the difference between the repulsion and attraction contributions, plays an important role in problem (1). Indeed, the sign of $\xi\gamma - \chi\alpha$ (positive, repulsion prevails over attraction, negative attraction prevails over repulsion) establishes whether the system has unbounded solutions or all solutions are bounded: see [TW13, HGZ17, GJZ18, LL16, Vig19] for some details on the issue.

Conditions on the data	**Property of the solution**	**Reference**
∘ $n \geq 2$ and any $u_0 \geq 0$ ∘ $\alpha,\beta,\gamma,\delta,\chi > 0$ ∘ $r > s \geq 1$ ∘ ξ large enough	Global and bounded	Theorem 4.4
∘ $n \geq 2$ and $u_0 \geq 0$ ∘ $\alpha,\beta,\gamma,\delta,\chi > 0$ ∘ $s > r \geq 1$ and $\bar{p} = ps$ with $p > 1$ ∘ $\lVert u_0\rVert_{L^{\bar{p}}(\Omega)}$ small enough ∘ ξ large enough	Global and bounded	Theorem 4.8
∘ $n \geq 2$ and any $u_0 \geq 0$ ∘ $r = 1$ and $s < 1$ ∘ $\alpha,\beta,\gamma,\delta,\chi,\xi > 0$	Global and bounded	Theorem 4.9
∘ $n \geq 2$ and any $u_0 \geq 0$ ∘ $\alpha,\beta,\gamma,\delta,\xi,\chi > 0$ ∘ $r = s = 1$ ∘ $\xi\gamma - \chi\alpha > 0$	Global and bounded	[TW13, Theorem 2.1]
∘ $n = 2$ and $u_0 \geq 0$ ∘ $\alpha,\beta,\gamma,\delta,\xi,\chi > 0$ ∘ $r = s = 1$ ∘ $\xi\gamma - \chi\alpha < 0$ and $\beta = \delta$ ∘ $\lVert u_0\rVert_{L^1(\Omega)}$ large enough	Blow-up at finite time	[TW13, Proposition 2.2]
∘ $n = 2$ and $u_0 \geq 0$ ∘ $\alpha,\beta,\gamma,\delta,\xi,\chi > 0$ ∘ $r = s = 1$ ∘ $\xi\gamma - \chi\alpha < 0$ and $\beta,\delta > 0$ ∘ $\lVert u_0\rVert_{L^1(\Omega)}$ large enough	Blow-up at finite time	[HGZ17, Theorem 1]

TABLE 1. Some known and new results concerning model (2).

Exactly in the perspective toward blow-up prevention, herein we consider this particular case (even proposed in [LCREKM03] to describe the aggregation of microglia observed in Alzheimer's disease) of problem (1),

$$
\begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{max}),\\
0 = \Delta v + f(u) - \beta v & \text{in } \Omega \times (0, T_{max}),\\
0 = \Delta w + \gamma u^r - \delta w & \text{in } \Omega \times (0, T_{max}),\\
u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}),\\
u_0(x) := u(x, 0) \geq 0 & x \in \Omega,
\end{cases} \tag{2}
$$

where, again, Ω is a smooth and bounded domain of $\mathbb{R}^n$, $n \geq 2$, $\gamma, \beta, \delta, \chi, \xi > 0$, $r \geq 1$ and

$$
f \in C^1([0, \infty)), \quad 0 \leq f(\theta) \leq \alpha \theta^s, \quad \alpha, s > 0. \tag{3}
$$

Moreover, $u_0(x)$ is a nonnegative and sufficiently regular function on $\bar{\Omega}$, whereas $T_{max} \in (0, \infty]$ obeys (in the sense of Lemma 2.1 below) the following dichotomy criterion: either $T_{max} = \infty$, and hence the solution to (2) is defined for all $x \in \Omega$ and all $t > 0$, or T_{max} is finite (the blow-up time), and the solution blows up at T_{max}.

From what we observed through our bibliographic research, so far no result concerning the effects produced by nonlinear production rates in attraction-repulsion models as that in (2) have been discussed at length. For instance, it is conceivable that a sublinear production law for the chemoattractant and a linear one for the chemorepellent suffice to guarantee globally bounded cells' distributions, despite any arbitrarily large initial distribution for the cells themselves and the size of the other parameters. Similarly, it can be expected that, even without inserting any logistic dampening source providing smoothness and equilibrium, magnifying the impact of high values of the third solution component, associated to the repulsion effect, on the evolution of the first should enforce global existence of solutions. But to what extent does it have to be magnified? What happens if the chemorepellent signal is produced by the bacterial density with a superlinear rate, whilst that of the chemoattractant with a (sub)linear one? Does it prevent any gathering of cell density as expected in the opposite case? And, even more, how do the cells react when attracted by a substance produced superlinearly and repelled by another also produced superlinearly, but more weakly than the first? And viceversa?

According to the previous considerations, we summarize our main results dealing with model (2) as follows (and in Table 1 we also frame them in terms of other known facts):

i) If the production rate of the repulsive signal surpasses the attractive counterpart, which is linear or superlinear, then any initial cell distribution spreads throughout time and remains bounded, provided the coefficient of the chemical signal responsible of the repulsion is sufficiently large (Theorem 4.4);

ii) If the production rate of the attractive stimulus surpasses the repulsive counterpart, which is linear or superlinear, then any sufficiently small initial cells' distribution expands for all times and remains bounded, as long as the coefficient of the chemical signal responsible of the repulsion is sufficiently large (Theorem 4.8);

iii) If the production rate of the chemoattractant is sublinear then any initial initial cells' distribution diffuses throughout the time and remains bounded for linear chemorepellent signals (Theorem 4.9), even for arbitrarily small (large) repulsive (attractive) coefficients.

Remark 1. *We believe that it is worthwhile spending some words on the results presented in the items above. Taking the limit case $s = 1$ and comparing the analysis in the first and third arrow of Table 1, we essentially observe that when $s < 1$ and $r = 1$, the sublinearity effect of the attraction in the first equation of system (2) suffices to prevail on the linear repulsion and the corresponding conclusion in iii) is obtained. Conversely the opposite situation, i.e. $s = 1$ and $r > 1$ (repulsion stronger than attraction), does not seem to paint the same picture: indeed in the case i) another condition enforcing the action of the repulsion is required. Also item ii) is suggestive: as far as the attraction is superior to a linear or superlinear repulsion, only small initial distributions give rise to global situations without explosion instabilities, but always provided large repulsive coefficients influence the dynamics. This is surprising, or somewhat interesting, since it seems to indicate that in attraction-repulsion chemotaxis models the impact of the repulsive and attractive agents is not specular: the effect of the firsts is weaker than that coming from the seconds.*

2. From local to globally bounded solutions

One of the first steps in dealing with solutions of system (2) is showing that they actually exist, at least locally.

Lemma 2.1. *Let Ω be a bounded and smooth domain in $\mathbb{R}^n$, $n \geq 2$. Assume $\gamma, \beta, \delta, \chi > 0$, $r \geq 1$ and let $0 \leq u_0(x) \in C^0(\bar{\Omega})$ be any nontrivial initial datum. Then for any $\xi > 0$ and f satisfying (3), problem (2) admits a unique classical solution (u, v, w) of nonnegative functions, precisely in the class*

$$C^0([0, T_{max}); C^0(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \times C^{2,0}(\bar{\Omega} \times (0, T_{max})) \times C^{2,0}(\bar{\Omega} \times (0, T_{max})).$$

Here $T_{max} \in (0, \infty]$, denoting the maximal existence time, is such that (dichotomy criterion) either $T_{max} = \infty$ (global-in-time classical solution) or if $T_{max} < \infty$ (local-in-time classical solution) then necessarily

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{4}$$

Moreover,

$$\int_\Omega u(\cdot, t) = m := \int_\Omega u_0 > 0 \quad \textit{for all} \quad t \in (0, T_{max}). \tag{5}$$

Proof. The first statement can be shown by straightforward adaptations of well-established methods involving an appropriate fixed point framework and standard parabolic and elliptic regularity theory (see, for instance, [Cie07], [HW05] and [FWY15]), as well as related comparison principles. On the other hand, relation (5) directly follows by integrating over Ω the equation for u in (2). $\square$

Once the (at worst local) solvability for problem (2) is ensured, the bridge establishing the globability and boundedness is achieved throughout some specific L^p estimates for these solutions. To be precise we have this

Lemma 2.2. *Under the assumptions of Lemma 2.1, $\xi > 0$ and f satisfying (3), let (u, v, w) be the classical solution to problem (2). If for some $\frac{n}{2} < p < n$ the functions $u, f(u)$ and u^r belong to $L^\infty((0, T_{max}); L^p(\Omega))$, then $T_{max} = \infty$ and u, v and w are uniformly bounded in $\Omega \times (0, \infty)$.*

Proof. Of course once it is shown that $u \in L^\infty((0, \infty); L^\infty(\Omega))$, both $f(u)$ and u^r are in this same space, so the same property (due to elliptic regularity results on the second and third equation of (2)) is enjoyed by v and w. From the hypotheses $f, u^r \in L^\infty((0, T_{max}); L^p(\Omega))$, the classical regularity theory on elliptic equations in conjunction with Sobolev embedding theorems infer that

$$v, w \in L^\infty((0, T_{max}); W^{2,p}(\Omega)) \text{ and } \nabla v, \nabla w \in L^\infty((0, T_{max}); W^{1,p}(\Omega)),$$

and so

$$v, w \in L^\infty((0, T_{max}); C^{[2-(n/p)]}(\bar{\Omega})) \text{ and } \nabla v, \nabla w \in L^\infty((0, T_{max}); L^q(\Omega)),$$

for all $n < q < p^* := \frac{np}{n-p}$. In particular, by posing $\tilde{v} = \chi u - \xi w$ we have that for some positive constant C_q

$$\|\tilde{v}(\cdot, t)\|_{L^q(\Omega)} + \|\nabla \tilde{v}(\cdot, t)\|_{L^q(\Omega)} \leq C_q \quad \text{for all} \quad t \in (0, T_{max}). \tag{6}$$

Additionally, for any $(x, t) \in \Omega \times (0, T_{max})$, the first equation of (2) reads $u_t = \Delta u - \nabla \cdot (u \nabla \tilde{v})$ so that for $t_0 := \max\{0, t-1\}$ the representation formula yields

$$u(\cdot, t) \leq e^{(t-t_0)\Delta} u(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla \tilde{v}(\cdot, s)) ds =: u_1(\cdot, t) + u_2(\cdot, t).$$

Under these circumstances, the rest of the proof follows that given in [VW27, Lemma 4.1]; precisely, in order to control the $L^\infty(\Omega)$-norm of u on $(0, T_{max})$, first one controls (also relying on $u \in L^\infty((0, T_{max}); L^p(\Omega))$) a suitable norm of the cross diffusion term $u \nabla \tilde{v}$ by replacing relation (24) therein with bound (6), then applications of known smoothing estimates for the Neumann heat semigroup entail this uniform bound on $(0, T_{max})$. Finally, the conclusion $u \in L^\infty((0, \infty); L^\infty(\Omega))$ is readily achieved from the dichotomy criterion (4). □

Remark 2 (On the question in [Vig19, Remark 1]). *From the above lemma, the open question posed in [Vig19, Remark 1] has a response: indeed, in the context of [Vig19, Theorem 3.1], if the u-component of the solution (u, v, w) to the bi-dimensional version of problem (2) with $r = s = 1$ becomes unbounded at some finite time t^* (in the sense of the $L^\infty(\Omega)$-norm) it also blows-up in the $L^p(\Omega)$-norm for any $p > 1$, since otherwise from Lemma 2.2 with $n = 2$ it would be globally bounded. In particular, that theorem continues to be valid also without the extra assumption that $\int_\Omega u^p \nearrow \infty$ as $t \nearrow t^*$.*

3. Preparatory Lemmas

With the crucial implication of Lemma 2.2 in our hands, in this section we aim at bounding on $(0, T_{max})$ the functional $\phi(t) := \int_\Omega u^{\bar{p}}$, for $\bar{p} > 1$, by means of a time independent constant. This will be obtained by deriving a proper absorptive differential inequality for $\phi(t)$, exactly with the aid of a series of lemmas, which we present according to our aims and bias.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and smooth domain and $\delta > 0$. Then for any nonnegative $h \in C^2(\bar{\Omega})$, the solution $0 \leq \psi \in C^{3,\kappa}(\bar{\Omega})$, $0 < \kappa < 1$, of the problem*

$$\begin{cases} 0 = \Delta\psi + h - \delta\psi & \text{in } \Omega, \\ \psi_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

has the following property: For any $\hat{c}, \sigma > 0$ and $p > 1$, there exists $\tilde{c} = \tilde{c}(\sigma, p) > 0$ such that

$$\hat{c}\int_\Omega \psi^{p+1} \leq (\sigma + \tilde{c})\int_\Omega h^{p+1}. \tag{8}$$

or

$$\hat{c}\int_\Omega \psi^{p+1} \leq \sigma\int_\Omega h^{p+1} + \frac{\tilde{c}}{|\Omega|^p}\Big(\int_\Omega h\Big)^{p+1}. \tag{9}$$

Proof. Retracing what has been presented in [Win14, Lemma 2.2] and [Lan15, Lemma 2.2], we give herein some details there omitted. From (7), a direct integration over Ω produces, for any $p > 1$,

$$\Big(\int_\Omega \psi\Big)^{p+1} = \Big(\frac{1}{\delta}\Big)^{p+1}\Big(\int_\Omega h\Big)^{p+1}, \tag{10}$$

whilst testing procedures and Young's inequality yield

$$p\int_\Omega \psi^{p-1}|\nabla\psi|^2 + \delta\int_\Omega \psi^{p+1} = \int_\Omega \psi^p h \leq \frac{4p}{(p+1)^2}\int_\Omega \psi^{p+1} + \frac{1}{4^p}(p+1)^{p-1}\int_\Omega h^{p+1}.$$

This, through the identity $|\nabla\psi^{\frac{p+1}{2}}|^2 = \frac{(p+1)^2}{4}\psi^{p-1}|\nabla\psi|^2$, reads for all $\eta \in (0, \frac{1}{2})$

$$\eta\int_\Omega |\nabla\psi^{\frac{p+1}{2}}|^2 \leq \eta\int_\Omega \psi^{p+1} + \eta\frac{(p+1)^{p+1}}{4^{p+1}p}\int_\Omega h^{p+1}. \tag{11}$$

On the other hand, for the same $\eta \in (0, \frac{1}{2})$, by virtue of the inclusions

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow L^{\frac{2}{p+1}}(\Omega),$$

Ehrling's Lemma (see [Sho97, Lemma 1.1]) yields a constant $c_E(\eta) > 0$ such that

$$\|V\|^2_{L^2(\Omega)} \leq \eta\|V\|^2_{W^{1,2}(\Omega)} + c_E(\eta)\|V\|^2_{L^{\frac{2}{p+1}}(\Omega)} \quad \text{for all } V \in W^{1,2}(\Omega);$$

subsequently, posing in this last relation $V = \psi^{\frac{p+1}{2}}$ and making use of (10) and (11), we obtain

$$(1-2\eta)\int_\Omega \psi^{p+1} \leq \eta\frac{(p+1)^{p+1}}{4^{p+1}p}\int_\Omega h^{p+1} + \Big(\frac{1}{\delta}\Big)^{p+1} c_E(\eta)\Big(\int_\Omega h\Big)^{p+1}.$$

Finally, for any $\hat{c} > 0$ we introduce the increasing function $\sigma : (0, \frac{1}{2}) \to (0, \infty)$ defined as $\sigma(\eta) = \frac{\eta}{1-2\eta}\frac{(p+1)^{p+1}\hat{c}}{4^{p+1}p}$, and statement (8) follows with the choice

$$\tilde{c} = \tilde{c}(\sigma, p) := \frac{\hat{c}|\Omega|^p}{\delta^{p+1}}\frac{c_E(\eta^{-1}(\sigma))}{1 - 2\eta^{-1}(\sigma)},$$

once Hölder's inequality provides

$$\left(\frac{1}{\delta}\right)^{p+1} c_E(\eta)\Big(\int_\Omega h\Big)^{p+1} \leq \frac{|\Omega|^p}{\delta^{p+1}} c_E(\eta)\int_\Omega h^{p+1}.$$

If, indeed, we refrain from using such an inequality, we directly get relation (9). □

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and smooth domain. Then we have these estimates:*

- *For any $\bar{p} > 1$ and $0 < \theta_1 = \frac{\frac{\bar{p}}{2}-\frac{1}{2}}{\frac{\bar{p}}{2}+\frac{1}{n}-\frac{1}{2}} < 1$, there is a constant $c_* > 0$ such that all functions $0 \leq \psi \in L^1(\Omega)$, with $m := \int_\Omega \psi$ and $\nabla\psi^{\frac{\bar{p}}{2}} \in L^2(\Omega)$, fulfill*

$$\int_\Omega \psi^{\bar{p}} \leq \frac{4(\bar{p}-1)}{\bar{p}}\int_\Omega |\nabla \psi^{\frac{\bar{p}}{2}}|^2 + c_* \quad \textit{for all } t \in (0, T_{max}). \tag{12}$$

Herein $c_* = c_*(m) > 0$, and $c_*(m) \nearrow 0$ as $m \nearrow 0$.

- *For any arbitrary reals $\epsilon_1 > 0$, $s \geq 1$ and $\bar{p} > \frac{ns}{2} \geq 1$, there exist computable and m-independent constants $d_1(\epsilon_1), c_1 > 0$ such that all functions $0 \leq \psi \in L^{\bar{p}}(\Omega)$, with $m := \int_\Omega \psi$ and $\nabla\psi^{\frac{\bar{p}}{2}} \in L^2(\Omega)$, comply with*

$$\int_\Omega \psi^{\bar{p}+s} \leq \epsilon_1 \int_\Omega |\nabla \psi^{\frac{\bar{p}}{2}}|^2 + d_1(\epsilon_1)\left(\int_\Omega \psi^{\bar{p}}\right)^{\frac{2\bar{p}+2s-sn}{2\bar{p}-ns}} + c_1 m^{\bar{p}+s} \quad \textit{on } (0, T_{max}). \tag{13}$$

Proof. The proof of (12) comes from an application of a general case of the Gagliardo–Nirenberg inequality: in particular, for any $\bar{p} > 1$, we can use [Vig 0, (22) of Lemma 4] with $f = u^{\frac{\bar{p}}{2}}$, $\mathfrak{p} = \mathfrak{q} = 2$ and $\mathfrak{r} = \frac{2}{\bar{p}}$ so to explicitly have (recall $(a+b)^2 \leq 2(a^2+b^2)$, for all $a, b \in \mathbb{R}$) for some $C_{GN} > 0$

$$\begin{aligned}\int_\Omega \psi^{\bar{p}} &= \|\psi^{\frac{\bar{p}}{2}}\|^2_{L^2(\Omega)} \leq C_{GN}^2(\|\nabla\psi^{\frac{\bar{p}}{2}}\|^{\theta_1}_{L^2(\Omega)}\|\psi^{\frac{\bar{p}}{2}}\|^{1-\theta_1}_{L^{\frac{2}{\bar{p}}}(\Omega)} + \|\psi^{\frac{\bar{p}}{2}}\|_{L^{\frac{2}{\bar{p}}}(\Omega)})^2\\ &= 2C_{GN}^2 m^{\bar{p}(1-\theta_1)}\left(\int_\Omega |\nabla\psi^{\frac{\bar{p}}{2}}|^2\right)^{\theta_1} + 2C_{GN}^2 m^{\bar{p}} \quad \text{on } (0, T_{max});\end{aligned}$$

hence, we conclude invoking the Young inequality with exponents θ_1 and $(1-\theta_1)$ and with $c_* = 2C_{GN}^2 m^{\bar{p}}[(1-\theta_1)(\frac{2(\bar{p}-1)}{\bar{p}\theta_1 C_{GN}^2})^{\frac{\theta_1}{\theta_1-1}} + 1]$.

To prove bound (13), the classical Gagliardo–Nirenberg inequality (see [Nir59, p. 126]) infers another $\tilde{C}_{GN} > 0$ by means of which we can control $\int_\Omega \psi^{\bar{p}+s}$ on $(0, T_{max})$ as follows:

$$\int_\Omega \psi^{\bar{p}+s} = \|\psi^{\frac{\bar{p}}{2}}\|^{2\frac{\bar{p}+s}{\bar{p}}}_{L^{2\frac{\bar{p}+s}{\bar{p}}}(\Omega)} \leq \left(\tilde{C}_{GN}(\|\nabla\psi^{\frac{\bar{p}}{2}}\|^{\theta_2}_{L^2(\Omega)}\|\psi^{\frac{\bar{p}}{2}}\|^{1-\theta_2}_{L^2(\Omega)} + \tilde{C}_{GN}\|\psi^{\frac{\bar{p}}{2}}\|_{L^{\frac{2}{\bar{p}}}(\Omega)}\right)^{2\frac{\bar{p}+s}{\bar{p}}},$$

with, since $\bar{p} > \frac{ns}{2} \Rightarrow \bar{p} > (n-2)\frac{s}{2}$, $\theta_2 = \frac{ns}{2(\bar{p}+s)} \in (0,1)$. Hence by using

$$(a+b)^{2\frac{\bar{p}+s}{\bar{p}}} \leq 2^{2\frac{\bar{p}+s}{\bar{p}}-1}\left(a^{2\frac{\bar{p}+s}{\bar{p}}} + b^{2\frac{\bar{p}+s}{\bar{p}}}\right) \quad \text{with} \quad a, b \geq 0,$$

we get for some m-independent $c_0 > 0$

$$\int_\Omega \psi^{\bar{p}+s} \leq c_0\left(\int_\Omega |\nabla\psi^{\frac{\bar{p}}{2}}|^2\right)^{\frac{ns}{2\bar{p}}}\left(\int_\Omega \psi^{\bar{p}}\right)^{\frac{2\bar{p}+2s-ns}{2\bar{p}}} + c_0 m^{\bar{p}+s} \quad \text{on } (0, T_{max}).$$

Finally, an application of the Young inequality with exponents $\frac{ns}{2\bar{p}}$ and $1 - \frac{ns}{2\bar{p}}$ and supported by the introduction of an arbitrarily positive ϵ_1, infers the statement. □

Lemma 3.3. *Let $\gamma_0 > 1$ and $l, L, C > 0$ fulfill the strict inequality*

$$C < \left(\frac{l^{\gamma_0}}{L\gamma_0}\right)^{\frac{1}{\gamma_0 - 1}} \left(\frac{\gamma_0 - 1}{\gamma_0}\right). \tag{14}$$

Then there exists $\phi_* > 0$ such that solutions of the initial problem

$$\begin{cases} \phi'(t) \leq -l\phi(t) + L\phi(t)^{\gamma_0} + C & \text{for } t > 0, \\ \phi(0) \leq \phi_*, \end{cases} \tag{15}$$

satisfy $\phi(t) \leq \phi_*$ for all $t \in (0, \infty)$.

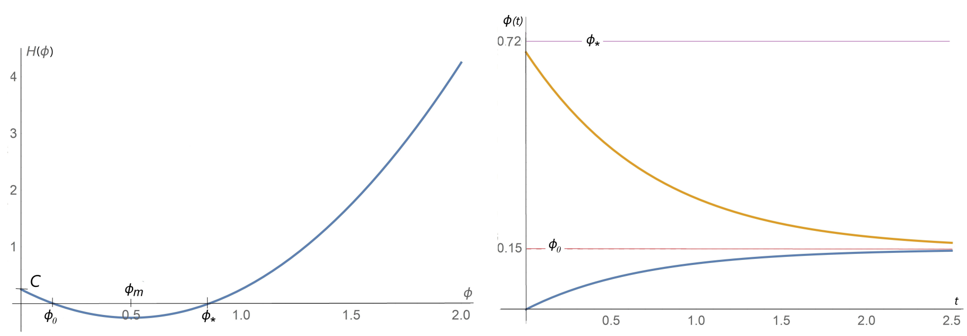
Proof. For any $\phi \geq 0$, let us define the regular function $H(\phi) := -l\phi + L\phi^{\gamma_0} + C$. Hence it is straightforwardly seen that

$$H(0) = C > 0, \qquad \lim_{\phi \to +\infty} H(\phi) = +\infty,$$

and that for $\phi_m := (l/L\gamma_0)^{\frac{1}{\gamma_0 - 1}}$ it holds that $H'(\phi_m) = 0$; specifically the point $(\phi_m, H(\phi_m))$ represents the absolute minimum of H. On the other hand, in view of relation (14), we can check that

$$H(\phi_m) = C - \left(\frac{l^{\gamma_0}}{L\gamma_0}\right)^{\frac{1}{\gamma_0 - 1}} \left(\frac{\gamma_0 - 1}{\gamma_0}\right) < 0.$$

In this way, the existence of $0 < \phi_0 < \phi_m < \phi_*$ such that $H(\phi_0) = H(\phi_*) = 0$ is guaranteed (see subfigure 1(a)). In particular ϕ_* itself is a (global) supersolution of the equation $\phi' = H(\phi)$ (like ϕ_0, but it is smaller than ϕ_*) satisfying the initial condition $\phi_*(0) = \phi_*$. We finally conclude by comparison arguments, since ϕ is a subsolution (subfigure 1(b)) to problem (15). □



(a) The smallest (ϕ_0) and the largest (ϕ_*) roots of the equation $H(\phi) = 0$. The value ϕ_m is the minimum point of H and the absolute minimum $H(\phi_m)$ is such that $H(\phi_m) < 0$.

(b) The funtions $\phi(t) = \phi_0$ and $\phi(t) = \phi_*$ are, respectively, the (gloabl) stable and unstable supersolutions of the equation $\phi' = H(\phi)$; hence any subsolution of $\phi' = H(\phi)$ complying with $\phi(0) \leq \phi_*$ remains below ϕ_*.

FIGURE 1. Interpretation of comparison principles.

Remark 3. *Relation (8) of Lemma 3.1 is essential to estimate in the proofs of the three theorems integrals involving the w-component of the solution (u, v, w) to problem (2) in terms of others involving the u-component. A similar role is played* *by (12) of Lemma 3.2, by means of which certain bounds on $(0, T_{max})$ of $\int_\Omega \nabla u^{\frac{\bar{p}}{2}}$ are expressed as linear combinations of powers of $\phi(t)$. Conversely Lemma 3.3 and (13) of Lemma 3.2 are employed only during the proof of Theorem 4.8, for which the desired absorptive differential inequality for $\phi(t)$ can be achieved under precise technical conditions (as for instance (14) of Lemma 3.3).*

4. Some properties of classical solutions: Proofs of the theorems

The succeeding three subsections, §4.1, §4.2 and §4.3, include the formal proofs of the conceptual results presented in items i), ii) and iii) of §1. The contents of these subsections are all concerned with the derivations of a priori estimates on $(0, T_{max})$ for the component u of the solution (u, v, w) to problem (2) provided by Lemma 2.1. To be precise, the evolutive analysis on $(0, T_{max})$ of the functional $\phi(t) := \int_\Omega u^{\bar{p}}$ is the instrument implying the desired bound for u in $L^p(\Omega)$; in turn, such a bound is sufficient to guarantee the uniform boundedness of u. Henceforth, let us continue our computation toward such a derivation.

4.1. The case $r > s \geq 1$. Proof of Theorem 4.4.

Lemma 4.1. *Under the assumptions of Lemma 2.1, $\xi > 0$ and f satisfying (3) with the precise choice $r > s \geq 1$, let (u, v, w) be the classical solution to problem (2). Then for any $\epsilon_2, \epsilon_3, \sigma > 0$ and $p > 1$, there exists $\tilde{c} > 0$ and $d_3(\epsilon_3) > 0$ such that for $\bar{p} = pr$ the u-component satisfies for all $t \in (0, T_{max})$*

$$\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq -\frac{4(\bar{p}-1)}{\bar{p}}\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 + \left[\epsilon_2 + \epsilon_3 + \gamma^{p+1}(\sigma + \tilde{c}) - \xi\gamma(\bar{p}-1)\right]\int_\Omega u^{\bar{p}+r} + d_3(\epsilon_3).$$

Proof. We set $\bar{p} = pr$ and compute $\frac{d}{dt}\int_\Omega u^{\bar{p}}$. Using problem (2) and the divergence theorem (applied twice in both cross-diffusion terms), thanks to the hypotheses (3) on f, we have that for all $t \in (0, T_{max})$

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^{\bar{p}} &= \bar{p}\int_\Omega u^{\bar{p}-1}u_t = \bar{p}\int_\Omega u^{\bar{p}-1}[\Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w)] \\ &\leq -4\frac{(\bar{p}-1)}{\bar{p}}\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 + \alpha\chi(\bar{p}-1)\int_\Omega u^{\bar{p}+s} \\ &\quad + \xi\delta(\bar{p}-1)\int_\Omega u^{\bar{p}}w - \gamma\xi(\bar{p}-1)\int_\Omega u^{\bar{p}+r} - \chi\beta(\bar{p}-1)\int_\Omega u^{\bar{p}}v.\end{aligned} \tag{16}$$

For any $\epsilon_2 > 0$ and some $d_2(\epsilon_2) > 0$, by relying first on Young's inequality and then on (8) with $h = \gamma u^r$, $\psi = w$ and $\hat{c} = d_2(\epsilon_2)$, we can write on $(0, T_{max})$

$$\begin{aligned}\xi\delta(\bar{p}-1)\int_\Omega u^{\bar{p}}w &\leq \epsilon_2\int_\Omega u^{\bar{p}+r} + d_2(\epsilon_2)\int_\Omega w^{p+1} \\ &\leq \epsilon_2\int_\Omega u^{\bar{p}+r} + \gamma^{p+1}(\sigma+\tilde{c})\int_\Omega u^{\bar{p}+r},\end{aligned} \tag{17}$$

and, again by Young's inequality for any $\epsilon_3 > 0$ and some $d_3(\epsilon_3) > 0$

$$\alpha\chi(\bar{p}-1)\int_\Omega u^{\bar{p}+s} \leq \epsilon_3\int_\Omega u^{\bar{p}+r} + d_3(\epsilon_3) \quad \text{on } (0, T_{max}). \tag{18}$$

We henceforth have the claim by plugging these two gained estimates in (16) and neglecting the nonpositive term $-\chi\beta(p-1)\int_\Omega u^p v$. ◻

Lemma 4.2. *Under the assumptions of Lemma 2.1, and f satisfying (3) with the precise choice $r > s \geq 1$, the following holds: For any $\sigma > 0$ and $p > 1$, there exists $\xi^*(\sigma, p) > 0$ such that if $\xi > \xi^*(\sigma, p)$ and (u, v, w) is the classical solution to problem (2), we obtain for $\bar{p} = pr$ and some $c_2 > 0$*

$$\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq -4\frac{\bar{p}-1}{\bar{p}}\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 + c_2 \quad \textit{for all} \quad t \in (0, T_{max}).$$

Proof. For any $\sigma > 0$ and $p > 1$, let us set $\xi^*(\sigma, p) = \gamma^p \frac{\sigma+\tilde{c}}{rp-1}$, where $\tilde{c} = \tilde{c}(\sigma, p)$ was introduced in Lemma 3.1. Assumption $\xi > \xi^*(\sigma, p)$ allows us to choose in Lemma 4.1 these values: $\epsilon_2 = \epsilon_3 = \frac{k}{2}$, with $k = \gamma(\bar{p}-1)(\xi - \xi^*(\sigma, p)) > 0$. Thereafter, we see that $\epsilon_2 + \epsilon_3 + \gamma^{p+1}(\sigma + \tilde{c}) - \xi\gamma(\bar{p}-1) = 0$, and this yields the conclusion with $c_2 = d_3\left(\frac{k}{2}\right)$. □

Lemma 4.3. *Under the assumptions of Lemma 2.1, $\sigma > 0$, $p > 1$ and f satisfying (3) with the precise choice $r > s \geq 1$, let $\xi^*(\sigma, p)$ be the constant provided in Lemma 4.1. Then if $\xi > \xi^*(\sigma, p)$, the classical solution (u, v, w) to problem (2) is such that for $\bar{p} = pr$ and some $D > 0$*

$$\int_\Omega u^{\bar{p}} \leq D \quad \textit{for all } t \in (0, T_{max}).$$

Proof. For $\bar{p} = pr > 1$, by employing (12) of Lemma 3.2 with $\psi = u$ we have the inequality

$$-4\frac{\bar{p}-1}{\bar{p}}\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 \leq -\int_\Omega u^{\bar{p}} + c_* \quad \text{for all } t \in (0, T_{max}),$$

which in conjunction with Lemma 4.2 produce for for $\phi(t) = \int_\Omega u^{\bar{p}}$ the absorptive differential inequality $\phi'(t) \leq c_3 - \phi(t)$ on $(0, T_{max})$, with $c_3 = c_* + c_2$. By complementing this inequality with the natural initial condition $\phi(0) = \int_\Omega u_0^{\bar{p}}$, we immediately have $\phi(t) \leq \max\{\phi(0), c_3\} =: D$, for all $t \in (0, T_{max})$. □

We can hence show the first

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and smooth domain. Assume $\beta, \gamma, \delta, \chi > 0$, $r > 1$, f satisfying (3) for $r > s \geq 1$, and let $0 \leq u_0(x) \in C^0(\bar{\Omega})$ be any nontrivial initial datum. Then there exists ξ^* such that, if $\xi > \xi^*$, problem (2) admits a unique solution (u, v, w) of nonnegative and bounded functions in the class*

$$C^0([0,\infty); C^0(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \times C^{2,0}(\bar{\Omega} \times (0,\infty)) \times C^{2,0}(\bar{\Omega} \times (0,\infty)).$$

Proof. For any $n \geq 2$, let $\sigma = 1$, $p = \frac{3n}{4} \in (\frac{n}{2}, n)$ and $\xi^* = \xi^*(1, \frac{3n}{4})$ from Lemma 4.2. Additionally for $\xi > \xi^*$, let (u, v, w) be the classical solution to (2) provided by Lemma 2.1. The assumption $\xi > \xi^*$ allows us to apply Lemma 4.3; hence, for any $r > 1$, $u^r \in L^\infty((0, T_{max}); L^p(\Omega))$ and in turn $u, u^s \in L^\infty((0, T_{max}); L^p(\Omega))$. In particular, from (3), we have $f \in L^\infty((0, T_{max}); L^p(\Omega))$ as well, and an application of Lemma 2.2 immediately concludes the proof.

□

4.2. The case $s > r \geq 1$. Proof of Theorem 4.8.

Lemma 4.5. *Under the assumptions of Lemma 2.1, and f satisfying (3) with the precise choice $s > r \geq 1$, the following holds: For any $\sigma > 0$ and $p > 1$, there exists $\xi_*(\sigma, p) > 0$ such that if $\xi \geq \xi_*(\sigma, p)$ and (u, v, w) is the classical solution to problem (2), we obtain for $\bar{p} = ps$*

$$\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq -4\frac{\bar{p}-1}{\bar{p}}\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 + c_4\int_\Omega u^{\bar{p}+s} \quad \text{on } (0, T_{max}).$$

with some computable $c_4 > 0$.

Proof. Let $\bar{p} = ps$; the Young inequality and (8), with $h = \gamma u^r$, $\psi = w$ and $\hat{c} = c_5$ (with some $c_5 > 0$), entail

$$\begin{aligned}\xi\delta(\bar{p}-1)\int_\Omega u^{\bar{p}}w &\leq \chi\alpha(\bar{p}-1)\int_\Omega u^{\bar{p}+s} + c_5\int_\Omega w^{p+1}\\ &\leq \chi\alpha(\bar{p}-1)\int_\Omega u^{\bar{p}+s} + \gamma^{p+1}(\sigma+\tilde{c})\int_\Omega u^{\bar{p}+r} \quad \text{on } (0, T_{max}),\end{aligned}$$

so that from relation (16) we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq& -4\frac{(\bar{p}-1)}{\bar{p}}\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 + 2\alpha\chi(\bar{p}-1)\int_\Omega u^{\bar{p}+s}\\ &+ [\gamma^{p+1}(\sigma+\tilde{c}) - \gamma\xi(\bar{p}-1)]\int_\Omega u^{\bar{p}+r} \quad \text{on } (0, T_{max}).\end{aligned} \tag{19}$$

Taking $\xi_*(\sigma, p) = \gamma^p\frac{\sigma+\tilde{c}}{ps-1}$ (naturally $\tilde{c}(\sigma, p)$ again from Lemma 3.1), the assumption $\xi \geq \xi_*(\sigma, p)$ implies what we claimed for $c_4 = 2\alpha\chi(\bar{p}-1)$. □

Lemma 4.6. *Under the assumptions of Lemma 2.1, $\sigma > 0$, $p > \frac{n}{2}$ and f satisfying (3) with the precise choice $s > r \geq 1$, let $\xi_*(\sigma, p)$ be the constant provided in Lemma 4.5. Then if $\xi \geq \xi_*(\sigma, p)$, the classical solution (u, v, w) to problem (2) is such that for $\bar{p} = ps$ and $\gamma_0 = \frac{2p+2-n}{2p-n} > 1$*

$$\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq -l\int_\Omega u^{\bar{p}} + L\left(\int_\Omega u^{\bar{p}}\right)^{\gamma_0} + C \quad \text{on } (0, T_{max}),$$

with $l, L, C > 0$ computable and m-independent constants. Additionally, $C = C(m)$ is such that $C(m) \nearrow 0$ as $m \nearrow 0$.

Proof. The assumption on p implies $ps = \bar{p} > \frac{ns}{2}$. In order to estimate $\int_\Omega u^{\bar{p}+s}$ appearing in the conclusion of Lemma 4.5, we use (13) with $\psi = u$ and set $\gamma_0 = \frac{2p+2-n}{2p-n}$ so having

$$\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq \left(-4\frac{\bar{p}-1}{\bar{p}} + c_4\epsilon_1\right)\int_\Omega |\nabla u^{\frac{\bar{p}}{2}}|^2 + c_4 d_1(\epsilon_1)\left(\int_\Omega u^{\bar{p}}\right)^{\gamma_0} + c_4 c_1 m^{\bar{p}+s},$$

for all $t \in (0, T_{max})$. On the other hand, for $\epsilon_1 = 2\frac{\bar{p}-1}{\bar{p}c_4}$, recalling the expression of c_* in relation (12), the previous estimate reads

$$\frac{d}{dt}\int_\Omega u^{\bar{p}} \leq -\frac{1}{2}\int_\Omega u^{\bar{p}} + c_4 d_1\Big(2\frac{\bar{p}-1}{\bar{p}c_4}\Big)\left(\int_\Omega u^{\bar{p}}\right)^{\gamma_0} + m^{\bar{p}}\left(c_7 + c_8 m^s\right) \quad t \in (0, T_{max}),$$

for computable (and m-independent) $c_7, c_8 > 0$, so that we conclude with an evident choice of l, L and $C = C(m)$. □

Lemma 4.7. *Under the assumptions of Lemma 2.1, $\sigma > 0$, $p > \frac{n}{2}$ and f satisfying (3) with the precise choice $s > r \geq 1$, let $\xi_*(\sigma, p)$ be the constant provided in Lemma 4.5. Then for $\bar{p} = ps$ there exist $m_* > 0$ and $\phi_* > 0$ with the following property: For any $0 \leq u_0 \in C^0(\bar{\Omega})$ complying with*

$$\int_\Omega u_0 \leq m_* \quad \textit{and} \quad \int_\Omega u_0^{\bar{p}} \leq \phi_*, \tag{20}$$

the classical solution (u, v, w) to problem (2) emanating from u_0 satisfies

$$\int_\Omega u^{\bar{p}} \leq \phi_* \quad \textit{on } (0, \infty).$$

Proof. For l, L, γ_0 taken from Lemma 4.6, $\sigma > 0$, $p > \frac{n}{2}$ and $\xi_*(\sigma, p)$ as in Lemma 4.5, we choose m_* sufficiently small as to ensure for $C = C(m_*)$ (from the same Lemma 4.6) the validity of bound (14) in Lemma 3.3. In turn, for $\phi_* > 0$ provided by this last lemma, let $0 \leq u_0 \in C^0(\bar{\Omega})$ fulfill assumptions (20), and (u, v, w) be the classical solution to problem (2) established in Lemma 2.1. Under these circumstances, since from Lemma 4.6 it is seen that $\phi(t) = \int_\Omega u^{\bar{p}}$ satisfies the initial problem (15) of Lemma 3.3, we have the claim. □

We so have the second

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and smooth domain. Assume $\beta, \gamma, \delta, \chi > 0$, $r \geq 1$, f satisfying (3) for $s > r \geq 1$, and let $0 \leq u_0(x) \in C^0(\bar{\Omega})$ be any nontrivial initial datum. There exist $\xi_*, m_* > 0$ and $\phi_* > 0$ such that, if $\xi \geq \xi_*$, then for any $0 \leq u_0(x) \in C^0(\bar{\Omega})$ fulfilling $\int_\Omega u_0 \leq m_*$ as well as $\int_\Omega u_0^{\frac{3ns}{4}} \leq \phi_*$, problem (2) admits a unique solution (u, v, w) of nonnegative and bounded functions in the class*

$$C^0([0, \infty); C^0(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \times C^{2,0}(\bar{\Omega} \times (0, \infty)) \times C^{2,0}(\bar{\Omega} \times (0, \infty)).$$

Proof. For any $n \geq 2$, let $\sigma = 1$, $p = \frac{3n}{4} \in (\frac{n}{2}, n)$, $\xi_* = \xi_*(1, \frac{3n}{4})$ as in Lemma 4.5 and m_* and ϕ_* taken according to Lemma 4.7. In this way, for u_0 complying with our assumptions, the u-component of the classical solution to problem (2) is such that $u^s \in L^p((0, T_{max}); L^\infty(\Omega))$. Subsequently, and reasoning exactly as in the proof of Theorem 4.4, we rely on Lemma 2.2 and conclude. □

4.3. **The case $s < 1$ and $r = 1$. Proof of Theorem 4.9.** The next theorem closes our paper.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and smooth domain. Assume $\beta, \gamma, \delta, \xi, \chi > 0$, f satisfying (3) for $s < 1$ and $r = 1$, and let $0 \leq u_0(x) \in C^0(\bar{\Omega})$ be any nontrivial initial datum. Then problem (2) admits a unique solution (u, v, w) of nonnegative and bounded functions in the class*

$$C^0([0, \infty); C^0(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \times C^{2,0}(\bar{\Omega} \times (0, \infty)) \times C^{2,0}(\bar{\Omega} \times (0, \infty)).$$

Proof. In light of all of the above (particularly taking in mind proof the of Theorem 4.9), it is sufficient to show that for $\bar{p} = pr = p$, and some $\frac{n}{2} < p < n$ and $\underline{c} > 0$, we arrive at the crucial inequality

$$\frac{d}{dt} \int_\Omega u^p \leq -\frac{4(p-1)}{p} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \underline{c} \quad \text{for all} \quad t \in (0, T_{max}).$$

This is simply attained by applying (9) with $h = \gamma u^r = \gamma u$, $\psi = w$ and $\hat{c} = d_2(\epsilon_2)$, invoking the conservation of mass $\int_\Omega u = m$ for all $t \in (0, T_{max})$, so to reduce relation (17) into

$$\xi\delta(\bar{p}-1)\int_\Omega u^{\bar{p}} w \le \epsilon_2 \int_\Omega u^{\bar{p}+r} + \gamma^{p+1}\sigma \int_\Omega u^{\bar{p}+r} + \frac{\tilde{c}}{|\Omega|^p}(m\gamma)^{p+1} \quad \text{on } (0, T_{max}).$$

Choosing ϵ_2 and σ sufficiently small, as well as ϵ_3 in estimate (18), there is $\underline{c} > 0$ such that bound (16) becomes exactly as our desired inequality.

□

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