FINE BOUNDARY REGULARITY FOR THE DEGENERATE FRACTIONAL p-LAPLACIAN

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Abstract. We consider a nonlocal equation driven by the fractional p-Laplacian $(-\Delta)_p^s$ with $s \in]0,1[$ and $p \geqslant 2$ (degenerate case), with a bounded reaction f and Dirichlet type conditions in a smooth domain Ω. By means of barriers, a nonlocal superposition principle, and the comparison principle, we prove that any weak solution u of such equation exhibits a weighted Hölder regularity up to the boundary, that is, $u/\mathrm{d}_\Omega^s \in C^\alpha(\overline{\Omega})$ for some $\alpha \in]0,1[$, d_Ω being the distance from the boundary.

1. Introduction and main result

This paper is devoted to the study of some fine boundary regularity properties of the weak solution to the following problem:

$$\begin{cases} (-\Delta)_p^s\, u = f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.1}$$

Here, and throughout the paper, $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$, $\Omega^c = \mathbb{R}^N \setminus \Omega$, $s \in]0,1[$, $p \in]1,\infty[$ are real numbers, and $f \in L^\infty(\Omega)$. The leading operator is the s-fractional p-Laplacian, defined as the gradient of the energy

$$J(u) = \frac{1}{p} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy$$

in the space

$$W_0^{s,p}(\Omega) = \big\{u \in L^p(\mathbb{R}^N) : J(u) < \infty,\, u = 0 \quad \text{in } \Omega^c\big\}.$$

When restricted to conveniently smooth u's, such operator can be rephrased pointwisely as

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\, dy,$$

i.e., as a singular integral operator of fractional order s and summability power p, which for $p = 2$ reduces to the Dirichlet fractional Laplacian $(-\Delta)^s$ (up to a multiplicative constant). For a deep discussion on various notions (weak, viscous and strong) of solutions to (1.1), see [19]. A useful comparison principle for $(-\Delta)_p^s$ has been proved in [24], a Hopf's lemma in [5] and some strong comparison principles in [18], while its spectral properties are studied in [9, 17, 24].

The interior regularity theory for problem (1.1) is well developed. The linear case $p = 2$ is quite classical and Schauder estimates are available in the form $f \in C^\alpha \Rightarrow u \in C^{2s+\alpha}$ whenever $2s + \alpha$ is not an integer (see [28]). In the general case $p \neq 2$ the situation is more involved. The first results are [6, 7], dealing with local regularity and Harnack inequalities when $f = 0$ in (1.1). In the inhomogeneous case [3, 14, 15, 21, 23] contain local Hölder regularity estimates under various integrability assumptions on f, however the dependance of the Hölder exponent is not specified and not optimal. The papers [1, 30] deal with the degenerate case $p \geqslant 2$ and show higher fractional differentiability of u when fractional differentiability of the forcing term is assumed. In [26] higher

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fractional differentiability is obtained for any $p > 1$ under summability assumptions on f. Finally, still in the case $p \geqslant 2$, the optimal Hölder exponent for the solution of (1.1) is obtained in [2], giving e.g. $u \in C^{p's}_{\rm loc}(\Omega)$ when $f \in L^\infty(\Omega)$ and $p's < 1$.

The boundary regularity for problem (1.1) is more delicate. As a comparison, consider its local counterpart

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

(formally obtained by letting $s \to 1^-$ in (1.1)). It is well known that, for example, $u \in C^{1,\alpha}_{\rm loc}(\Omega)$ whenever f is bounded, and nothing more can be expected, regardless of the smoothness of f. This regularity can easily be extended up to the boundary, as follows. One straightens the boundary near $x_0 \in \partial\Omega$ and consider the odd reflection of the resulting u: as it turns out, it solves a similar equation in a larger domain containing x_0 in its interior, therefore satisfying the previous local regularity estimates. The odd reflection trick then shows that in general the interior and boundary regularity for (1.2) coincide. Boundary regularity for a wider class of nonlinear *local* operators is proved in [22].

This is no longer true for the fractional problem (1.1). For instance, the function $u(x) = (1-|x|^2)^s_+$ solves (1.1) for $\Omega = B_1$, $p = 2$ and $f =$ const. in Ω. Its interior regularity is C^∞ (as the Schauder theory *a priori* forces for C^∞ right-hand sides), but its boundary regularity is only C^s. Thus, we see that there is no obvious way to reproduce the odd reflection trick to deduce boundary regularity for (1.1), since actually boundary and interior regularity are quantitatively different.

The first result dealing with the boundary regularity for problem (1.1) is contained in [28] for $p = 2$, where it is proved that $u \in C^s(\mathbb{R}^N)$ whenever the non-homogeneous term is bounded. In the nonlinear case, [14, 15] contain a global Hölder continuity result, with an unspecified Hölder exponent (see also [20] for a refinement and generalisation when $f = 0$). Coupling the barrier argument contained in [14] with the optimal interior regularity of [2] provides the optimal regularity $u \in C^s(\mathbb{R}^N)$ when $p \geqslant 2$. Notice that the construction of the barrier in [14] only requires that $\partial\Omega$ is Lipschitz continuous and satisfies the exterior ball condition, matching the probably minimal regularity of the boundary assumed in [28] in the linear case. The same is expected to be true in the case $p \in \,]1, 2[$, but the optimal (or, at least C^s) interior regularity in this framework is missing.

Still, even in the linear case, there is much more to be said. Despite the optimal regularity $u \in C^s(\overline{\Omega})$ rules out in general the existence of the classical normal derivative, in the seminal paper [28] a regularity result for the s-normal derivative

$$\frac{\partial u}{\partial \nu^s}(x_0) := \lim_{t \to 0^+} \frac{u(x_0 + t\nu_{x_0})}{t^s},$$

where ν_{x_0} denotes the inner normal to $\partial\Omega$ at $x_0 \in \partial\Omega$. More precisely, they proved that, if when $p = 2$ and $\partial\Omega$ is $C^{1,1}$, then any solution u of (1.1) satisfies

$$\Big\| \frac{u}{{\rm d}_\Omega^s} \Big\|_{C^\alpha(\overline{\Omega})} \le C \|f\|_{L^\infty(\Omega)}, \quad {\rm d}_\Omega(x) := {\rm dist}(x, \partial\Omega)$$

for some $\alpha = \alpha(N, s, \Omega) \in \,]0, 1[$, $C = C(N, s, \Omega) > 0$.

The latter can also be seen as a weighted Hölder regularity result and it provided several applications to overdetermined problems [8], nonlinear analysis [12, 13], free boundary problems [4] and integration by parts formula [29]. For further references and related results we refer to the survey article [27].

Our main contribution is an analogous fine boundary regularity result for the weak solution to (1.1) in the degenerate case $p \geqslant 2$.

Theorem 1.1. *Let $p \geqslant 2$, Ω be a bounded domain with $C^{1,1}$ boundary and $\mathrm{d}_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$. Then there exist $\alpha \in]0, s]$ and $C > 0$, depending on N, p, s, and Ω, s.t. for all $f \in L^\infty(\Omega)$ the weak solution $u \in W_0^{s,p}(\Omega)$ to problem (1.1) satisfies $u/\mathrm{d}_\Omega^s \in C^\alpha(\overline{\Omega})$ and*

$$\Big\|\frac{u}{\mathrm{d}_\Omega^s}\Big\|_{C^\alpha(\overline{\Omega})} \leqslant C\|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}.$$

With the result above we hope to provide nonlocal regularity theory with an analog of Lieberman's $C^{1,\alpha}(\overline{\Omega})$ regularity theorem for the (local) p-Laplacian [22]. We privilege weak solutions (e.g., with respect to viscosity solutions, see [23]) mainly because we consider problem (1.1) in a variational perspective. A useful application of Theorem 1.1 is given in [16], yielding the equivalence of Sobolev and weighted Hölder local minimizers for the energy functional of a nonlinear boundary value problem driven by $(-\Delta)_p^s$ (similar results are proved in [13] for the linear case $p = 2$, and in [10] for the local p-Laplacian).
The singular case $p \in]1, 2[$ of Theorem 1.1 remains open, but it can probably be dealt with through suitable variations of the techniques presented here. Another interesting issue is related to the case of unbounded reactions. In fact, the $C^{1,\alpha}(\overline{\Omega})$ result for problem (1.2) can be achieved even when $f \in L^q(\Omega)$ for some $q > N$, so one can conjecture that Theorem 1.1 above also holds for sufficiently summable right hand side. However, our approach extensively uses the boundedness of the reaction and it is not apparent how to deal with unbounded f's.

Sketch of proof. Our aim is a weak Harnack inequality for the function u/d_Ω^s, and in particular a pointwise control of u/d_Ω^s in terms of an integral quantity. Our strategy is to exploit the nonlocality of the operator and define the following nonlocal excess:

$$\mathrm{Ex}(u, k, R, x_0) = \fint_{\tilde{B}_{R,x_0}} \Big|\frac{u}{\mathrm{d}_\Omega^s} - k\Big|\, dx,$$

with $k \in \mathbb{R}$, $R > 0$, and $\tilde{B}_{R,x_0}$ being a small ball of radius comparable to R, placed at distance greater than R in the inner normal direction from $x_0 \in \partial\Omega$ (see figure 1 and properties (2.2) for a precise definition). We call it nonlocal because it turns out that, given a bound on $(-\Delta)_p^s u$, the pointwise behaviour of u/d_Ω^s *inside* $B_R(x_0) \cap \Omega$ is controlled by the magnitude of the excess of u in $\tilde{B}_{R,x_0}$, which takes into account the behaviour of u/d_Ω^s *outside of* $B_R(x_0) \cap \Omega$.
In order to describe the scheme of the proof, consider the case of Ω being the half-space $\mathbb{R}^N_+ = \{x_N > 0\}$, $x_0 = 0$, $R = 1$, and $D_1 = B_1 \cap \mathbb{R}^N_+$. We are going to prove two types of weak Harnack inequalities. The first one is for supersolutions and reads

$$\begin{cases} (-\Delta)_p^s u \geqslant 0 & \text{in } D_1 \\ u \geqslant \mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N_+ \end{cases} \quad \Longrightarrow \quad \inf_{B_{1/4}\cap\mathbb{R}^N_+} \Big(\frac{u}{\mathrm{d}_\Omega^s} - 1\Big) \geqslant \sigma\, \mathrm{Ex}(u). \tag{1.3}$$

Here $e_N = (0, \dots, 1)$, $B_{1/4}$ is centered at 0 and σ is a positive constant depending only on N, p, and s. Besides, the translated ball $e_N + B_{1/4}$ corresponds to $\tilde{B}_1$ and we have set

$$\mathrm{Ex}(u) = \mathrm{Ex}(u, 1, 1, 0) = \fint_{e_N + B_{1/4}} \Big(\frac{u}{\mathrm{d}_\Omega^s} - 1\Big)\, dx.$$

The second one regards subsolutions and is

$$\begin{cases} (-\Delta)_p^s u \leqslant 0 & \text{in } D_1 \\ u \leqslant \mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N_+ \end{cases} \quad \Longrightarrow \quad \inf_{B_{1/4}\cap\mathbb{R}^N_+} \Big(1 - \frac{u}{\mathrm{d}_\Omega^s}\Big) \geqslant \sigma\, \mathrm{Ex}(u). \tag{1.4}$$

Note that in both cases we have a precise sign information on the difference $u/\mathrm{d}_\Omega^s - 1$ in the translated ball. The similarity of the two statements is misleading, since, as will be seen later, the second one is actually considerably more difficult to prove than the first one.

The reason why these kind of nonlocal weak Harnack inequalities hold lies in the following nonlocal superposition principle, which in a different form was proved in [14]. Given a regular function w and a perturbation u, define

$$\widetilde{w}_u = w + (u - w)\chi_{\tilde{B}_1}$$

Then, under some mild control of w in terms of d_Ω^s on $\tilde{B}_1$, we have

$$\begin{cases} u \geqslant w \text{ in } \tilde{B}_1 & \Longrightarrow \quad (-\Delta)^s \widetilde{w} \leqslant (-\Delta)^s w - c\,\mathrm{Ex}(u) \quad \text{in } D_1 \\ u \leqslant w \text{ in } \tilde{B}_1 & \Longrightarrow \quad (-\Delta)^s \widetilde{w} \geqslant (-\Delta)^s w + c\,\mathrm{Ex}(u) \quad \text{in } D_1 \end{cases} \tag{1.5}$$

for some $c = c(N, p, s) > 0$.

Our strategy for proving, e.g., (1.3) can then be roughly described as follows:

(i) Build a one parameter family of basic barrier w_λ ($\lambda \in \mathbb{R}$) obeying the bounds

$$\begin{cases} |(-\Delta)_p^s\, w_\lambda| \leqslant C\lambda & \text{in } \tilde{B}_1 \\ w_\lambda \geqslant (1+\lambda)\mathrm{d}_\Omega^s & \text{in } D_{1/4} \\ w_\lambda \leqslant \mathrm{d}_\Omega^s & \text{in } D_1^c \end{cases} \tag{1.6}$$

(ii) Choose $\lambda \simeq \mathrm{Ex}(u)$ so that the nonlocal superposition principle (1.5) ensures

$$(-\Delta)_p^s\, \widetilde{w}_\lambda \leqslant 0 \leqslant (-\Delta)_p^s\, u \quad \text{in } D_1.$$

and thanks to the global control $w_\lambda \leqslant u$ in D_1^c, deduce that $\widetilde{w}_\lambda$ is an actual lower barrier for u. Thus, by comparison, $w = w_\lambda \leqslant u$ in $D_{1/4}$.

(iii) Conclude from the second condition in (1.6) that

$$\frac{u}{\mathrm{d}_\Omega^s} - 1 \geqslant \frac{w}{\mathrm{d}_\Omega^s} - 1 \geqslant \lambda \simeq \mathrm{Ex}(u) \quad \text{in } D_{1/4}.$$

Most of the paper will thus be devoted to the construction of the family of basic barriers satisfying (1.6). As it turns out, the construction will depend on the size of $\mathrm{Ex}(u)$, and we will need three different kinds of barriers. More precisely, for small values of $\mathrm{Ex}(u)$ (and thus of λ), we will build the barrier w_λ starting from d_Ω^s (which in the case of a half-space obeys $(-\Delta)_p^s\, \mathrm{d}_\Omega^s = 0$) and performing a $C^{1,1}$-small diffeomorphism of the domain supported in D_1, to get the first condition in (1.6). A similar construction yields the upper barrier to prove (1.4) in the case of small excess. For large values of $\mathrm{Ex}(u)$, the lower barrier will be a multiple (of order $\simeq \lambda$) of the torsion function

$$\begin{cases} (-\Delta)_p^s\, v = 1 & \text{in } D_{1/2} \\ v = 0 & \text{in } D_{1/2}^c, \end{cases}$$

which, thanks to a Hopf type lemma and the size of $\mathrm{Ex}(u) \simeq \lambda$, fulfills the second bound in (1.6). Unfortunately, when we are looking for the corresponding basic *upper* barrier w_λ for large $\mathrm{Ex}(u) \simeq \lambda$, namely

$$\begin{cases} |(-\Delta)_p^s\, w_\lambda| \leqslant C\lambda & \text{in } \tilde{B}_1 \\ w_\lambda \leqslant (1-\lambda)\mathrm{d}_\Omega^s & \text{in } D_{1/4} \\ w_\lambda \geqslant \mathrm{d}_\Omega^s & \text{in } D_1^c \end{cases}$$

(in order to prove the weak Harnack inequality for subsolutions (1.4)), the previous construction fails. Indeed, when $\lambda > 1$, $w_\lambda/\mathrm{d}_\Omega^s$ must change sign near $\partial\Omega \cap (D_1 \setminus D_{1/4})$ and, even in the case of a half-space, we lack explicit examples of functions with bounded $(-\Delta)_p^s$ having such behaviour. To get around this difficulty we employ an abstract construction chiefly based on the Lewy-Stampacchia inequality, building an upper barrier which solves a double obstacle problem. This ensures that, for large excess, the solution u is nonpositive in $D_{1/2}$, and now the torsion function argument applies providing the desired bounds.

Finally, we localize (1.3) and (1.4), requiring the pointwise bounds to hold only in D_2. This is done by looking at the truncations of u below or above d_Ω^s and, due to the nonlocality of the operator, it produces additional non-homogeneous terms (usually called *tails* in the literature) which in the case $p \geqslant 2$ are quite delicate to care of (see Remark 2.8 in this respect). Having the local version of the weak Harnack inequality finally gives the desired Hölder continuity through well known techniques, originally developed in [28] for the linear case.

Notation. Throughout the paper, dependence on N, p, s will often be omitted. Positive constants will be denoted by $C_1, C_2, \dots$ When measurable functions are involved, the expression 'in Ω' will always mean 'a.e. in Ω' (and similar). We will regularly set $a^{p-1} = |a|^{p-2}a$ for all $a \in \mathbb{R}$. The positive order cone of a function space X is denoted X_+. For all function f, we denote by f_+ its positive part. Functions defined in a domain $U \subset \mathbb{R}^N$ will be identified with their extensions to $\mathbb{R}^N$ vanishing in U^c. The minimum (resp. maximum) of two functions f, g is denoted by $f \wedge g$ (resp. $f \vee g$). Though our main theorem is only proved for $p \geqslant 2$, all the intermediate results will, unless otherwise stated, hold for any $p > 1$.

2. Preliminaries

As we said in Section 1, $\Omega \subset \mathbb{R}^N$ will always be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. For all $x \in \mathbb{R}^N$ and $R > 0$ we set

$$B_R(x) = \{y \in \mathbb{R}^N : |x-y| < R\}, \quad D_R(x) = B_R(x) \cap \Omega$$

(we omit the x-dependence if $x = 0$, i.e., we set $B_R(0) = B_R$, $D_R(0) = D_R$). We define a distance function by setting for all $x \in \mathbb{R}^N$

$$\mathrm{d}_\Omega(x) = \inf_{y \in \Omega^c} |x-y|.$$

Clearly $\mathrm{d}_\Omega : \mathbb{R}^N \to \mathbb{R}_+$ is 1-Lipschitz continuous. By the $C^{1,1}$-regularity of $\partial\Omega$, Ω has the *interior sphere property*, namely there exists $R > 0$ s.t. for all $x \in \partial\Omega$ we can find $y \in \Omega$ s.t. $B_{2R}(y) \subseteq \Omega$ is tangent to $\partial\Omega$ at x (in some results this weaker property alone will suffice). We denote by $\rho > 0$ the supremum of such R's, i.e.

$$\rho = \rho(\Omega) = \sup\{R : \forall x \in \partial\Omega\ \exists B_{2R} \subseteq \Omega \text{ s.t. } x \in \partial B_{2R}\} > 0 \tag{2.1}$$

and define the neighborhood of $\partial\Omega$ by setting

$$\Omega_\rho = \{x \in \Omega : \mathrm{d}_\Omega(x) < \rho\}.$$

By the choice of ρ, the metric projection $\Pi_\Omega : \Omega_\rho \to \partial\Omega$ is well defined and is $C^{1,1}$ if $\partial\Omega$ is $C^{1,1}$. Moreover (see figure 1), for all $x \in \partial\Omega$ and $R \in]0, \rho[$ there exists a ball $\tilde{B}_{x,R}$ of radius $R/4$ s.t.

$$\tilde{B}_{x,R} \subset D_{2R}(x) \setminus D_{3R/2}(x), \qquad \inf_{y \in \tilde{B}_{x,R}} \mathrm{d}_\Omega(y) \geqslant \frac{3R}{2}. \tag{2.2}$$

We recall now the definitions of the main function spaces that we shall use in this paper. For all measurable $u : \mathbb{R}^N \to \mathbb{R}$ we set

$$[u]_{s,p}^p = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy,$$

and we define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

which is a Banach space with respect to the norm $\|u\|_{s,p} = [u]_{s,p} + \|u\|_{L^p(\mathbb{R}^N)}$, with $C_c^\infty(\mathbb{R}^N)$ as a dense subspace. We also set

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \quad \text{in } \Omega^c\}$$

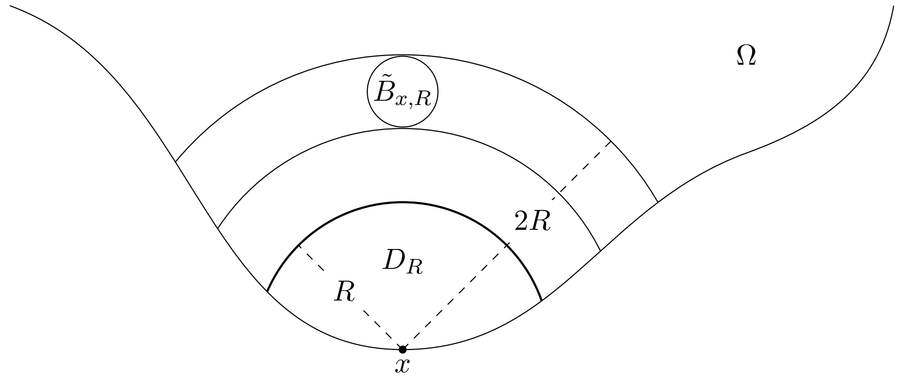

FIGURE 1. The ball $\tilde{B}_{x,R}$, with center in the normal direction.

(equivalent to the definition given in Section 1), the latter being a uniformly convex, separable Banach space with the norm $[u]_{s,p}$. The dual space of $W^{s,p}_0(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$. We will also use the following function space:

$$\widetilde{W}^{s,p}(\Omega) = \Big\{u \in L^p_{\rm loc}(\mathbb{R}^N) : \ \exists\, \Omega' \Supset \Omega \text{ s.t. } u \in W^{s,p}(\Omega') \text{ and } \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx < \infty\Big\}.$$

Such space plays an important rôle in the study of our problem, since by [14, Lemma 2.3] for all $u \in \widetilde{W}^{s,p}(\Omega)$ we have $(-\Delta)^s_p\, u \in W^{-s,p'}(\Omega)$. We also set, for any open subset $U \subset \Omega$,

$$\widetilde{W}^{s,p}_0(U) = \big\{u \in \widetilde{W}^{s,p}(U) : \ u(x) = 0 \text{ in } \Omega^c\big\}$$

(note that u does not necessarily vanish in all of U^c). We define a notion of *nonlocal tail* (slightly different from that introduced in [6]) by setting for all measurable $u : \mathbb{R}^N \to \mathbb{R}$, $R > 0$, and $q \geqslant 1$

$$\mathrm{tail}_q(u,R) = \Big[\int_{\Omega \cap B_R^c} \frac{|u(x)|^q}{|x|^{N+s}}\,dx\Big]^{\frac{1}{q}}. \tag{2.3}$$

All equations and inequalities involving $(-\Delta)^s_p$ are meant in the *weak* sense, unless explicitly stated: e.g., for any $u \in W^{s,p}_0(\Omega)$ and $f \in L^\infty(\Omega)$, we say that $(-\Delta)^s_p\, u = f$ in Ω, if

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy = \int_\Omega f(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in W^{s,p}_0(\Omega).$$

Similarly, we say that $(-\Delta)^s_p\, u \leqslant f$ in Ω if for all $\varphi \in W^{s,p}_0(\Omega)_+$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \leqslant \int_\Omega f(x)\varphi(x)\,dx.$$

We will also use the space of α-Hölder continuous functions

$$C^\alpha(\overline{\Omega}) = \big\{u \in C(\overline{\Omega}) : \ [u]_{C^\alpha(\overline{\Omega})} < \infty\big\},$$

where

$$[u]_{C^\alpha(\overline{\Omega})} = \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

which is a Banach space endowed with the norm

$$\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^\alpha(\overline{\Omega})}.$$

In the rest of the section we will list some useful technical results on solutions to (1.1) type problems on several domains: *for simplicity, we always denote the domain by Ω, but in the forthcoming sections these results will also be applied to different domains.*

We begin with the following weak comparison principle (see [24, Lemma 9], [14, Proposition 2.10]):

Proposition 2.1. (Comparison principle) *Let $u, v \in \widetilde{W}^{s,p}(\Omega)$ satisfy*

$$\begin{cases} (-\Delta)_p^s\, u \leqslant (-\Delta)_p^s\, v & \text{in } \Omega \\ u \leqslant v & \text{in } \Omega^c. \end{cases}$$

Then $u \leqslant v$ in $\mathbb{R}^N$.

Our first result is a simple estimate on the solution to the torsion equation in a ball: for all $R > 0$, we denote by $u_R \in W_0^{s,p}(B_R)$ the (unique) solution to

$$\begin{cases} (-\Delta)_p^s\, u_R = 1 & \text{in } B_R \\ u_R = 0 & \text{in } B_R^c. \end{cases} \tag{2.4}$$

Lemma 2.2. *There exists $C_1 = C_1(N, p, s) > 1$ s.t. for all $R > 0$, $x \in \mathbb{R}^N$*

$$\frac{R^{\frac{s}{p-1}}}{C_1} \mathrm{d}_{B_R}^s(x) \leqslant u_R(x) \leqslant C_1 R^{\frac{s}{p-1}} \mathrm{d}_{B_R}^s(x).$$

Proof. First assume $R = 1$. By the strong maximum principle (see [25, Lemma 2.3]), we have $u_1 > 0$ in B_1, while by [14, Theorem 4.4] there exists $C_1 > 0$ s.t.

$$u_1 \leqslant C_1 \mathrm{d}_{B_1}^s \quad \text{in } \mathbb{R}^N. \tag{2.5}$$

By [14, Theorem 3.6] we can find $r \in]0, 1[$, $M > 0$ s.t. $|(-\Delta)_p^s\, \mathrm{d}_{B_1}^s| \leqslant M$ in $B_1 \setminus \overline{B}_r$. Set $m = \inf_{B_r} u_1 > 0$ and for all $x \in \mathbb{R}^N$

$$w(x) = \min\big\{m, M^{-\frac{1}{p-1}}\big\} \mathrm{d}_{B_1}^s(x).$$

Then we have

$$\begin{cases} (-\Delta)_p^s\, w \leqslant (-\Delta)_p^s\, u_1 & \text{in } B_1 \setminus \overline{B}_r \\ w \leqslant u_1 & \text{in } (B_1 \setminus \overline{B}_r)^c. \end{cases}$$

Proposition 2.1 yields $w \leqslant u_1$ in $\mathbb{R}^N$. So, for C_1 even bigger if necessary in (2.5), we improve to

$$\frac{\mathrm{d}_{B_1}^s}{C_1} \leqslant u_1 \leqslant C_1 \mathrm{d}_{B_1}^s \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

Now take an arbitrary $R > 0$ and set for all $x \in \mathbb{R}^N$

$$v(x) = \frac{u_R(Rx)}{R^{p's}}.$$

Then $v \in W_0^{s,p}(B_1)$ and by the homogeneity and scaling properties of $(-\Delta)_p^s$ (see [14, Proposition 2.9 (i) (ii)]) we have

$$\begin{cases} (-\Delta)_p^s\, v = 1 & \text{in } B_1 \\ v = 0 & \text{in } B_1^c. \end{cases}$$

By uniqueness $v = u_1$. Since $\mathrm{d}_{B_R}(Rx) = R\mathrm{d}_{B_1}(x)$, by (2.6) we have for all $x \in \mathbb{R}^N$

$$\frac{\mathrm{d}_{B_R}^s(Rx)}{C_1 R^s} \leqslant \frac{u_R(Rx)}{R^{p's}} \leqslant \frac{C_1 \mathrm{d}_{B_R}^s(Rx)}{R^s},$$

hence the conclusion. $\square$

The previous estimate allows us to use u_R as a barrier to prove a Hopf type lemma for the torsion equation in a general domain:

$$\begin{cases} (-\Delta)_p^s\, u = 1 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.7}$$

Lemma 2.3. (Hopf's lemma) *Let $u \in W_0^{s,p}(\Omega)$ solve (2.7) and Ω satisfy the interior sphere property (2.1). Then*

$$u(x) \geqslant \frac{1}{C_1}\rho^{\frac{s}{p-1}}\mathrm{d}_\Omega^s(x) \qquad \text{for all } x \in \mathbb{R}^N,$$

where $C_1 = C_1(N,p,s) > 1$ is given in the previous Lemma.

Proof. First, fix $x \in \Omega_\rho$. Then we can find a ball $B \subseteq \Omega$ of radius 2ρ, tangent to $\partial\Omega$ at $\Pi_\Omega(x)$ and s.t. $\mathrm{d}_\Omega(x) = \mathrm{d}_B(x)$. Let $v \in W_0^{s,p}(B)$ solve

$$\begin{cases} (-\Delta)_p^s\, v = 1 & \text{in } B \\ v = 0 & \text{in } B^c. \end{cases}$$

So we have

$$\begin{cases} (-\Delta)_p^s\, v \leqslant (-\Delta)_p^s\, u & \text{in } B \\ v \leqslant u & \text{in } B^c. \end{cases}$$

By Proposition 2.1 we have $v \leqslant u$ in $\mathbb{R}^N$. By Lemma 2.2 and $\mathrm{d}_\Omega(x) = \mathrm{d}_B(x)$, we infer

$$u(x) \geqslant v(x) \geqslant \frac{(2\rho)^{\frac{s}{p-1}}}{C_1}\mathrm{d}_\Omega^s(x). \tag{2.8}$$

Now assume $x \in \Omega \setminus \overline{\Omega}_\rho$, and set $R = \mathrm{d}_\Omega(x) \geqslant \rho$. The ball $B' = B_R(x)$ is contained in Ω and $\mathrm{d}_{B'}(x) = R = \mathrm{d}_\Omega(x)$. Considering the torsion function v' of B' and applying Proposition 2.1, we deduce through Lemma 2.2

$$u(x) \geqslant v'(x) \geqslant \frac{R^{\frac{s}{p-1}}}{C_1}\mathrm{d}_{B'}^s(x) = \frac{R^{\frac{s}{p-1}}}{C_1}R^s \geqslant \frac{\rho^{\frac{s}{p-1}}}{C_1}\mathrm{d}_\Omega^s(x). \tag{2.9}$$

From (2.8) and (2.9) we conclude. □

Another property of problem (2.7) is that its solution is a subsolution all over $\mathbb{R}^N$:

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded and $u \in W_0^{s,p}(\Omega)$ solve (2.7). Then $(-\Delta)_p^s\, u \leqslant 1$ in $\mathbb{R}^N$.*

Proof. Set for all $v \in W^{s,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$

$$J_1(v) = \frac{[v]_{s,p}^p}{p} - \int_{\mathbb{R}^N} v(x)\,dx,$$

(achieving its minimum on $W_0^{s,p}(\Omega)$ at u) and consider the constrained minimization problem

$$\inf_{v \leqslant u} J_1(v). \tag{2.10}$$

For any v it holds $J_1(v_+) \leqslant J_1(v)$. Moreover, $v \leqslant u$ implies $v_+ \in W_0^{s,p}(\Omega)$, hence

$$J_1(u) \geqslant \inf_{v \leqslant u} J_1(v) = \inf_{v \leqslant u} J_1(v_+) \geqslant \inf_{W_0^{s,p}(\Omega)} J_1 = J_1(u).$$

Thus u solves (2.10) as well. The variational inequality associated to (2.10) reads

$$\langle J_1'(u), v - u\rangle \geqslant 0 \quad \text{for all } v \in W^{s,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),\ v \leqslant u$$

and setting $v = u - \varphi$, we get

$$\langle J_1'(u), \varphi\rangle \leqslant 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N),\ \varphi \geqslant 0,$$

i.e., $(-\Delta)_p^s\, u \leqslant 1$ in all of $\mathbb{R}^N$, concluding the proof. □

We introduce a partial ordering on the dual space $W^{-s,p'}(\Omega)$ by defining the positive cone

$$W^{-s,p'}(\Omega)_+ = \{L \in W^{-s,p'}(\Omega) : \langle L, \varphi\rangle \geqslant 0 \text{ for all } \varphi \in W^{s,p}_0(\Omega)_+\}.$$

By the Riesz theorem and the density of $C^\infty_c(\Omega)$ in $W^{s,p}_0(\Omega)$, any $L \in W^{-s,p'}(\Omega)_+$ can be faithfully represented as a (positive) Radon measure on Ω (see the discussion in [11, p. 265]). Then, the *order dual* of w is defined as

$$W^{-s,p'}_{\leqslant}(\Omega) = \{L_1 - L_2 : L_1, L_2 \in W^{-s,p'}(\Omega)_+\}.$$

Such space inherits a lattice structure defined by duality through the lattice structure of $W^{s,p}_0(\Omega)$, as shown in [11, p. 260]. We now give a slight generalization of the Lewy-Stampacchia type inequality [11, Theorem 2.4] which is needed to treat double obstacle problems with obstacle not lying in $W^{s,p}_0(\Omega)$. The proof is well known and we describe it for sake of completeness, specializing to the case of the operator $(-\Delta)^s_p$.

Lemma 2.5. (Lewy-Stampacchia) *Let $\Omega \subseteq \mathbb{R}^N$ be bounded, $\varphi, \psi \in W^{s,p}_{\rm loc}(\mathbb{R}^N)$ be s.t.*

(i) $(-\Delta)^s_p\,\varphi, (-\Delta)^s_p\,\psi \in W^{-s,p'}_{\leqslant}(\Omega)$
(ii) $[\varphi, \psi] := \{v \in W^{s,p}_0(\Omega) : \varphi \leqslant v \leqslant \psi\} \neq \emptyset$

Then there exists a unique solution $u \in W^{s,p}_0(\Omega)$ to the problem

$$\min_{v\in[\varphi,\psi]} \frac{[v]^p_{s,p}}{p},$$

and it satisfies

$$0 \wedge (-\Delta)^s_p\,\psi \leqslant (-\Delta)^s_p\,u \leqslant 0 \vee (-\Delta)^s_p\,\varphi \quad \text{in } \Omega.$$

Proof. The existence and uniqueness statements for the minimization problem follow from the strict convexity and coercivity of $v \mapsto [v]^p_{s,p}$. The function $u \in [\varphi, \psi]$ is a minimizer iff it satisfies for all $v \in [\varphi, \psi]$

$$\langle (-\Delta)^s_p\,u, v - u\rangle \geqslant 0. \tag{2.11}$$

We prove now that

$$(-\Delta)^s_p\,u \leqslant 0 \vee (-\Delta)^s_p\,\varphi \quad \text{in } \Omega. \tag{2.12}$$

Recall from [11, Remark 3.3 and p. 261] that $v \mapsto [v]^p_{s,p}/p$ is sub-modular and strictly convex, hence its differential $(-\Delta)^s_p$ is a strictly $\mathcal{T}$-monotone map, i.e.

$$\langle (-\Delta)^s_p\,u - (-\Delta)^s_p\,v, (u - v)_+\rangle > 0 \quad \text{unless } v \leqslant u. \tag{2.13}$$

By condition (i), the strictly convex, coercive functional

$$J_2 : W^{s,p}_0(\Omega) \to \mathbb{R}, \quad J_2(v) = \frac{[v]^p_{s,p}}{p} - \langle (-\Delta)^s_p\,\varphi \vee 0, v\rangle,$$

is well defined, and we thus let w be the unique solution of the following problem

$$\min_{v\in\,(\infty,u]} J_2(v), \quad]-\infty, u] := \{v \in W^{s,p}_0(\Omega) : v \leqslant u\},$$

which therefore solves for all $v \in]-\infty, u]$

$$\langle J_2'(w), v - w\rangle \geqslant 0. \tag{2.14}$$

We claim that $u \geqslant w$, and then necessarily $u = w$. Condition (ii) forces $\varphi \leqslant 0$ in Ω^c, therefore $w \vee \varphi \in W^{s,p}_0(\Omega)$. Choosing $v = w \vee \varphi = w + (\varphi - w)_+$ gives

$$0 \leqslant \langle J_2'(w), (\varphi - w)_+\rangle = \langle (-\Delta)^s_p\,w - (0 \vee (-\Delta)^s_p\,\varphi), (\varphi - w)_+\rangle \leqslant \langle (-\Delta)^s_p\,w - (-\Delta)^s_p\,\varphi, (\varphi - w)_+\rangle.$$

By (2.13), this implies $\varphi \leqslant w$ and, by $w \leqslant u$, *a fortiori* $w \in [\varphi, \psi]$. Choosing $v = w \vee u = w + (u-w)_+$ as a test function in (2.14) gives

$$0 \leqslant \langle J_2'(w), (u-w)_+\rangle = \langle (-\Delta)_p^s\, w - (0 \vee (-\Delta)_p^s\, \varphi), (u-w)_+\rangle \leqslant \langle (-\Delta)_p^s\, w, (u-w)_+\rangle,$$

while letting $v = w \wedge u = u - (u-w)_+$ in (2.11), provides

$$0 \leqslant \langle (-\Delta)_p^s\, u, -(u-w)_+\rangle.$$

Summing up we obtain

$$0 \leqslant \langle (-\Delta)_p^s\, w - (-\Delta)_p^s\, u, (u-w)_+\rangle,$$

thus (2.13) entails $u \leqslant w$ and therefore $w = u$. This enforces (2.14) for u, then for all $z \in W_0^{s,p}(\Omega)_+$, setting $v = u - z \in]-\infty, u]$ we get

$$\langle (-\Delta)_p^s\, u, z\rangle \leqslant \langle (-\Delta)_p^s\, \varphi \vee 0, z\rangle,$$

proving (2.12). The first inequality of the thesis is achieved through a similar argument. □

A major tool in our proofs is the following nonlocal superposition principle:

Proposition 2.6. (Superposition principle) *Let Ω be bounded, $u \in \widetilde{W}^{s,p}(\Omega)$, $v \in L^1_{\rm loc}(\mathbb{R}^N)$, $V = \mathrm{supp}(u-v)$ satisfy*

(i) $\Omega \Subset \mathbb{R}^N \setminus V$;

(ii) $\displaystyle\int_V \frac{|v(x)|^{p-1}}{(1+|x|)^{N+ps}}\, dx < \infty.$

Set for all $x \in \mathbb{R}^N$

$$w(x) = \begin{cases} u(x) & \text{if } x \in V^c \\ v(x) & \text{if } x \in V. \end{cases}$$

Then $w \in \widetilde{W}^{s,p}(\Omega)$ and satisfies in Ω

$$(-\Delta)_p^s\, w(x) = (-\Delta)_p^s\, u(x) + 2\int_V \frac{(u(x)-v(y))^{p-1} - (u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\, dy.$$

Proof. We can rephrase $w = u + (v-u)\chi_V$, which implies $w \in \widetilde{W}^{s,p}(\Omega)$. By [14, Lemmas 2.3, 2.8] we have $(-\Delta)_p^s\, w \in W^{-s,p'}(\Omega)$, moreover for all $\varphi \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, w, \varphi\rangle = \langle (-\Delta)_p^s\, u, \varphi\rangle + \int_\Omega h(x)\varphi(x)\, dx,$$

where for all Lebesgue point $x \in V$ of u we have set

$$h(x) = 2\int_V \frac{(u(x)-v(y))^{p-1} - (u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\, dy.$$

This concludes the proof. □

We conclude this section with a key estimate for a function which is *locally* bounded by a suitable multiple of d_Ω^s (here we first require that $p \geqslant 2$). The passage from a global bound to a local bound can be delicate for a nonlocal operator such as $(-\Delta)_p^s$. While technical, the next proposition shows the main reason why the degeneracy of the operator forces, in the following sections, a peculiar decomposition of the right hand side (see Remark 2.8 below).

Proposition 2.7. *Let Ω be bounded, $p \geqslant 2$ and $u \in \widetilde{W}_0^{s,p}(D_R)$ satisfy $(-\Delta)_p^s\, u \in W_{\leqslant}^{-s,p'}(D_R)$:*

(*i*) *if there exists* $m \in \mathbb{R}$ *s.t.* $u \geqslant m\mathrm{d}_\Omega^s$ *in* D_{2R}*, then for all* $\varepsilon > 0$ *there exist* $C_\varepsilon = C_\varepsilon(N,p,s,\varepsilon) > 0$ *and another constant* $C_2 = C_2(N,p,s) > 0$ *s.t. in* D_R

$$(-\Delta)_p^s\left(u \vee m\mathrm{d}_\Omega^s\right) \geqslant (-\Delta)_p^s\, u - \frac{\varepsilon}{R^s}\Big\|\frac{u}{\mathrm{d}_\Omega^s} - m\Big\|_{L^\infty(D_R)}^{p-1} - C_\varepsilon \mathrm{tail}_{p-1}\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)^{p-1} - C_2|m|^{p-2}\mathrm{tail}_1\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big);$$

(*ii*) *if there exists* $M \in \mathbb{R}$ *s.t.* $u \leqslant M\mathrm{d}_\Omega^s$ *in* D_{2R}*, then for all* $\varepsilon > 0$ *there exist* $C'_\varepsilon = C'_\varepsilon(N,p,s,\varepsilon) > 0$ *and another constant* $C'_2 = C'_2(N,p,s) > 0$ *s.t. in* D_R

$$(-\Delta)_p^s\left(u \wedge M\mathrm{d}_\Omega^s\right) \leqslant (-\Delta)_p^s\, u + \frac{\varepsilon}{R^s}\Big\|M - \frac{u}{\mathrm{d}_\Omega^s}\Big\|_{L^\infty(D_R)}^{p-1} + C'_\varepsilon \mathrm{tail}_{p-1}\Big(\Big(\frac{u}{\mathrm{d}_\Omega^s} - M\Big)_+, 2R\Big)^{p-1} + C'_2|M|^{p-2}\mathrm{tail}_1\Big(\Big(\frac{u}{\mathrm{d}_\Omega^s} - M\Big)_+, 2R\Big).$$

Proof. We prove (*i*). We may assume $u/\mathrm{d}_\Omega^s - m \in L^\infty(D_R)$, otherwise there is nothing to prove. We will use the following elementary inequality: since $p \geqslant 2$, there exists $C_p > 0$ s.t.

$$(a-b)^{p-1} - (c-b)^{p-1} \leqslant C_p(|a|^{p-2} + |b|^{p-2})|a-c| + C_p|a-c|^{p-1} \qquad \text{for all } a,b,c \in \mathbb{R}. \tag{2.15}$$

Indeed, by Lagrange's theorem and convexity, we have

$$\begin{aligned}(a-b)^{p-1} - (c-b)^{p-1} &\leqslant C_p(|a|^{p-2} + |b|^{p-2} + |c|^{p-2})|a-c| \\ &\leqslant C_p\big(|a|^{p-2} + |b|^{p-2} + C'_p(|c-a|^{p-2} + |a|^{p-2})\big)|a-c|,\end{aligned}$$

which implies (2.15). Set $w = u \vee m\mathrm{d}_\Omega^s$. Since $\{u < m\mathrm{d}_\Omega^s\} \subseteq D_{2R}^c$ is bounded away from D_R, we can apply Proposition 2.6 and get for all $x \in D_R$

$$\begin{aligned}(-\Delta)_p^s\, w(x) &= (-\Delta)_p^s\, u(x) + 2\int_{\{u<m\mathrm{d}_\Omega^s\}} \frac{(u(x) - m\mathrm{d}_\Omega^s(y))^{p-1} - (u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy \\ &= (-\Delta)_p^s\, u(x) - 2\int_{\{u<m\mathrm{d}_\Omega^s\}} \frac{(m\mathrm{d}_\Omega^s(y) - u(x))^{p-1} - (u(y)-u(x))^{p-1}}{|x-y|^{N+ps}}\,dy.\end{aligned} \tag{2.16}$$

We use (2.15) to estimate the numerator of the integrand, recalling also that $\mathrm{d}_\Omega(x) \leqslant R$, $u(x) \geqslant m\mathrm{d}_\Omega^s(x)$, $R < \mathrm{d}_\Omega(y) \leqslant |y|$, and $u(y) < m\mathrm{d}_\Omega^s(y)$:

$$\begin{aligned}&(m\mathrm{d}_\Omega^s(y) - u(x))^{p-1} - (u(y)-u(x))^{p-1} \\ &\quad\leqslant C_p\big(|m\mathrm{d}_\Omega^s(y)|^{p-2} + |u(x)|^{p-2}\big)|m\mathrm{d}_\Omega^s(y) - u(y)| + C_p|m\mathrm{d}_\Omega^s(y) - u(y)|^{p-1} \\ &\quad\leqslant C_p\big(|m|^{p-2}\mathrm{d}_\Omega^{(p-2)s}(y) + |m|^{p-2}R^{(p-2)s} + (u(x) - m\mathrm{d}_\Omega^s(x))^{p-2}\big)(m\mathrm{d}_\Omega^s(y) - u(y)) \\ &\qquad + C_p(m\mathrm{d}_\Omega^s(y) - u(y))^{p-1} \\ &\quad\leqslant C|m|^{p-2}|y|^{(p-2)s}(m\mathrm{d}_\Omega^s(y) - u(y)) + \varepsilon(u(x) - m\mathrm{d}_\Omega^s(x))^{p-1} + C_\varepsilon(m\mathrm{d}_\Omega^s(y) - u(y))^{p-1},\end{aligned}$$

where in the end we have also used Young's inequality with exponents $q = (p-1)(p-2)^{-1}$ and $q' = p-1$. Here $C > 0$ depends only on N, p, s, while $C_\varepsilon > 0$ also depends on $\varepsilon > 0$. Now, by means of the inequality above and the relations $|x-y| \geqslant |y|/2 \geqslant R$, we can estimate the integral in (2.16), getting

$$\begin{aligned}
&\int_{\{u<m\mathrm{d}_\Omega^s\}} \frac{(m\mathrm{d}_\Omega^s(y)-u(x))^{p-1}-(u(y)-u(x))^{p-1}}{|x-y|^{N+ps}}\,dy \\
&\quad\leqslant \varepsilon\int_{\{u<m\mathrm{d}_\Omega^s\}} \frac{(u(x)-m\mathrm{d}_\Omega^s(x))^{p-1}}{|x-y|^{N+ps}}\,dy + C_\varepsilon\int_{\{u<m\mathrm{d}_\Omega^s\}} \frac{(m\mathrm{d}_\Omega^s(y)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy \\
&\qquad + C|m|^{p-2}\int_{\{u<m\mathrm{d}_\Omega^s\}} \frac{|y|^{(p-2)s}(m\mathrm{d}_\Omega^s(y)-u(y))}{|x-y|^{N+ps}}\,dy \\
&\quad\leqslant \varepsilon\Big\|m-\frac{u}{\mathrm{d}_\Omega^s}\Big\|_{L^\infty(D_R)}^{p-1}\int_{D_{2R}^c}\frac{R^{(p-2)s}}{|y|^{N+ps}}\,dy + C_\varepsilon\int_{D_{2R}^c}\Big(m-\frac{u(y)}{\mathrm{d}_\Omega^s(y)}\Big)_+^{p-1}\frac{dy}{|y|^{N+s}} \\
&\qquad + C|m|^{p-2}\int_{D_{2R}^c}\Big(m-\frac{u(y)}{\mathrm{d}_\Omega^s(y)}\Big)_+\frac{dy}{|y|^{N+s}} \\
&\quad\leqslant \frac{\varepsilon}{R^s}\Big\|\frac{u}{\mathrm{d}_\Omega^s}-m\Big\|_{L^\infty(D_R)}^{p-1} + C_\varepsilon \mathrm{tail}_{p-1}\Big(\Big(m-\frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)^{p-1} + C_2\mathrm{tail}_1\Big(\Big(m-\frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big),
\end{aligned}$$

where we may take, if necessary, $\varepsilon>0$ even smaller and $C_\varepsilon>0$ even bigger, plus some $C_2(N,p,s)$. Plugging the last inequality into (2.16) (and replacing ε with $\varepsilon/2$), we achieve *(i)*.

The argument for *(ii)* is immediate, by replacing u with $-u$ and m with $-M$. □

Remark 2.8. Before going further, a short discussion is in order. Proposition 2.7 provides bounds of the fractional p-Laplacians of truncated functions, which involve *two* tail terms with different exponents, namely tail_{p-1} and tail_1. One of the main issues in the forthcoming sections will be to estimate inductively such tail terms, taking into account that they behave differently when $R\to 0^+$, with tail_1 being asymptotically larger than tail_{p-1}. In adjusting those estimates, the quantities $|m|^{p-2}$, $|M|^{p-2}$ multiplying the term tail_1 in *(i)*, *(ii)* respectively, will play a fundamental rôle. That is why we will emphasize the m-dependence of the right hand side for supersolutions (respectively, its M-dependence for subsolutions). Precisely, we shall prove a lower bound for a function u satisfying

$$\begin{cases} (-\Delta)_p^s\, u \geqslant -K - m^{p-2}H & \text{in } D_R \\ u \geqslant m\mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N, \end{cases}$$

and an upper bound for a function u satisfying

$$\begin{cases} (-\Delta)_p^s\, u \leqslant K + M^{p-2}H & \text{in } D_R \\ u \leqslant M\mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N, \end{cases}$$

respectively, with convenient $K,H,m,M\geqslant 0$. As we will see, the upper and lower bounds require substantially different approaches.

3. The lower bound

This section is devoted to the study of supersolutions of (1.1) type problems on special domains, locally bounded from below by a multiple of d_Ω^s. For such supersolutions we aim at proving a lower bound for the quotient u/d_Ω^s near the boundary (see Proposition 3.7 below).

First, we assume that the supersolution u is *globally* bounded from below by $m\mathrm{d}_\Omega^s$ and rephrase the lower bound on $(-\Delta)_p^s\, u$ as $-K-m^{p-2}H$. Precisely, we assume $p\geqslant 2$, $0\in\partial\Omega$ (for simplicity of notation), $R\in\,]0,\rho/4[$, and consider $u\in\widetilde{W}^{s,p}(D_R)$ satisfying for some $K,H,m\geqslant 0$

$$\begin{cases} (-\Delta)_p^s\, u \geqslant -K - m^{p-2}H & \text{in } D_R \\ u \geqslant m\mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

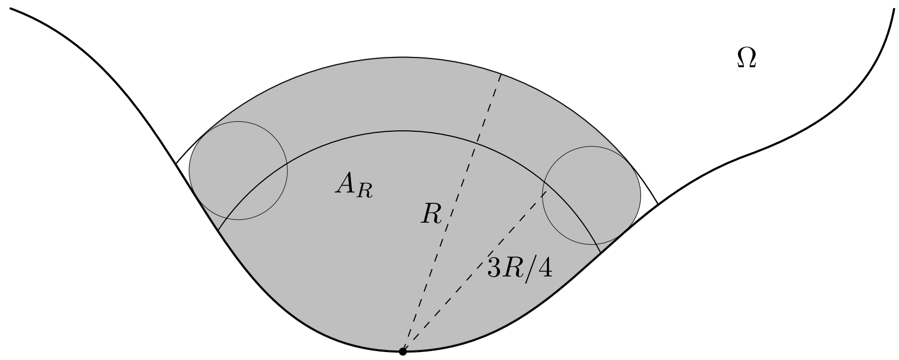

FIGURE 2. The regularized set A_R in gray.

A major rôle in determining the behavior of u/d_Ω^s in a semi-disc $D_R(x_0)$ is played by the following nonlocal excess

$$\mathrm{Ex}(u,k,R,x_0) = \fint_{\tilde{B}_{x_0,R}} \left| \frac{u(x)}{\mathrm{d}_\Omega^s(x)} - k \right| dx, \tag{3.2}$$

where $\tilde{B}_{x_0,R}$ is defined as in (2.2). As we will frequently assume $x_0 = 0$, the dependence on the latter will be omitted. We begin by proving a lower bound for the case of large values of the excess, which highlights the nonlocal feature of the equation.

Lemma 3.1. *Let $u \in \widetilde{W}^{s,p}(D_R)$ solve (3.1), $p \geqslant 2$ and Ω satisfy (2.1). Then there exist $\theta_1 = \theta_1(N,p,s) \geqslant 1$, $C_3 = C_3(N,p,s) > 1$, $\sigma_1 = \sigma_1(N,p,s) \in]0,1]$ s.t. for all $R \in]0, \rho/4[$*

$$\mathrm{Ex}(u,m,R) \geqslant m\theta_1 \implies \inf_{D_{R/2}} \left(\frac{u}{\mathrm{d}_\Omega^s} - m \right) \geqslant \sigma_1 \mathrm{Ex}(u,m,R) - C_3 (KR^s)^{\frac{1}{p-1}} - C_3 H R^s.$$

Proof. Set

$$A_R = \bigcup \left\{ B_r(y) : y \in \mathbb{R}^N,\, r \geqslant \frac{R}{8},\, B_r(y) \subset D_R \right\}.$$

By the regularity of $\partial\Omega$ stated in (2.1) and $R < \rho/4$, $A_R \subset \mathbb{R}^N$ is a bounded domain satisfying the interior sphere condition with radius $\rho_{A_R} \geqslant R/16$ (see figure 2). Moreover we claim that

$$\mathrm{d}_\Omega \leqslant C \mathrm{d}_{A_R} \quad \text{in } D_{R/2}. \tag{3.3}$$

First note that $D_{3R/4} \subseteq A_R$ implies $\mathrm{d}_{D_{3R/4}} \leqslant \mathrm{d}_{A_R}$ in $\mathbb{R}^N$. Furthermore, for all $x \in D_{R/2}$ we have

$$\mathrm{d}_\Omega(x) \leqslant |x - \Pi_\Omega(\Pi_{D_{3R/4}}(x))| \leqslant |x - \Pi_{D_{3R/4}}(x)| + |\Pi_{D_{3R/4}}(x) - \Pi_\Omega(\Pi_{D_{3R/4}}(x))|.$$

To proceed, we distinguish two cases:

(*a*) if $\Pi_{D_{3R/4}}(x) \in \partial\Omega$, then $\Pi_\Omega(\Pi_{D_{3R/4}}(x)) = \Pi_{D_{3R/4}}(x)$ and so

$$\mathrm{d}_\Omega(x) \leqslant \mathrm{d}_{D_{3R/4}}(x) \leqslant \mathrm{d}_{A_R}(x);$$

(*b*) if $\Pi_{D_{3R/4}}(x) \notin \partial\Omega$, then we have $|\Pi_{D_{3R/4}}(x)|$, $|\Pi_\Omega(\Pi_{D_{3R/4}}(x))| \leqslant R$ and $\mathrm{d}_{D_{3R/4}}(x) \geqslant R/4$, which in turn implies

$$\mathrm{d}_\Omega(x) \leqslant \mathrm{d}_{D_{3R/4}}(x) + |\Pi_{D_{3R/4}}(x)| + |\Pi_\Omega(\Pi_{D_{3R/4}}(x))| \leqslant 9\mathrm{d}_{D_{3R/4}}(x) \leqslant 9\mathrm{d}_{A_R}(x).$$

Both cases lead to (3.3). We will also use the following elementary inequality from [14, Eq. (2.7)]: since $p \geqslant 2$, for all $a \in \mathbb{R}$, $b \geqslant 0$ we have

$$(a+b)^{p-1} - a^{p-1} \geqslant 2^{2-p} b^{p-1}. \tag{3.4}$$

Let $v \in W_0^{s,p}(A_R)$ be the solution of the torsion problem

$$\begin{cases} (-\Delta)_p^s\, v = 1 & \text{in } A_R \\ v = 0 & \text{in } A_R^c. \end{cases}$$

By Lemma 2.4 we have $(-\Delta)_p^s\, v \leqslant 1$ in $\mathbb{R}^N$. Besides, by Lemma 2.3 and (3.3) we have

$$v \geqslant \frac{R^{\frac{s}{p-1}}}{C}\mathrm{d}_\Omega^s \quad \text{in } D_{R/2}. \tag{3.5}$$

Pick $\lambda > 0$ (to be determined later) and set

$$w(x) = \begin{cases} \dfrac{\lambda}{R^{\frac{s}{p-1}}}v(x) & \text{if } x \in \tilde{B}_R^c \\ u(x) & \text{if } x \in \tilde{B}_R, \end{cases}$$

where $\tilde{B}_R$ is defined as in (2.2). We note that $\mathrm{dist}\,(\tilde{B}_R,\, D_R) > 0$, so we can apply Proposition 2.6. Also using homogeneity of $(-\Delta)_p^s$, (3.4), and the relations $\mathrm{d}_\Omega(y) < 3R/2$, $|x-y| > 3R/4$, we get for all $x \in D_R$

$$\begin{aligned} (-\Delta)_p^s\, w(x) &= (-\Delta)_p^s \Big(\frac{\lambda}{R^{\frac{s}{p-1}}}v(x)\Big) + 2\int_{\tilde{B}_R} \frac{(w(x)-u(y))^{p-1} - w^{p-1}(x)}{|x-y|^{N+ps}}\,dy \\ &\leqslant \frac{\lambda^{p-1}}{R^s} - \frac{1}{C}\int_{\tilde{B}_R} \frac{u(y)^{p-1}}{|x-y|^{N+ps}}\,dy \\ &\leqslant \frac{\lambda^{p-1}}{R^s} - \frac{1}{C}\int_{\tilde{B}_R} \frac{(u(y)-m\mathrm{d}_\Omega^s(y))^{p-1}}{|x-y|^{N+ps}}\,dy. \end{aligned}$$

Observe that by the property (2.2) of $\tilde{B}_R$

$$\begin{aligned} \int_{\tilde{B}_R} \frac{(u(y)-m\mathrm{d}_\Omega^s(y))^{p-1}}{|x-y|^{N+ps}}\,dy &\geqslant \int_{\tilde{B}_R} \Big(\frac{u(y)}{\mathrm{d}_\Omega^s(y)} - m\Big)^{p-1} \frac{\mathrm{d}_\Omega^{s(p-1)}(y)}{|x-y|^{N+ps}}\,dy \\ &\geqslant \frac{(3/2R)^{s(p-1)}}{(R/2)^{N+ps}} \int_{\tilde{B}_R} \Big(\frac{u(y)}{\mathrm{d}_\Omega^s(y)} - m\Big)^{p-1} dy, \end{aligned}$$

and thus by Hölder inequality and the fact that $u \geqslant m\mathrm{d}_\Omega^s$ in $\tilde{B}_R$,

$$(-\Delta)_p^s\, w(x) \leqslant \frac{\lambda^{p-1}}{R^s} - \frac{1}{CR^s}\fint_{\tilde{B}_R} \Big(\frac{u(y)}{\mathrm{d}_\Omega^s(y)} - m\Big)^{p-1} dy \leqslant \frac{\lambda^{p-1}}{R^s} - \frac{\mathrm{Ex}(u,m,R)^{p-1}}{CR^s}.$$

Choosing

$$\lambda = \frac{\mathrm{Ex}(u,m,R)}{(2C^2)^{\frac{1}{p-1}}}, \tag{3.6}$$

we have

$$(-\Delta)_p^s\, w \leqslant -\frac{\mathrm{Ex}(u,m,R)^{p-1}}{2CR^s} \quad \text{in } D_R. \tag{3.7}$$

Now we choose the constants, setting

$$\theta_1 = \frac{1}{\sigma_1} = 2C(2C^2)^{\frac{1}{p-1}}, \quad C_3 = \sigma_1 \max\big\{(4C)^{\frac{1}{p-1}},\, 4C\theta_1^{2-p}\big\}.$$

Clearly θ_1, $C_3 \geqslant 1 \geqslant \sigma_1 > 0$ only depend on N, p, s. Assuming

$$\mathrm{Ex}(u,m,R) \geqslant m\theta_1, \tag{3.8}$$

we claim that

$$\inf_{D_{R/2}} \Big(\frac{u}{\mathrm{d}_\Omega^s} - m\Big) \geqslant \sigma_1 \mathrm{Ex}(u,m,R) - C_3(KR^s)^{\frac{1}{p-1}} - C_3 HR^s.$$

Two cases may occur:

(a) If $\sigma_1 \mathrm{Ex}(u,m,R) \leqslant C_3(KR^s)^{\frac{1}{p-1}} + C_3HR^s$, then the claim is immediate being the left hand side non-negative.

(b) If $\sigma_1 \mathrm{Ex}(u,m,R) > C_3(KR^s)^{\frac{1}{p-1}} + C_3HR^s$, then from the definitions above and (3.8) we have

$$\mathrm{Ex}(u,m,R)^{p-1} \geqslant \begin{cases} \Big(\dfrac{C_3}{\sigma_1}\Big)^{p-1} KR^s \geqslant 4CKR^s \\ (m\theta_1)^{p-2}\mathrm{Ex}(u,m,R) \geqslant (m\theta_1)^{p-2}\dfrac{C_3}{\sigma_1}HR^s \geqslant 4Cm^{p-2}HR^s, \end{cases}$$

and by summing up

$$\mathrm{Ex}(u,m,R)^{p-1} \geqslant 2CR^s(K + m^{p-2}H).$$

Now by (3.1), (3.7), and recalling that $w = \chi_{\tilde{B}_R} u$ in D_R^c, we have

$$\begin{cases} (-\Delta)_p^s\, w \leqslant -K - m^{p-2}H \leqslant (-\Delta)_p^s\, u & \text{in } D_R \\ w \leqslant u & \text{in } D_R^c. \end{cases}$$

By Proposition 2.1 we have $w \leqslant u$ in $\mathbb{R}^N$. In particular, for all $x \in D_{R/2}$, recalling (3.5) and the definition of λ in (3.6), we have

$$u(x) \geqslant \frac{\lambda}{R^{\frac{s}{p-1}}} v(x) \geqslant \frac{\mathrm{Ex}(u,m,R)}{C(2C^2)^{\frac{1}{p-1}}} \mathrm{d}_\Omega^s(x).$$

Thus, by (3.8) again

$$\inf_{D_{R/2}} \Big(\frac{u}{\mathrm{d}_\Omega^s} - m\Big) \geqslant \mathrm{Ex}(u,m,R)\Big(\frac{1}{C(2C^2)^{\frac{1}{p-1}}} - \frac{1}{\theta_1}\Big) = \sigma_1 \mathrm{Ex}(u,m,R).$$

In both cases the proof is concluded. □

Remark 3.2. In Lemma 3.1 we bound u/d_Ω^s from below by means of the sum of three terms, one of which depends on u while the others do not, and the latter are in fact dropped unless the sum is negative. This strategy will be used several times in the following results.

The next result is a change of variables lemma for $(-\Delta)_p^s$, strictly related to the discussion on the boundedness of the fractional p-Laplacians of distance functions developed in [14, Section 3]. Here $\mathcal{GL}_N$ denotes the group of all invertible matrices in $\mathbb{R}^{N\times N}$, and $|A|$ denotes any matrix norm. For all $A \in \mathcal{GL}_N$, $x \in \Omega$, and $\varepsilon > 0$ we set

$$g_\varepsilon(A,x) = \int_{B_\varepsilon^c(x)} \frac{(\mathrm{d}_\Omega^s(x) - \mathrm{d}_\Omega^s(y))^{p-1}}{|A(x-y)|^{N+ps}}\, dy. \tag{3.9}$$

We need some more notation for this result: for all $U, V \subset \mathbb{R}^N$ we denote the Hausdorff distance between U and V by

$$\mathrm{dist}_{\mathcal{H}}(U,V) = \max\Big\{\sup_{x\in U} \mathrm{dist}(x,V),\, \sup_{y\in V} \mathrm{dist}(y,U)\Big\},$$

and the symmetric difference by

$$U\Delta V = (U \setminus V) \cup (V \setminus U).$$

Finally, for all $U \subset \mathbb{R}^N$ we denote by $\mathcal{H}^{N-1}(U)$ the $(N-1)$-dimensional Hausdorff measure of U.

Lemma 3.3. *If $\partial\Omega$ is $C^{1,1}$, there exist $\delta = \delta(N) > 0$, $C_4 = C_4(N,p,s,\Omega) > 0$ and g_0 s.t.*

(i) $g_\varepsilon \to g_0$ *in* $L^\infty_{\mathrm{loc}}(B_\delta(I) \times \Omega_{\rho/2})$, *as* $\varepsilon \to 0^+$;

(ii) $\|g_0\|_{L^\infty(B_\delta(I)\times\Omega_{\rho/2})} \leqslant C_4$.

Proof. Since $\mathcal{GL}_N$ is an open subset of $\mathbb{R}^{N\times N}$, we can find $\delta > 0$ (only depending on N) s.t. $B_{2\delta}(I) \subset \mathcal{GL}_N$. Choose $A \in B_\delta(I)$, $C = C(N) > 0$ s.t. $|A|, |A^{-1}| \leqslant C$. By translation invariance and boundedness of $\Omega_{\rho/2}$, we may assume $0 \in \partial\Omega$ and prove that $g_\varepsilon \to g_0$ locally uniformly in $B_\delta(I) \times D_{\rho/2}$ as $\varepsilon \to 0^+$, for some g_0 with $\|g_0\|_{L^\infty(B_\delta(I)\times D_{\rho/2})} \leqslant C$ (allowing $C > 0$ to grow bigger and eventually depend on N, p, s, and Ω). As the estimates will be uniform with respect to $A \in B_\delta(I)$, we will omit the dependence on A for simplicity.

Observe that restricting the domain of integration in (3.9) to $D_{3\rho/4} \cap B^c_\varepsilon(x)$ has the sole effect of adding an equi-bounded term to both g_ε and g_0, so that we can actually prove the statement for

$$\tilde{g}_\varepsilon(x) = \int_{D_{3\rho/4}\cap B^c_\varepsilon(x)} \frac{(\mathrm{d}^s_\Omega(x) - \mathrm{d}^s_\Omega(y))^{p-1}}{|A(x-y)|^{N+ps}}\, dy.$$

Since $\partial\Omega$ is of class $C^{1,1}$, there exists a diffeomorphism $\Phi \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ s.t. $\Phi(0) = 0$, $\mathrm{d}_\Omega(x) = (x'_N)_+$ for all $x \in D_{\rho/2}$, $x' = \Phi(x)$, and $\Phi(D_{3\rho/4}) \subseteq D'_{\rho'}$ where $\rho' = \rho'(\Omega) > 0$ and

$$D'_{\rho'} = \big\{x' \in \mathbb{R}^N : |x'| < \rho',\ x'_N > 0\big\}.$$

Moreover we may assume (taking $C > 0$ bigger if necessary) that for all $x \in D_\rho$, $x' \in D'_{\rho'}$

$$\frac{1}{C} \leqslant |D\Phi(x)|,\ |D\Phi^{-1}(x')| \leqslant C. \tag{3.10}$$

Now fix $x \in D_{\rho/2}$ and set

$$x' = \Phi(x), \quad M_x = D\Phi^{-1}(x') = (D\Phi(x))^{-1}.$$

Fix as well $\varepsilon \in]0, \rho/4[$. We act on $\tilde{g}_\varepsilon(x)$ with the change of variables $y' = \Phi(y)$ and we get

$$\begin{aligned}
\tilde{g}_\varepsilon(x) &= \int_{\Phi(D_{3\rho/4}\cap B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|A(\Phi^{-1}(x') - \Phi^{-1}(y'))|^{N+ps}} |\mathrm{det} D\Phi^{-1}(y')|\, dy' \\
&= \int_{\Phi(D_{3\rho/4}\cap B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} K(x', y')\, dy',
\end{aligned}$$

where we have set

$$K(x', y') = \frac{|AM_x(x'-y')|^{N+ps} |\mathrm{det} D\Phi^{-1}(y')|}{|A(\Phi^{-1}(x') - \Phi^{-1}(y'))|^{N+ps}}.$$

Again we can add a bounded term to $\tilde{g}_\varepsilon$ and instead consider

$$h_\varepsilon(x) = \int_{\Phi(B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} K(x', y')\, dy'. \tag{3.11}$$

By (3.10) we have for all $x' \in D'_{\rho'}$, $y' \in \Phi(B^c_\varepsilon(x))$

$$\frac{1}{C} \leqslant K(x', y') \leqslant C.$$

We introduce a linearized operator $L_x : \mathbb{R}^N \to \mathbb{R}^N$ by setting for all $y \in \mathbb{R}^N$

$$L_x(y) = \Phi(x) + D\Phi(x)(y - x),$$

which by Taylor expansion and $\Phi \in C^{1,1}(\mathbb{R}^N)$ satisfies for all $y \in \mathbb{R}^N$

$$|L_x(y) - \Phi(y)| \leqslant C|x-y|^2.$$

In turn, this implies the geometric inequality

$$d_\varepsilon := \mathrm{dist}_{\mathcal{H}}\big(\Phi(B_\varepsilon(x)),\, L_x(B_\varepsilon(x))\big) \leqslant C\varepsilon^2.$$

Set

$$\Delta_\varepsilon(x) = \Phi(B^c_\varepsilon(x)) \Delta L_x(B^c_\varepsilon(x)),$$

then by the inequality above and $\Phi(B_\varepsilon(x))\Delta L_x(B_\varepsilon(x)) \subseteq B_{d_\varepsilon}(\partial L_x(B_\varepsilon(x)))$ we have

$$|\Delta_\varepsilon(x)| \leqslant C\mathcal{H}^{N-1}(\partial L_x(B^c_\varepsilon(x)))\varepsilon^2 \leqslant C\varepsilon^{N+1}. \tag{3.12}$$

We split (3.11) as:

$$\begin{aligned} h_\varepsilon(x) &= \Big[\int_{\Phi(B^c_\varepsilon(c))\setminus L_x(B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} K(x',y')\,dy' \\ &\quad - \int_{L_x(B^c_\varepsilon(x))\setminus\Phi(B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} K(x',y')\,dy'\Big] \\ &\quad + \int_{L_x(B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} K(x',y')\,dy' \\ &= h^1_\varepsilon(x) + h^2_\varepsilon(x), \end{aligned}$$

and we estimate separately $h^1_\varepsilon(x)$ and $h^2_\varepsilon(x)$. By s-Hölder continuity of the function $y' \to (y'_N)^s_+$, estimates on $|M_x|$, (3.12), and direct integration we have

$$\begin{aligned} |h^1_\varepsilon(x)| &\leqslant C\int_{\Delta_\varepsilon(x)} \frac{|(x'_N)^s_+ - (y'_N)^s_+|^{p-1}}{|AM_x(x'-y')|^{N+ps}}\,dy' \\ &\leqslant C|A^{-1}|^{N+ps}\int_{\Delta_\varepsilon(x)} \frac{dy'}{|x'-y'|^{N+s}} \leqslant \frac{C|\Delta_\varepsilon(x)|}{\varepsilon^{N+s}} \leqslant C\varepsilon^{1-s}, \end{aligned}$$

which by $s \in]0,1[$ implies

$$h^1_\varepsilon \to 0 \quad \text{in } L^\infty(D_{\rho/2}) \text{ as } \varepsilon \to 0^+. \tag{3.13}$$

Now we turn to $h^2_\varepsilon(x)$, which we split further:

$$\begin{aligned} h^2_\varepsilon(x) &= \int_{L_x(B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} |\mathrm{det}M_x|\,dy' \\ &\quad + \int_{L_x(B^c_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} (K(x',y') - |\mathrm{det}M_x|)\,dy' \\ &= h^3_\varepsilon(x) + h^4_\varepsilon(x). \end{aligned}$$

We first deal with $h^3_\varepsilon(x)$, using the change of variables $z' = M_x(x'-y')$ and setting $x'' = M_x x'$, $\lambda = |M_x^{-T}e_N|$, and $v' = \lambda^{-1}M_x^{-T}e_N$:

$$\begin{aligned} h^3_\varepsilon(x) &= \int_{B^c_\varepsilon} \frac{\big(((M_x^{-1}x'')\cdot e_N)^s_+ - (M_x^{-1}(x''-z')\cdot e_N)^s_+\big)^{p-1}}{|Az'|^{N+ps}}\,dz' \\ &= \lambda^{(p-1)s}\int_{B^c_\varepsilon} \frac{\big((x'\cdot v')^s_+ - ((x'-z')\cdot v')^s_+\big)^{p-1}}{|Az'|^{N+ps}}\,dz'. \end{aligned}$$

By rotational invariance and [14, Lemma 3.2], we have

$$h^3_\varepsilon \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}(D_{\rho/2}) \text{ as } \varepsilon \to 0^+ \tag{3.14}$$

(this is where the convergence turns locally uniform instead of uniform). To estimate $h^4_\varepsilon(x)$, we can again add a bounded term and consider instead

$$h^5_\varepsilon(x) := \int_{L_x(B_1(x))\setminus L_x(B_\varepsilon(x))} \frac{((x'_N)^s_+ - (y'_N)^s_+)^{p-1}}{|AM_x(x'-y')|^{N+ps}} (K(x',y') - |\mathrm{det}M_x|)\,dy'.$$

By [14, Eq. (3.7)] we have

$$\big|K(x',y') - |\mathrm{det}M_x|\big| \leqslant C|x'-y'| \quad \text{for all } x' \in D'_{\rho'},\ y' \in L_x(B^c_\varepsilon(x))$$

and using Hölder continuity, we have

$$\frac{|(x'_N)_+^s-(y'_N)_+^s|^{p-1}}{|AM_x(x'-y')|^{N+ps}}|K(x',y')-|\mathrm{det}M_x||\leqslant C\frac{|x'-y'|^{(p-1)s+1}}{|AM_x(x'-y')|^{N+ps}}\leqslant\frac{C}{|x'-y'|^{N+s-1}},$$

and the latter function lies in $L^1(L_x(B_1(x)))$. Now, letting

$$h^5(x):=\int_{L_x(B_1(x))\setminus L_x(B_\varepsilon(x))}\frac{((x'_N)_+^s-(y'_N)_+^s)^{p-1}}{|AM_x(x'-y')|^{N+ps}}(K(x',y')-|\mathrm{det}M_x|)\,dy'.$$

we have, via direct integration and $L_x(B_1(x))\subseteq B_C(x')$,

$$|h^5(x)|\leqslant C\int_{L_x(B_1(x))\setminus L_x(B_\varepsilon(x))}\frac{dy'}{|x'-y'|^{N+s-1}},$$

and similarly, by $L_x(B_\varepsilon(x))\subseteq B_{C\varepsilon}(x')$ (see (3.10)),

$$|h^5_\varepsilon(x)-h^5(x)|\leqslant C\int_{L_x(B_\varepsilon(x))}\frac{dy'}{|x'-y'|^{N+s-1}}\leqslant C\varepsilon^{1-s}.$$

Again by $s\in]0,1[$, we deduce that $h^5_\varepsilon\to h^5$ in $L^\infty_{\rm loc}(D_{\rho/2})$ as $\varepsilon\to 0^+$. Taking into account the several splittings and (3.13), (3.14), we obtain the claim. ☐

By virtue of the previous result, we are able to construct our first barrier:

Lemma 3.4. (Barrier/1) *Let $\partial\Omega$ be $C^{1,1}$, $R\in]0,\rho/4[$, $\varphi\in C^\infty_c(B_1)$ s.t. $0\leqslant\varphi\leqslant 1$ in B_1, and for all $\lambda\in\mathbb{R}$, $x\in\mathbb{R}^N$ set*

$$w_\lambda(x)=\Big(1+\lambda\varphi\Big(\frac{2x}{R}\Big)\Big)\mathrm{d}^s_\Omega(x).$$

Then, there exist $\lambda_1=\lambda_1(N,p,s,\Omega,\varphi)>0$, $C_5=C_5(N,p,s,\Omega,\varphi)>0$ s.t. for all $|\lambda|\leqslant\lambda_1$

$$|(-\Delta)^s_p\,w_\lambda|\leqslant C_5\Big(1+\frac{|\lambda|}{R^s}\Big)\quad\text{in } D_{R/2}.$$

Proof. For $\lambda=0$, the conclusion follows from [14, Theorem 3.6]. So, let $\lambda\in\mathbb{R}$ satisfy

$$0<|\lambda|\leqslant\frac{1}{2\|\varphi\|_{L^\infty(B_1)}}.$$

Set for all $x\in\mathbb{R}^N$

$$\psi_\lambda(x)=\frac{1}{\lambda}\Big(\Big(1+\lambda\varphi\Big(\frac{2x}{R}\Big)\Big)^{\frac{1}{s}}-1\Big),$$

so $\psi_\lambda\in C^\infty_c(B_{R/2})$ and for all $x\in\mathbb{R}^N$

$$1+\lambda\varphi\Big(\frac{2x}{R}\Big)=(1+\lambda\psi_\lambda(x))^s.$$

Moreover, by the chain rule there exists $C>0$ (depending on N, p, s, Ω, and φ) s.t. for all $x\in\mathbb{R}^N$

$$|\psi_\lambda(x)|+R|D\psi_\lambda(x)|+R^2|D^2\psi_\lambda(x)|\leqslant C\chi_{B_{R/2}}(x).\tag{3.15}$$

Since $\Pi_\Omega\in C^{1,1}(B_R,\partial\Omega)$, by taking $|\lambda|>0$ even smaller if necessary, we may set for all $x\in\mathbb{R}^N$

$$\Phi_\lambda(x)=x+\lambda\psi_\lambda(x)(x-\Pi_\Omega(x)),$$

thus defining a diffeomorphism $\Phi_\lambda\in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$ s.t. $\Phi_\lambda(\Omega)=\Omega$, $\Phi_\lambda(x)=x$ for all $x\in B^c_{R/2}$, and $\Pi_\Omega(\Phi_\lambda(x))=\Pi_\Omega(x)$ for all $x\in D_R$. Besides we define $\Psi_\lambda=\Phi_\lambda^{-1}\in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$. The key point is that w_λ is actually equivalent to a distance function, up to the diffeomorphism Φ_λ of the domain. Indeed, with these notations, we have for all $x\in D_R$

$$\begin{aligned}w_\lambda(x)&=(1+\lambda\psi_\lambda(x))^s|x-\Pi_\Omega(x)|^s=|\Phi_\lambda(x)-\Pi_\Omega(x)|^s\\&=|\Phi_\lambda(x)-\Pi_\Omega(\Phi_\lambda(x))|^s=\mathrm{d}^s_\Omega(\Phi_\lambda(x)),\end{aligned}\tag{3.16}$$

We begin collecting some estimates on the first and second order derivatives of Φ_λ, Ψ_λ that will be used later. For all $x, x' \in \mathbb{R}^N$ we claim

$$|D\Phi_\lambda(x) - I| \leqslant C|\lambda|\chi_{B_{R/2}}(x), \quad |D\Psi_\lambda(x') - I| \leqslant C|\lambda|\chi_{B_{R/2}}(x'). \tag{3.17}$$

Indeed, recall that $\Phi_\lambda = \Psi_\lambda = I$ in $B^c_{R/2}$. Instead, for all $x \in B_{R/2}$, $i, k \in \{1, \dots N\}$ we have

$$\partial_i \Phi^k_\lambda(x) - \delta_{ik} = \lambda\Big[\partial_i\psi_\lambda(x)(x - \Pi_\Omega(x))^k + \psi_\lambda(x)(\delta_{ik} - \partial_i\Pi^k_\Omega(x))\Big]$$

(where ξ^k denotes the k-th component of $\xi \in \mathbb{R}^N$, δ_{ik} is the Kronecker symbol and ∂_i is the partial derivative with respect to x_i). By (3.15), this implies the first inequality in (3.17). By further reducing $|\lambda| > 0$ if necessary, the latter yields $|(D\Phi_\lambda(x))^{-1}| \leqslant C$, hence, setting $x' = \Phi_\lambda(x)$,

$$|D\Psi_\lambda(x') - I| = |(D\Phi_\lambda(x))^{-1}(I - D\Phi_\lambda(x))| \leqslant C|\lambda|\chi_{B_{R/2}}(x) = C|\lambda|\chi_{B_{R/2}}(x'),$$

(since $\Phi_\lambda(B_{R/2}) = B_{R/2}$), which concludes the proof of (3.17).
Regarding the second-order derivatives, for a.e. $x, x' \in \mathbb{R}^N$ we claim

$$|D^2\Phi_\lambda(x)| \leqslant \frac{C|\lambda|}{R}\chi_{B_{R/2}}(x), \quad |D^2\Psi_\lambda(x')| \leqslant \frac{C|\lambda|}{R}\chi_{B_{R/2}}(x'). \tag{3.18}$$

Indeed, for a.e. $x \in B_{R/2}$, $i, j, k \in \{1, \dots N\}$ we have

$$\begin{aligned}\partial_{ij}\Phi^k_\lambda(x) = \lambda\Big[&\partial_{ij}\psi_\lambda(x)(x - \Pi_\Omega(x))^k + \partial_i\psi_\lambda(x)(\delta_{jk} - \partial_j\Pi^k_\Omega(x)) \\ &+ \partial_j\psi_\lambda(x)(\delta_{ik} - \partial_i\Pi^k_\Omega(x)) + \psi_\lambda(x)\partial_{ij}\Pi^k_\Omega(x)\Big],\end{aligned}$$

which by (3.15) implies the first estimate in (3.18). Regarding the second one, observe that $D^2\Psi_\lambda = 0$ in $B^c_{R/2}$, while for $\Phi_\lambda(x) = x' \in B_{R/2}$, the chain rule gives, almost everywhere,

$$\partial_{ij}\Psi^k_\lambda(x') = -\partial_{\beta\gamma}\Phi^\alpha_\lambda(x)\partial_i\Psi^\beta_\lambda(x')\partial_j\Psi^\gamma_\lambda(x')\partial_\alpha\Psi^k_\lambda(x'),$$

with the sum over repeated indexes convention. Due to the estimate for $D^2\Phi_\lambda$ and (from (3.17)) $\|D\Psi_\lambda\|_\infty \leqslant C$ when $|\lambda|$ is sufficiently small, we infer the second inequality in (3.18).
Now set for all $\varepsilon > 0$, $x \in \mathbb{R}^N$

$$g_{\varepsilon,\lambda}(x) = \int_{\{|\Phi_\lambda(x) - \Phi_\lambda(y)| \geqslant \varepsilon\}} \frac{(\mathrm{d}^s_\Omega(\Phi_\lambda(x)) - \mathrm{d}^s_\Omega(\Phi_\lambda(y)))^{p-1}}{|x - y|^{N+ps}}\, dy.$$

We claim that there exist $\lambda_1, C > 0$, depending only N, p, s, Ω, and φ, s.t. for every $0 < |\lambda| < \lambda_1$ there exists $g_{0,\lambda} \in L^\infty(D_{R/2})$ s.t.

$$\|g_{0,\lambda}\|_{L^\infty(D_{R/2})} \leqslant C\Big(1 + \frac{|\lambda|}{R^s}\Big), \quad g_{\varepsilon,\lambda} \to g_{0,\lambda} \text{ in } L^\infty_{\rm loc}(D_{R/2}), \text{ as } \varepsilon \to 0^+. \tag{3.19}$$

The path to (3.19) begins with the change of variables $x' = \Phi_\lambda(x)$, $y' = \Phi_\lambda(y)$ (note that by the previous discussion $x' \in D_{R/2}$ whenever $x \in D_{R/2}$) and defining

$$K(x', y') := \frac{|D\Psi_\lambda(x')(x' - y')|^{N+ps}}{|\Psi_\lambda(x') - \Psi_\lambda(y')|^{N+ps}}|\det D\Psi_\lambda(y')|,$$

so that for all $x \in D_{R/2}$ we can rephrase

$$\begin{aligned}
g_{\varepsilon,\lambda}(x) &= \int_{B^c_\varepsilon(x')} \frac{(\mathrm{d}^s_\Omega(x') - \mathrm{d}^s_\Omega(y'))^{p-1}}{|D\Psi_\lambda(x')(x'-y')|^{N+ps}} K(x',y')\,dy' \\
&= \int_{B^c_\varepsilon(x')} \frac{(\mathrm{d}^s_\Omega(x') - \mathrm{d}^s_\Omega(y'))^{p-1}}{|D\Psi_\lambda(x')(x'-y')|^{N+ps}} |\mathrm{det} D\Psi_\lambda(x')|\,dy' \\
&\quad + \int_{B^c_\varepsilon(x')} \frac{(\mathrm{d}^s_\Omega(x') - \mathrm{d}^s_\Omega(y'))^{p-1}}{|D\Psi_\lambda(x')(x'-y')|^{N+ps}} (K(x',y') - |\mathrm{det} D\Psi_\lambda(x')|)\,dy' \\
&= g^1_{\varepsilon,\lambda}(x) + g^2_{\varepsilon,\lambda}(x).
\end{aligned}$$

By (3.17) and Lemma 3.3, taking if necessary $|\lambda| > 0$ even smaller, the claim (3.19) is true for $g^1_{\varepsilon,\lambda}$, with corresponding $g^1_{0,\lambda}$ obeying $\|g^1_{0,\lambda}\|_{L^\infty(D_{R/2})} \leqslant C$. Regarding $g^2_{\varepsilon,\lambda}$, we split as follows:

$$\begin{aligned}
K(x',y') - |\mathrm{det} D\Psi_\lambda(x')| &= \frac{|D\Psi_\lambda(x')(x'-y')|^{N+ps}}{|\Psi_\lambda(x') - \Psi_\lambda(y')|^{N+ps}} \big(|\mathrm{det} D\Psi_\lambda(y')| - |\mathrm{det} D\Psi_\lambda(x')|\big) \\
&\quad + \Big(\frac{|D\Psi_\lambda(x')(x'-y')|^{N+ps}}{|\Psi_\lambda(x') - \Psi_\lambda(y')|^{N+ps}} - 1\Big) |\mathrm{det} D\Psi_\lambda(x')| \\
&= K_1(x',y') + K_2(x',y').
\end{aligned}$$

We first estimate $K_1(x',y')$, by applying the triangle inequality, Jacobi's formula for the derivative of a determinant, and estimates (3.17), (3.18):

$$\begin{aligned}
|K_1(x',y')| &\leqslant \|D\Phi_\lambda\|^{N+ps}_{L^\infty(\mathbb{R}^N)} \|D\Psi_\lambda\|^{N+ps}_{L^\infty(\mathbb{R}^N)} \big|\mathrm{det} D\Psi_\lambda(y') - \mathrm{det} D\Psi_\lambda(x')\big| \\
&\leqslant C \Big| \int_0^1 \frac{d}{dt} \mathrm{det} D\Psi_\lambda(x' + t(y'-x'))\,dt \Big| \\
&\leqslant C \int_0^1 \|D\Psi_\lambda\|^{N-1}_{L^\infty(\mathbb{R}^N)} |D^2\Psi_\lambda(x' + t(y'-x'))| |x'-y'|\,dt \\
&\leqslant \frac{C|\lambda|}{R} |x'-y'| \int_0^1 \chi_{B_{R/2}}(x' + t(y'-x'))\,dt \\
&\leqslant \frac{C|\lambda|}{R} |x'-y'| \min\Big\{1, \frac{R}{|x'-y'|}\Big\} \leqslant C|\lambda| \min\Big\{\frac{|x'-y'|}{R}, 1\Big\},
\end{aligned}$$

where the calculations above are justified for a.e. $y' \in \mathbb{R}^N$ since, by a well known property, Sobolev functions ($\mathrm{det} D\Psi_\lambda$ in our case) are absolutely continuous on almost every line. Similarly, to estimate $K_2(x',y')$ we argue as in [14, Lemma 3.4], applying (3.17), (3.18), and Taylor's expansion with integral remainder:

$$\begin{aligned}
|K_2(x',y')| &\leqslant C\Big(\frac{|D\Psi_\lambda(x')(x'-y')|^2}{|\Psi_\lambda(x') - \Psi_\lambda(y')|^2} - 1\Big) \\
&\leqslant C \frac{\big|\Psi_\lambda(x') - \Psi_\lambda(y') + D\Psi_\lambda(x')(x'-y')\big| \, \big|\Psi_\lambda(x') - \Psi_\lambda(y') - D\Psi_\lambda(x')(x'-y')\big|}{|\Psi_\lambda(x') - \Psi_\lambda(y')|^2} \\
&\leqslant \frac{C\|D\Phi_\lambda\|^2_{L^\infty(\mathbb{R}^N)}}{|x'-y'|^2} \big(2\|D\Psi_\lambda\|_{L^\infty(\mathbb{R}^N)} |x'-y'|\big) \Big| \int_0^1 (1-t) \frac{d^2}{dt^2} \Psi_\lambda(x' + t(y'-x'))\,dt \Big| \\
&\leqslant \frac{C}{|x'-y'|} \int_0^1 \frac{|\lambda||x'-y'|^2}{R} \chi_{B_{R/2}}(x' + t(y'-x'))\,dt \\
&\leqslant \frac{C|\lambda||x'-y'|}{R} \min\Big\{1, \frac{R}{|x'-y'|}\Big\} \leqslant C|\lambda| \min\Big\{\frac{|x'-y'|}{R}, 1\Big\}.
\end{aligned}$$

Summing up the last relations, we have for all $x' \in D_{R/2}$ and a.e. $y' \in \mathbb{R}^N$

$$\big|K(x',y') - |\mathrm{det}D\Psi_\lambda(x')|\big| \leqslant C|\lambda| \min\Big\{\frac{|x'-y'|}{R}, 1\Big\}.$$

Set

$$h(x',y') = \frac{|\mathrm{d}_\Omega^s(x') - \mathrm{d}_\Omega^s(y')|^{p-1}}{|D\Psi_\lambda(x')(x'-y')|^{N+ps}}\big|K(x',y') - |\mathrm{det}D\Psi_\lambda(x')|\big|,$$

so that

$$g^2_{\varepsilon,\lambda}(x) = \int_{B^c_\varepsilon(x')} h(x',y')\, dy'.$$

Using the previous estimate and the s-Hölder continuity of d_Ω^s, we get for all $x' \in D_{R/2}$

$$\begin{aligned}\|h(x',\cdot)\|_{L^1(\mathbb{R}^N)} &\leqslant C|\lambda| \int_{\mathbb{R}^N} \frac{|\mathrm{d}_\Omega^s(x') - \mathrm{d}_\Omega^s(y')|^{p-1}}{|D\Psi_\lambda(x')(x'-y')|^{N+ps}} \min\Big\{\frac{|x'-y'|}{R}, 1\Big\}\, dy' \\ &\leqslant C|\lambda| \|D\Phi_\lambda\|^{N+ps}_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{1}{|x'-y'|^{N+s}} \min\Big\{\frac{|x'-y'|}{R}, 1\Big\}\, dy' \\ &\leqslant C|\lambda|\Big[\int_{\{|z'|<R\}} \frac{dz'}{|z'|^{N+s-1}} + \int_{\{|z'|\geqslant R\}} \frac{dz'}{|z'|^{N+s}}\Big] \leqslant \frac{C|\lambda|}{R^s}.\end{aligned}$$

By an entirely similar argument to the one used to deal with h^5 in the previous Lemma, we obtain the claim (3.19) for $g^2_{\varepsilon,\lambda}$ as well, with corresponding $g^2_{0,\lambda}$ obeying $\|g^2_{0,\lambda}\|_{L^\infty(D_{R/2})} \leqslant C|\lambda|/R^s$. Finally, recalling (3.16) and applying [14, Corollary 2.7], we conclude that, whenever $|\lambda| \leqslant \lambda_1$,

$$(-\Delta)_p^s\, w_\lambda(x) = \lim_{\varepsilon\to 0^+} g_{\varepsilon,\lambda}(x),$$

and therefore

$$|(-\Delta)_p^s\, w_\lambda| \leqslant \|g_{0,\lambda}\|_{L^\infty(D_{R/2})} \leqslant C_5\Big(1 + \frac{|\lambda|}{R^s}\Big) \quad \text{in } D_{R/2},$$

for convenient $\lambda_1, C_5 > 0$ depending on N, p, s, Ω, and φ. □

Remark 3.5. In the case when Ω is a half space, we get the cleaner estimate

$$|(-\Delta)_p^s\, w_\lambda| \leqslant \frac{C}{R^s}|\lambda| \quad \text{in } D_{R/2},$$

for all sufficiently small $|\lambda|$ depending on φ.

The next result yields a lower bound on the supersolution of (3.1) similar to that given in Lemma 3.1, but for small excess (defined in (3.2)):

Lemma 3.6. *Let $\partial\Omega$ be $C^{1,1}$, $u \in \widetilde{W}^{s,p}(D_R)$ solve (3.1) and $p \geqslant 2$. Then, for all $\theta \geqslant 1$ there exist $C_\theta = C_\theta(N,p,s,\Omega,\theta) > 1$, $\sigma_\theta = \sigma_\theta(N,p,s,\Omega,\theta) \in]0,1]$ s.t. for all $R \in\ ]0,\rho/4[$*

$$\mathrm{Ex}(u,m,R) \leqslant m\theta \quad \Longrightarrow \quad \inf_{D_{R/2}} \Big(\frac{u}{\mathrm{d}_\Omega^s} - m\Big) \geqslant \sigma_\theta \mathrm{Ex}(u,m,R) - C_\theta(m^{p-1}+K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} - C_\theta H R^s.$$

Proof. Let $\varphi \in C_c^\infty(B_1)$ be s.t. $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $B_{1/2}$, and set for all $\lambda > 0$, $x \in \mathbb{R}^N$

$$w_\lambda(x) = m\Big(1 + \lambda\varphi\Big(\frac{x}{R}\Big)\Big)\mathrm{d}_\Omega^s(x).$$

Then $w_\lambda \in W^{s,p}(D_R)$ and satisfies

$$\inf_{\tilde{B}_R} w_\lambda \geqslant m\Big(\frac{3R}{2}\Big)^s, \quad \sup_{D_R} w_\lambda \leqslant m(1+\lambda)R^s, \tag{3.20}$$

where $\tilde{B}_R$ is defined as in (2.2). By homogeneity and Lemma 3.4 (with R in the place of $R/2$) we can find $\lambda_1 > 0$ and $C_5 > 1$ s.t. for all $\lambda \in]0,\lambda_1]$

$$(-\Delta)_p^s\, w_\lambda \leqslant C_5 m^{p-1}\Big(1 + \frac{\lambda}{R^s}\Big) \quad \text{in } D_R. \tag{3.21}$$

With no loss of generality we may assume

$$0 < \lambda_1 \leqslant \min\Big\{1, \frac{(3/2)^s - 1}{2}\Big\}.$$

Now set for all $x \in \mathbb{R}^N$

$$v_\lambda(x) = \begin{cases} w_\lambda(x) & \text{if } x \in \tilde{B}_R^c \\ u(x) & \text{if } x \in \tilde{B}_R. \end{cases}$$

Clearly, since $\tilde{B}_R$ is bounded and at a positive distance from D_R, we can apply Proposition 2.6 and deduce that $v_\lambda \in \widetilde{W}^{s,p}(D_R)$ and for all $x \in D_R$

$$(-\Delta)_p^s\, v_\lambda(x) = (-\Delta)_p^s\, w_\lambda(x) + 2\int_{\tilde{B}_R} \frac{(w_\lambda(x) - u(y))^{p-1} - (w_\lambda(x) - w_\lambda(y))^{p-1}}{|x-y|^{N+ps}}\, dy. \tag{3.22}$$

We need to estimate the integral in (3.22). We note that, for all $x \in D_R$ and $y \in \tilde{B}_R$, by (3.1) we have $u(y) \geqslant m\mathrm{d}_\Omega^s(y) \geqslant w_\lambda(y)$. Using (3.20), we have as well

$$u(y) - w_\lambda(x) \geqslant w_\lambda(y) - w_\lambda(x) \geqslant \frac{mR^s}{2}\Big(\Big(\frac{3}{2}\Big)^s - 1\Big).$$

By Lagrange's theorem we deduce

$$(u(y) - w_\lambda(x))^{p-1} - (w_\lambda(y) - w_\lambda(x))^{p-1} \geqslant \frac{m^{p-2}R^{(p-2)s}}{C}(u(y) - m\mathrm{d}_\Omega^s(y)).$$

Plugging (3.21) and these estimates into (3.22) and recalling the properties (2.2) of $\tilde{B}_R$, we get

$$\begin{aligned}(-\Delta)_p^s\, v_\lambda(x) &\leqslant C_5 m^{p-1}\Big(1 + \frac{\lambda}{R^s}\Big) - 2\frac{m^{p-2}R^{(p-2)s}}{C}\int_{\tilde{B}_R} \frac{u(y) - m\mathrm{d}_\Omega^s(y)}{|x-y|^{N+ps}}\, dy \\ &\leqslant C_5 m^{p-1}\Big(1 + \frac{\lambda}{R^s}\Big) - \frac{2m^{p-2}}{CR^s}\int_{\tilde{B}_R} \Big(\frac{u(y)}{\mathrm{d}_\Omega^s(y)} - m\Big)\, dy \\ &\leqslant Cm^{p-1} + \frac{m^{p-2}}{R^s}\Big(C\lambda m - \frac{\mathrm{Ex}(u,m,R)}{C}\Big),\end{aligned}$$

for all $x \in D_R$. We then want to find suitable $\sigma_\theta, C_\theta, \lambda$ s.t. either the thesis is trivial, or

$$Cm^{p-1} + \frac{m^{p-2}}{R^s}\Big(C\lambda m - \frac{\mathrm{Ex}(u,m,R)}{C}\Big) \leqslant -K - m^{p-2}H,$$

allowing by comparison to infer $u \geqslant w_\lambda$ in D_R. As it turns out, this reduces to an elementary set of inequalities, which can be solved for λ being the right quantity to get the conclusion.
We thus fix $\theta \geqslant 1$, set

$$\sigma_\theta = \frac{\lambda_1}{2\theta C^2}, \quad C_\theta = \sigma_\theta \max\big\{4C,\, (4C^2\theta^{p-2})^{\frac{1}{p-1}}\big\}, \quad \lambda = \frac{\sigma_\theta \mathrm{Ex}(u,m,R)}{m},$$

and assume

$$\mathrm{Ex}(u,m,R) \leqslant m\theta. \tag{3.23}$$

By the choice of constants and (3.23) we have

$$\lambda \leqslant \frac{\lambda_1}{2C^2} < \lambda_1, \quad C\lambda m \leqslant \frac{\mathrm{Ex}(u,m,R)}{2C},$$

so by the estimate above

$$(-\Delta)_p^s\, v_\lambda \leqslant Cm^{p-1} - \frac{m^{p-2}\mathrm{Ex}(u,m,R)}{2CR^s} \quad \text{in } D_R. \tag{3.24}$$

Being the left-hand side of the thesis non-negative by assumption, we can suppose

$$\sigma_\theta \mathrm{Ex}(u,m,R) \geqslant C_\theta(m^{p-1} + K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} + C_\theta H R^s.$$

In particular, by the choice of C_θ and (3.23) (recall that $C \geqslant 1 \geqslant \sigma_\theta$)

$$\begin{aligned}\mathrm{Ex}(u,m,R)^{p-1} &\geqslant \frac{C_\theta^{p-1}(m^{p-1}+K)R^s}{\sigma_\theta^{p-1}} \geqslant 4C^2\theta^{p-2}(m^{p-1}+K)R^s \\ &\geqslant \frac{4C\mathrm{Ex}(u,m,R)^{p-2}}{m^{p-2}}(Cm^{p-1}+K)R^s.\end{aligned}$$

The last two inequalities lead to

$$m^{p-2}\mathrm{Ex}(u,m,R) \geqslant \begin{cases} 4C(Cm^{p-1}+K)R^s \\ \dfrac{C_\theta}{\sigma_\theta}m^{p-2}HR^s \geqslant 4Cm^{p-2}HR^s, \end{cases}$$

and by summing up to

$$m^{p-2}\mathrm{Ex}(u,m,R) \geqslant 2C(Cm^{p-1}+K+m^{p-2}H)R^s.$$

Thus, by (3.24) and (3.1) we have

$$\begin{cases} (-\Delta)_p^s\, v_\lambda \leqslant -K - m^{p-2}H \leqslant (-\Delta)_p^s\, u & \text{in } D_R \\ v_\lambda \leqslant u & \text{in } D_R^c, \end{cases}$$

which by Proposition 2.1 implies $v_\lambda \leqslant u$ in $\mathbb{R}^N$. In particular, recalling the definitions of w_λ and λ, for all $x \in D_{R/2}$ we have

$$\frac{u(x)}{\mathrm{d}_\Omega^s(x)} - m \geqslant \frac{w_\lambda(x)}{\mathrm{d}_\Omega^s(x)} - m = m\lambda \geqslant \sigma_\theta \mathrm{Ex}(u,m,R),$$

which gives the conclusion. □

Finally, we localize the global bound from below in (3.1) and prove the main result of this section, i.e., the lower bound on supersolutions of (1.1) type problems *locally* bounded from below by a multiple of d_Ω^s. Precisely, we deal, for some $\tilde{K}, m \geqslant 0$, with the problem

$$\begin{cases} (-\Delta)_p^s\, u \geqslant -\tilde{K} & \text{in } D_R \\ u \geqslant m\mathrm{d}_\Omega^s & \text{in } D_{2R}. \end{cases} \tag{3.25}$$

Proposition 3.7. (Lower bound) *Let $\partial\Omega$ be $C^{1,1}$, $u \in \widetilde{W}_0^{s,p}(D_R)$ solve (3.25) and $p \geqslant 2$. Then, for all $\varepsilon > 0$ there exist $\tilde{C}_\varepsilon = \tilde{C}_\varepsilon(N,p,s,\Omega,\varepsilon) > 0$ and two more constants $\sigma_2 = \sigma_2(N,p,s,\Omega) \in]0,1]$, $C_6 = C_6(N,p,s,\Omega) > 1$ s.t. for all $R \in]0,\rho/4[$*

$$\begin{aligned}\inf_{D_{R/2}}\Big(\frac{u}{\mathrm{d}_\Omega^s} - m\Big) &\geqslant \sigma_2\mathrm{Ex}(u,m,R) - \varepsilon\Big\|\frac{u}{\mathrm{d}_\Omega^s} - m\Big\|_{L^\infty(D_R)} - C_6\mathrm{tail}_1\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)R^s \\ &\quad - \tilde{C}_\varepsilon\Big[m + \tilde{K}^{\frac{1}{p-1}} + \mathrm{tail}_{p-1}\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)\Big]R^{\frac{s}{p-1}}.\end{aligned}$$

Proof. We may assume $u/\mathrm{d}_\Omega^s - m \in L^\infty(D_R)$, otherwise there is nothing to prove. We set $v = u \vee m\mathrm{d}_\Omega^s$ and fix $\varepsilon > 0$. By (3.25) and Proposition 2.7 *(i)* (with ε^{p-1} replacing ε) there exists $C_\varepsilon, C_2 > 0$ with C_2 depending on N, p, s, and C_ε also depending on ε, s.t. in D_R

$$\begin{aligned}(-\Delta)_p^s\, v &\geqslant -\tilde{K} - \frac{\varepsilon^{p-1}}{R^s}\Big\|\frac{u}{\mathrm{d}_\Omega^s} - m\Big\|_{L^\infty(D_R)}^{p-1} - C_\varepsilon\mathrm{tail}_{p-1}\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)^{p-1} \\ &\quad - C_2 m^{p-2}\mathrm{tail}_1\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big) =: -K - m^{p-2}H,\end{aligned}$$

where we have set

$$K = \tilde{K} + \frac{\varepsilon^{p-1}}{R^s}\Big\|\frac{u}{\mathrm{d}_\Omega^s} - m\Big\|_{L^\infty(D_R)}^{p-1} + C_\varepsilon\mathrm{tail}_{p-1}\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)^{p-1},$$

$$H = C_2 \mathrm{tail}_1\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big).$$

Thus, v satisfies (3.1) with $K, H, m \geqslant 0$ defined as above. By Lemma 3.1 we can find constants $0 < \sigma_1 \leqslant 1 \leqslant \theta_1$, C_3 (depending on N, p, s) s.t.

$$\mathrm{Ex}(v, m, R) \geqslant m\theta_1 \implies \inf_{D_{R/2}} \Big(\frac{v}{\mathrm{d}_\Omega^s} - m\Big) \geqslant \sigma_1 \mathrm{Ex}(v, m, R) - C_3(KR^s)^{\frac{1}{p-1}} - C_3 H R^s.$$

Next, choose $\theta = \theta_1 \geqslant 1$ in Lemma 3.6. Then, there exist constants $0 < \sigma_{\theta_1} \leqslant 1 \leqslant C_{\theta_1}$ s.t.

$$\mathrm{Ex}(v, m, R) \leqslant m\theta_1 \implies \inf_{D_{R/2}} \Big(\frac{v}{\mathrm{d}_\Omega^s} - m\Big) \geqslant \sigma_{\theta_1} \mathrm{Ex}(v, m, R) - C_{\theta_1}(m^{p-1} + K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} - C_{\theta_1} H R^s.$$

Set $\sigma_2 \in \,]0, 1[$, $C > 1$ defined as

$$\sigma_2 = \min\big\{\sigma_1, \sigma_{\theta_1}\big\}, \quad C = \max\big\{C_3, C_{\theta_1}\big\},$$

hence depending only on N, p, s, and Ω. In both cases, since $v = u$ in $D_{2R} \supset \tilde{B}_R$, we have

$$\inf_{D_{R/2}} \Big(\frac{u}{\mathrm{d}_\Omega^s} - m\Big) \geqslant \sigma_2 \mathrm{Ex}(u, m, R) - C(m^{p-1} + K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} - CHR^s.$$

By (3.25) and the definitions of K, H, we have

$$\begin{aligned}
\inf_{D_{R/2}} \Big(\frac{u}{\mathrm{d}_\Omega^s} - m\Big) &\geqslant \sigma_2 \mathrm{Ex}(u, m, R) - C(m^{p-1} + K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} - CHR^s \\
&\geqslant \sigma_2 \mathrm{Ex}(u, m, R) - C\varepsilon \Big\| \frac{u}{\mathrm{d}_\Omega^s} - m \Big\|_{L^\infty(D_R)} - C \mathrm{tail}_1\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big) R^s \\
&\quad - C\Big[m + \tilde{K}^{\frac{1}{p-1}} + C_\varepsilon \mathrm{tail}_{p-1}\Big(\Big(m - \frac{u}{\mathrm{d}_\Omega^s}\Big)_+, 2R\Big)\Big] R^{\frac{s}{p-1}},
\end{aligned}$$

which gives the claim (by renaming ε and the constants involved) $\square$

4. The upper bound

This section is devoted to proving an upper bound for the quotient u/d_Ω^s, where u is a subsolution of a (1.1) type problem on a special domain, locally bounded from above by a multiple of d_Ω^s. The upper bound differs substantially from the lower one, as for large values of the corresponding nonlocal excess, the function u will change sign along the boundary, which of course agrees with u being bounded *from above* by a positive multiple of d_Ω^s. The difficulty comes then from the degeneracy of $(-\Delta)_p^s$, as u will have vanishing normal s-derivative at some boundary point, and any barrier for u forcing such transition will present the same phenomenon and thus require a more delicate construction.

Throughout, we will assume $0 \in \partial\Omega$, $R \in \,]0, \rho/4[$ with ρ defined in (2.1). As in Section 3, we first consider a function $u \in \widetilde{W}^{s,p}(D_R)$ satisfying

$$\begin{cases} (-\Delta)_p^s u \leqslant K + M^{p-2} H & \text{in } D_R \\ u \leqslant M \mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

for some $M, K, H \geqslant 0$. We begin by constructing an explicit barrier:

Lemma 4.1. (Barrier/2) *Let $\partial\Omega$ be $C^{1,1}$, $R \in \,]0, \rho/4[$ and $\bar{x} \in D_{R/2}$. Then there exist $v \in W_0^{s,p}(\Omega) \cap C(\mathbb{R}^N)$ and $C_3' = C_3'(N, p, s, \Omega) > 1$ s.t.*

(i) $|(-\Delta)_p^s v| \leqslant \dfrac{C_3'}{R^s}$ *in* D_{2R};

(ii) $v(\bar{x}) = 0$;

(iii) $v \geqslant \dfrac{\mathrm{d}_\Omega^s}{C_3'}$ *in* D_R^c;

(iv) $|v| \leqslant C_3' R^s$ *in* D_{2R}.

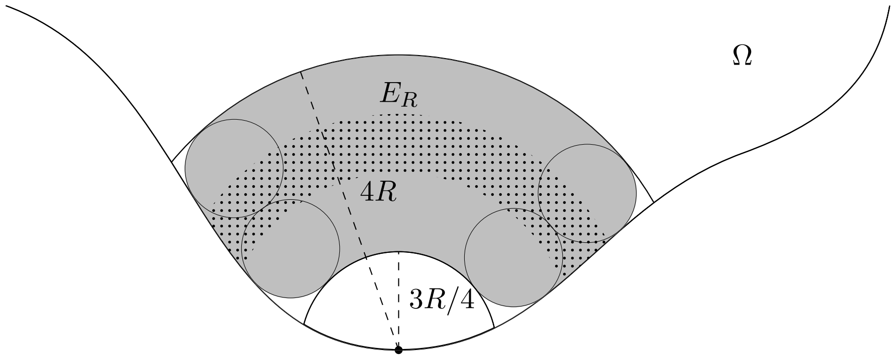

FIGURE 3. The regularized set E_R in gray; in the dotted part we have $\mathrm{d}_\Omega \leqslant C\mathrm{d}_{E_R}$.

Proof. We will construct the barrier as a solution of a double obstacle problem, and to this end we divide the proof in several steps.

Step 1 (geometry). Set

$$E_R = \bigcup \Big\{ B_r(y) : \, y \in \Omega, \, r \geqslant \frac{R}{8}, \, B_r(y) \subset D_{4R} \setminus D_{3R/4} \Big\}.$$

By the regularity of $\partial\Omega$ stated in (2.1) and $R < \rho/4$, $E_R \subset \Omega$ is a bounded domain with the interior sphere property with radius $\rho_{E_R} \geqslant R/16$ (see figure 3). We claim that

$$\mathrm{d}_\Omega \leqslant C\mathrm{d}_{E_R} \quad \text{in } D_{3R} \setminus D_R. \tag{4.2}$$

Indeed, fix a point $x \in D_{3R} \setminus D_R$. Since $\mathrm{d}_{E_R}(x) \geqslant R/8$ and

$$\mathrm{d}_{D^c_{7R/8}}(x) \leqslant 3R + \frac{R}{4} \leqslant 26\, \mathrm{d}_{E_R}(x),$$

we have $\mathrm{d}_{D^c_{7R/8}}(x) \leqslant C\mathrm{d}_{E_R}(x)$. By the triangle inequality and $R < \rho/4$ we have

$$\begin{aligned} \mathrm{d}_\Omega(x) &= |x - \Pi_\Omega(x)| \leqslant |x - \Pi_\Omega(\Pi_{D^c_{7R/8}}(x))| \\ &\leqslant |x - \Pi_{D^c_{7R/8}}(x)| + |\Pi_{D^c_{7R/8}}(x) - \Pi_\Omega(\Pi_{D^c_{7R/8}}(x))|. \end{aligned}$$

We distinguish two cases:

(a) if $\Pi_{D^c_{7R/8}}(x) \in \partial\Omega$, then

$$\mathrm{d}_\Omega(x) \leqslant \mathrm{d}_{D^c_{7R/8}}(x) \leqslant C\mathrm{d}_{E_R}(x);$$

(b) if $\Pi_{D^c_{7R/8}}(x) \notin \partial\Omega$, then $|\Pi_{D^c_{7R/8}}(x)| = 7R/8$, which in turn implies $|\Pi_\Omega(\Pi_{D^c_{7R/8}}(x))| \leqslant R$ and so

$$\mathrm{d}_\Omega(x) \leqslant \mathrm{d}_{D^c_{7R/8}}(x) + R + \frac{7R}{8} \leqslant C\mathrm{d}_{E_R}(x) + \frac{15R}{8} \leqslant C\mathrm{d}_{E_R}(x).$$

In both cases we get (4.2).

Step 2 (lower obstacle). Let $\tilde\varphi \in W^{s,p}_0(E_R)$ be the solution of the torsion problem

$$\begin{cases} (-\Delta)^s_p\, \tilde\varphi = 1 & \text{in } E_R \\ \tilde\varphi = 0 & \text{in } E^c_R. \end{cases}$$

By Lemma 2.4 we have $(-\Delta)^s_p\, \tilde\varphi \leqslant 1$ in $\mathbb{R}^N$, while Lemma 2.3 and the estimate on ρ_{E_R} imply

$$\tilde\varphi \geqslant \frac{R^{\frac{s}{p-2}}}{C}\mathrm{d}^s_{E_R} \quad \text{in } \mathbb{R}^N,$$

with some $C > 0$ depending on N, p, s. As in Section 2, we denote by $u_{4R} \in W_0^{s,p}(B_{4R})$ the solution to the torsion equation (2.4) in B_{4R}. So, since $E_R \subset B_{4R}$, we have

$$\begin{cases} (-\Delta)_p^s\,\tilde{\varphi} \leqslant (-\Delta)_p^s\,u_{4R} & \text{in } B_{4R} \\ \tilde{\varphi} \leqslant u_{4R} & \text{in } B_{4R}^c. \end{cases}$$

By Proposition 2.1 and Lemma 2.2 we have

$$\tilde{\varphi} \leqslant u_{4R} \leqslant CR^{\frac{s}{p-1}}\mathrm{d}_{B_{4R}}^s \leqslant CR^{p's} \quad \text{in } \mathbb{R}^N.$$

We set $\varphi = R^{-\frac{s}{p-1}}\tilde{\varphi} \in W_0^{s,p}(E_R)$, so by [14, Lemma 2.9 (i)] and the inequalities above we have

$$(-\Delta)_p^s\,\varphi = \frac{(-\Delta)_p^s\,\tilde{\varphi}}{R^s} \leqslant \frac{1}{R^s} \quad \text{in } \mathbb{R}^N, \tag{4.3}$$

as well as

$$\varphi \leqslant CR^{\left(p'-\frac{1}{p-1}\right)s} = CR^s \quad \text{in } \mathbb{R}^N. \tag{4.4}$$

Now, by (4.2) and Lemma 2.3 we have

$$\varphi \geqslant \frac{\mathrm{d}_{E_R}^s}{C} \geqslant \frac{\mathrm{d}_\Omega^s}{C} \quad \text{in } D_{3R} \setminus D_R. \tag{4.5}$$

Step 3 (upper obstacle). Pick $\lambda > 0$ (to be determined later) and set for all $x \in \mathbb{R}^N$

$$\psi(x) = \frac{\lambda}{R^{\frac{s}{p-1}}}\big(\max_{\mathbb{R}^N} u_{R/8} - u_{R/8}(x-\bar{x})\big),$$

where $u_{R/8} \in W_0^{s,p}(B_{R/8})$ solves (2.4) in $B_{R/8}$. Clearly $\psi \in \widetilde{W}^{s,p}(\Omega)$, $\psi \geqslant 0$ and $\psi(\bar{x}) = 0$ (since $u_{R/8}$ is radially decreasing in $B_{R/8}$). We claim that for all $\lambda(N,p,s,\Omega) > 0$ big enough

$$\psi \geqslant \varphi \quad \text{in } \mathbb{R}^N. \tag{4.6}$$

Indeed, fix $x \in \mathbb{R}^N$. Two cases may occur:

(a) if $x \in D_{3R/4}$, then $\varphi(x) = 0$, while $\psi(x) \geqslant 0$;
(b) if $x \in D_{3R/4}^c$, then $|x-\bar{x}| > R/8$, hence $u_{R/8}(x-\bar{x}) = 0$, while by Lemma 2.4

$$\max_{\mathbb{R}^N} u_{R/8} \geqslant \frac{R^{\frac{s}{p-1}}}{C}\max_{\mathbb{R}^N}\mathrm{d}_{B_{R/8}}^s \geqslant \frac{R^{p's}}{C},$$

which in turn implies $\psi(x) \geqslant \lambda R^s/C$. By using (4.4), we have $\varphi(x) \leqslant \psi(x)$ for large enough λ.

In both cases we have (4.6) for some $\lambda(N,p,s,\Omega) > 0$ which will be fixed henceforth. By [14, Lemma 2.9 (i)] and Lemma 2.4 we have

$$(-\Delta)_p^s\,\psi \geqslant -\frac{C}{R^s} \quad \text{in } \mathbb{R}^N. \tag{4.7}$$

One last property of ψ is that

$$\psi \leqslant CR^s \quad \text{in } \mathbb{R}^N, \tag{4.8}$$

which follows from the upper bound in Lemma 2.4 and

$$\psi \leqslant \frac{C}{R^{\frac{s}{p-1}}}\max_{\mathbb{R}^N} u_{R/8} \leqslant C\max_{\mathbb{R}^N}\mathrm{d}_{B_{R/8}}^s \leqslant CR^s.$$

Step 4 (the barrier). Consider the constrained minimization problem

$$\min\big\{[u]_{s,p}^p : u \in W_0^{s,p}(\Omega), \quad \varphi \leqslant u \leqslant \psi \text{ in } \mathbb{R}^N\big\}. \tag{4.9}$$

By Lemma 2.5, problem (4.9) has a solution $\tilde{v} \in W_0^{s,p}(\Omega)$, which satisfies

$$0 \wedge (-\Delta)_p^s\,\psi \leqslant (-\Delta)_p^s\,\tilde{v} \leqslant 0 \vee (-\Delta)_p^s\,\varphi \quad \text{in } \Omega.$$

By (4.3), (4.7) we have

$$|(-\Delta)_p^s\,\tilde v| \leqslant \frac{C}{R^s} \quad \text{in } D_{2R}. \tag{4.10}$$

Besides, since $\varphi(\bar x) = \psi(\bar x) = 0$ we deduce $\tilde v(\bar x) = 0$, while (4.8) implies

$$0 \leqslant \tilde v \leqslant CR^s \quad \text{in } \mathbb{R}^N. \tag{4.11}$$

Moreover, by (4.5) we have

$$\tilde v \geqslant \frac{\mathrm{d}_\Omega^s}{\tilde C} \quad \text{in } D_{3R}\setminus D_R, \tag{4.12}$$

for some $\tilde C = \tilde C(N,p,s,\Omega) > 0$. Still, $\tilde v$ is not the desired function as it only satisfies the lower bound (4.12) in $D_{3R}\setminus D_R$. So we need to extend (4.12) to the larger set D_R^c while keeping the other properties. Set for all $x \in \mathbb{R}^N$

$$v(x) = \begin{cases} \tilde v(x) & \text{if } x \in D_{3R} \\ \tilde v(x) \vee \dfrac{\mathrm{d}_\Omega^s(x)}{\tilde C} & \text{if } x \in D_{3R}^c. \end{cases}$$

Clearly $v \in W_0^{s,p}(\Omega)$ satisfies *(ii)* and *(iv)*, since, by (4.12), we are changing $\tilde v$ only outside of D_{3R}. Moreover, *(iii)* now holds by construction. So, it remains to check *(i)* for v. By Proposition 2.6 we have for all $x \in D_{2R}$

$$(-\Delta)_p^s\, v(x) = (-\Delta)_p^s\,\tilde v(x) + 2\int_{D_{3R}^c\cap\{\tilde v<\mathrm{d}_\Omega^s/\tilde C\}} \frac{(\tilde v(x)-\mathrm{d}_\Omega^s(y)/\tilde C)^{p-1} - (\tilde v(x)-\tilde v(y))^{p-1}}{|x-y|^{N+ps}}\,dy. \tag{4.13}$$

By the monotonicity of $t \mapsto t^{p-1}$ the integrand is negative and (4.10) yelds

$$(-\Delta)_p^s\, v \leqslant \frac{C_3'}{R^s} \quad \text{in } D_{2R}.$$

On the other hand, for all $x \in D_{2R}$, $y \in D_{3R}^c$ we have by (4.11)

$$\Big|\tilde v(x) - \frac{\mathrm{d}_\Omega^s(y)}{\tilde C}\Big| \leqslant C(R^s + |y|^s) \leqslant C|y|^s$$

and

$$|\tilde v(x) - \tilde v(y)| \leqslant CR^s \leqslant C|y|^s.$$

Since $|x-y| \geqslant |y|/3$ for all $x \in D_{2R}$, $y \in D_{3R}^c$, plugging these inequalities into (4.13) gives

$$(-\Delta)_p^s\, v \geqslant -\frac{C_3'}{R^s} - C\int_{D_{3R}^c} \frac{dy}{|y|^{N+s}}\,dy \geqslant -\frac{C_3'}{R^s} \quad \text{in } D_{2R},$$

for a possibly larger $C_3' > 1$ (depending on N, p, s, and Ω), which concludes the proof of *(i)*. □

The next result shows that, if a subsolution of (4.1) is small enough in $\tilde B_R$, then it is actually negative in $D_{R/2}$:

Lemma 4.2. *Let $\partial\Omega$ be $C^{1,1}$, $R \in\,]0,\rho/4[$, $p \geqslant 2$ and $u \in \widetilde W^{s,p}(D_R)$ satisfy (4.1). Then there exists $C_4' = C_4'(N,p,s,\Omega) > 1$ s.t.*

$$\mathrm{Ex}(u,M,R) \geqslant C_4'\big(M + (KR^s)^{\frac{1}{p-1}} + HR^s\big) \implies \sup_{D_{R/2}} u \leqslant 0.$$

Proof. Fix $\bar x \in D_{R/2}$, and let $v \in W_0^{s,p}(\Omega)$ be the barrier in the previous Lemma. Set

$$w(x) = \begin{cases} C_3' M v(x) & \text{if } x \in \tilde B_R^c \\ u(x) & \text{if } x \in \tilde B_R, \end{cases}$$

for all $x \in \mathbb{R}^N$, $C_3' > 1$ being as in Lemma 4.1. Recall that $\mathrm{dist}\,(D_R, \tilde{B}_R) > 0$. By Proposition 2.6, [14, Lemma 2.9 (i)], inequality (3.4), and Lemma 4.1 (i) (iii), for all $x \in D_R$ we have

$$\begin{aligned}(-\Delta)_p^s\, w(x) &= (-\Delta)_p^s\, (C_3' M v(x)) + 2\int_{\tilde{B}_r} \frac{(C_3' M v(x) - u(y))^{p-1} - (C_3' M v(x) - C_3' M v(y))^{p-1}}{|x-y|^{N+ps}}\, dy \\ &\geqslant (C_3' M)^{p-1} (-\Delta)_p^s\, v(x) + \frac{1}{C}\int_{\tilde{B}_R} \frac{(C_3' M v(y) - u(y))^{p-1}}{|x-y|^{N+ps}} \\ &\geqslant -\frac{C M^{p-1}}{R^s} + \frac{1}{C R^{ps}} ⨍_{\tilde{B}_R} (M \mathrm{d}_\Omega^s(y) - u(y))^{p-1}\, dy.\end{aligned}$$

By the properties (2.2) of $\tilde{B}_R$, Hölder's inequality (recall that $p \geqslant 2$), and $u \leqslant M\mathrm{d}_\Omega^s$ in $\tilde{B}_R$

$$⨍_{\tilde{B}_R} (M \mathrm{d}_\Omega^s(y) - u(y))^{p-1}\, dy \geqslant \frac{R^{s(p-1)}}{C} ⨍_{\tilde{B}_R} \Big(M - \frac{u(y)}{\mathrm{d}_\Omega^s(y)}\Big)^{p-1} dy \geqslant \frac{R^{s(p-1)}}{C} \mathrm{Ex}(u, M, R)^{p-1},$$

so that

$$(-\Delta)_p^s\, w \geqslant -\frac{C M^{p-1}}{R^s} + \frac{\mathrm{Ex}(u, M, R)^{p-1}}{C R^s} \quad \text{in } D_R, \tag{4.14}$$

for some $C \geqslant C_3'$. Now set

$$C_4' = (3C^2)^{\frac{1}{p-1}} \geqslant (3C)^{\frac{1}{p-1}},$$

which only depends on N, p, s, and Ω. Assume

$$\mathrm{Ex}(u, M, R) \geqslant C_4'\big(M + (K R^s)^{\frac{1}{p-1}} + H R^s\big). \tag{4.15}$$

A straightforward computation leads from (4.15) to the following inequalities

$$\mathrm{Ex}(u, M, R)^{p-1} \geqslant \begin{cases} (C_4' M)^{p-1} \geqslant 3C^2 M^{p-1} \\ (C_4')^{p-1} K R^s \geqslant 3 C K R^s \\ (C_4' M)^{p-2} \mathrm{Ex}(u, M, R) \geqslant 3 C M^{p-2} H R^s, \end{cases}$$

and hence to

$$\mathrm{Ex}(u, M, R)^{p-1} \geqslant C^2 M^{p-1} + C K R^s + C M^{p-2} H R^s.$$

So, by (4.14) we have

$$(-\Delta)_p^s\, w \geqslant K + M^{p-2} H \geqslant (-\Delta)_p^s\, u \quad \text{in } D_R.$$

Besides, we have $u \leqslant w$ in D_R^c: indeed, if $x \in \tilde{B}_R$ there is nothing to prove. If $x \in D_R^c \setminus \tilde{B}_R$, by (4.1) and Lemma 4.1 (iii) we have

$$u(x) \leqslant M \mathrm{d}_\Omega^s(x) \leqslant C_3' M v(x) = w(x).$$

Summarizing, we obtained

$$\begin{cases} (-\Delta)_p^s\, u \leqslant (-\Delta)_p^s\, w & \text{in } D_R \\ u \leqslant w & \text{in } D_R^c. \end{cases}$$

By Proposition 2.1 we have $u \leqslant w$ in $\mathbb{R}^N$. In particular, by Lemma 4.1 (ii) we get $u(\bar{x}) \leqslant 0$. By arbitrariness of $\bar{x} \in D_{R/2}$, the proof is concluded. □

Now we can prove our upper bounds on subsolutions. First we prove an upper bound for large values of $\mathrm{Ex}(u, M, R)$:

Lemma 4.3. *Let $\partial\Omega$ be $C^{1,1}$, $p \geqslant 2$ and $u \in \widetilde{W}^{s,p}(D_R)$ satisfy (4.1). Then there exist $\theta_1' = \theta_1'(N, p, s, \Omega) \geqslant 1$, $\sigma_1' = \sigma_1'(N, p, s, \Omega) \in \,]0, 1]$, and $C_5' = C_5'(N, p, s, \Omega) > 1$ s.t. for all $R \in \,]0, \rho/4[$*

$$\mathrm{Ex}(u, M, R) \geqslant M\theta_1' \implies \inf_{D_{R/4}} \Big(M - \frac{u}{\mathrm{d}_\Omega^s}\Big) \geqslant \sigma_1' \mathrm{Ex}(u, M, R) - C_5' (K R^s)^{\frac{1}{p-1}} - C_5' H R^s.$$

Proof. We set

$$H_R = \bigcup \Big\{ B_r(y) : \, y \in D_{3R/8}, \, r \geqslant \frac{R}{16}, \, B_r(y) \subset D_{3R/8} \Big\}.$$

By (2.1), H_R satisfies the interior sphere property with radius $\rho_{H_R} \geqslant R/32$. Moreover,

$$\mathrm{d}_\Omega \leqslant C \mathrm{d}_{H_R} \quad \text{in } D_{R/4} \tag{4.16}$$

for some $C > 1$ (this is proved exactly as (3.3), changing the radii). Let $\varphi \in W_0^{s,p}(H_R)$ solve

$$\begin{cases} (-\Delta)_p^s \, \varphi = 1 & \text{in } H_R \\ \varphi = 0 & \text{in } H_R^c. \end{cases} \tag{4.17}$$

By Lemma 2.4 we have $(-\Delta)_p^s \, \varphi \leqslant 1$ in $\mathbb{R}^N$. Besides we have

$$\frac{R^{\frac{s}{p-1}}}{C} \mathrm{d}_\Omega^s \leqslant \varphi \leqslant C R^{p's} \quad \text{in } D_{R/4}, \tag{4.18}$$

the first inequality coming from Lemma 2.3 and (4.16), while the second is proved as in Lemma 4.1 by comparing φ to $u_{R/2}$. Now pick $\lambda > 0$ (to be determined later) and set for all $x \in \mathbb{R}^N$

$$v(x) = \begin{cases} -\dfrac{\lambda}{R^{\frac{s}{p-1}}} \varphi(x) & \text{if } x \in D_{R/2} \\ M \mathrm{d}_\Omega^s(x) & \text{if } x \in D_{R/2}^c. \end{cases}$$

Clearly $v \in \widetilde{W}^{s,p}(H_R)$ and $\mathrm{dist}\,(D_{R/2}^c, H_R) > 0$. So we can apply Proposition 2.6, which along with [14, Lemma 2.9 (i)], (4.17) and some direct calculations yields for all $x \in H_R \subset D_{R/2}$

$$\begin{aligned} (-\Delta)_p^s \, v(x) &= -\frac{\lambda^{p-1}}{R^s} (-\Delta)_p^s \, \varphi(x) + 2 \int_{D_{R/2}^c} \frac{(-\lambda R^{-\frac{s}{p-1}} \varphi(x) - M \mathrm{d}_\Omega^s(y))^{p-1} - (-\lambda R^{-\frac{s}{p-1}} \varphi(x))^{p-1}}{|x-y|^{N+ps}} \, dy \\ &\geqslant -\frac{\lambda^{p-1}}{R^s} - C \int_{D_{R/2}^c} \frac{\lambda^{p-1} R^{-s} \varphi^{p-1}(x) + M^{p-1} \mathrm{d}_\Omega^{(p-1)s}(y)}{|x-y|^{N+ps}} \, dy. \end{aligned}$$

Therefore, using $C|y-x| > |y|$ for $x \in H_R$ and $y \in B_{R/2}^c$, (4.18) and $\mathrm{d}_\Omega(y) \leqslant |y|$, we get

$$\begin{aligned} (-\Delta)_p^s \, v(x) &\geqslant -\frac{\lambda^{p-1}}{R^s} - C(\lambda^{p-1} + M^{p-1}) \int_{B_{R/2}^c} \frac{R^{(p-1)s} + |y|^{(p-1)s}}{|y|^{N+ps}} \, dy \\ &\geqslant -C \, \frac{\lambda^{p-1} + M^{p-1}}{R^s}, \end{aligned} \tag{4.19}$$

for $x \in H_R$. Further, set for all $x \in \mathbb{R}^N$

$$w(x) = \begin{cases} v(x) & \text{if } x \in \tilde{B}_R^c \\ u(x) & \text{if } x \in \tilde{B}_R, \end{cases}$$

where $\tilde{B}_R$ is defined in (2.2). By Proposition 2.6, $w \in \widetilde{W}^{s,p}(H_R)$ and for all $x \in H_R$

$$\begin{aligned} (-\Delta)_p^s \, w(x) &= (-\Delta)_p^s \, v(x) + 2 \int_{\tilde{B}_R} \frac{(v(x) - u(y))^{p-1} - (v(x) - M \mathrm{d}_\Omega^s(y))^{p-1}}{|x-y|^{N+ps}} \, dy \\ &\geqslant -C \, \frac{\lambda^{p-1} + M^{p-1}}{R^s} + \frac{1}{C} \int_{\tilde{B}_R} \frac{(M \mathrm{d}_\Omega^s(y) - u(y))^{p-1}}{|x-y|^{N+ps}} \, dy \\ &\geqslant -C \, \frac{\lambda^{p-1} + M^{p-1}}{R^s} + \frac{1}{CR^s} \fint_{\tilde{B}_R} \Big(M - \frac{u(y)}{\mathrm{d}_\Omega^s(y)} \Big)^{p-1} \, dy \\ &\geqslant -C \, \frac{\lambda^{p-1} + M^{p-1}}{R^s} + \frac{\mathrm{Ex}(u, M, R)^{p-1}}{CR^s}, \end{aligned} \tag{4.20}$$

where we have also used (4.19), (3.4) and Hölder's inequality. So far, $C > 1$ has been chosen as big as necessary to satisfy all inequalities, depending only on N, p, s, and Ω. Now we can fix the constants in such a way that either the thesis is trivial or w is an upper barrier for u. Set

$$\lambda = \frac{\mathrm{Ex}(u, M, R)}{(4C^2)^{\frac{1}{p-1}}}, \qquad \theta_1' = \max\left\{2C_4', (4C^2)^{\frac{1}{p-1}}\right\},$$

$$\sigma_1' = \frac{1}{C(4C^2)^{\frac{1}{p-1}}}, \qquad C_5' = \sigma_1' \max\left\{2C_4', (4C)^{\frac{1}{p-1}}, \frac{4C}{(\theta_1')^{p-2}}\right\},$$

where $C_4' > 0$ is as in Lemma 4.2. Clearly $C_5' > 1$, and all these constants (except λ) only depend on N, p, s, and Ω. Now we prove the asserted implication. Assume

$$\mathrm{Ex}(u, M, R) \geqslant M\theta_1'. \tag{4.21}$$

Then, with the previous choices, (4.20) implies in H_R

$$\begin{aligned}(-\Delta)_p^s w &\geqslant \frac{C}{R^s}\left[-\frac{\mathrm{Ex}(u, M, R)^{p-1}}{4C^2} - \Big(\frac{\mathrm{Ex}(u, M, R)}{\theta_1'}\Big)^{p-1} + \frac{\mathrm{Ex}(u, M, R)^{p-1}}{C^2}\right] \\ &\geqslant \frac{\mathrm{Ex}(u, M, R)^{p-1}}{2CR^s}.\end{aligned} \tag{4.22}$$

We can also assume

$$\sigma_1'\mathrm{Ex}(u, M, R) \geqslant C_5'(KR^s)^{\frac{1}{p-1}} + C_5' HR^s, \tag{4.23}$$

otherwise there is nothing to prove (recall that u satisfies (4.1)). Such relation and (4.21) imply

$$\mathrm{Ex}(u, M, R)^{p-1} \geqslant \begin{cases} \Big(\dfrac{C_5'}{\sigma_1'}\Big)^{p-1} KR^s \geqslant 4CKR^s \\ (M\theta_1')^{p-2}\dfrac{C_5'}{\sigma_1'} HR^s \geqslant 4CM^{p-2}HR^s, \end{cases}$$

and in turn

$$\frac{\mathrm{Ex}(u, M, R)^{p-1}}{2CR^s} \geqslant K + M^{p-2}H.$$

Plugging the last inequality into (4.22), we get

$$(-\Delta)_p^s w \geqslant K + M^{p-2}H \geqslant (-\Delta)_p^s u \quad \text{in } H_R. \tag{4.24}$$

Let us now consider the pointwise estimates for $x \in H_R^c$. Three cases may occur:

(a) if $x \in \tilde{B}_R$, then $w(x) = u(x)$;

(b) if $x \in D_{R/2}^c \cap \tilde{B}_R^c$, then $w(x) = M\mathrm{d}_\Omega^s(x) \geqslant u(x)$ by assumption;

(c) if $x \in D_{R/2} \cap H_R^c$, by (4.23), (4.21) we also have

$$\mathrm{Ex}(u, M, R) \geqslant \begin{cases} \dfrac{C_5'}{\sigma_1'}(KR^s)^{\frac{1}{p-1}} + \dfrac{C_5'}{\sigma_1'}HR^s \geqslant 2C_4'(KR^s)^{\frac{1}{p-1}} + 2C_4'HR^s \\ M\theta_1' \geqslant 2C_4'M, \end{cases}$$

which summarizes as

$$\mathrm{Ex}(u, M, R) \geqslant C_4'\big(M + (KR^s)^{\frac{1}{p-1}} + HR^s\big),$$

and by Lemma 4.2 implies $u \leqslant 0$ in $D_{R/2}$, hence $w(x) = 0 \geqslant u(x)$.

Therefore $u \leqslant w$ in H_R^c, and recalling (4.24) we therefore have

$$\begin{cases} (-\Delta)_p^s u \leqslant (-\Delta)_p^s w & \text{in } H_R \\ u \leqslant w & \text{in } H_R^c. \end{cases}$$

By Proposition 2.1 we deduce $u \leqslant w$ in $\mathbb{R}^N$. In particular, for all $x \in D_{R/4}$ we have (recalling the definitions of φ, v, w, and of λ)

$$u(x) \leqslant -\frac{\lambda\varphi(x)}{R^{\frac{s}{p-1}}} \leqslant -\sigma_1'\mathrm{Ex}(u,M,R)\mathrm{d}_\Omega^s(x).$$

So we have

$$\inf_{D_{R/4}}\Big(M-\frac{u}{\mathrm{d}_\Omega^s}\Big) \geqslant -\sup_{D_{R/4}}\frac{u}{\mathrm{d}_\Omega^s} \geqslant \sigma_1'\mathrm{Ex}(u,M,R),$$

which readily yields the conclusion. □

Now we prove a similar upper bound for the case when $\mathrm{Ex}(u,M,R)$ is small:

Lemma 4.4. *Let $\partial\Omega$ be $C^{1,1}$, $p \geqslant 2$, $u \in \widetilde{W}^{s,p}(D_R)$ solve (4.1) and $R \in \,]0,\rho/4[$. Then, for all $\theta \geqslant 1$ there exist $\sigma_\theta' = \sigma_\theta'(N,p,s,\Omega,\theta) \in \,]0,1]$, $C_\theta' = C_\theta'(N,p,s,\Omega,\theta) > 1$ s.t.*

$$\mathrm{Ex}(u,M,R) \leqslant M\theta \implies \inf_{D_{R/2}}\Big(M-\frac{u}{\mathrm{d}_\Omega^s}\Big) \geqslant \sigma_\theta'\mathrm{Ex}(u,M,R) - C_\theta'(M^{p-1}+K)^{\frac{1}{p-1}}R^{\frac{s}{p-1}} - C_\theta' H R^s.$$

Proof. The proof is similar to the one of Lemma 3.6 and we only sketch it. Fix $\varphi \in C_c^\infty(B_1)$ s.t. $\varphi = 1$ in $B_{1/2}$ and $0 \leqslant \varphi \leqslant 1$ in B_1, let $\lambda_1 > 0$ be as in Lemma 3.4, and for all $\lambda \in \,]0,\lambda_1]$ set

$$w_\lambda(x) = M\Big(1-\lambda\varphi\Big(\frac{x}{R}\Big)\Big)\mathrm{d}_\Omega^s(x), \qquad x \in \mathbb{R}^N.$$

Without loss of generality we may assume $\lambda_1 \leqslant 1$. Then $w_\lambda \in W^{s,p}(D_R)$ and it satisfies

$$\begin{cases} (-\Delta)_p^s\, w_\lambda \geqslant -C_5 M^{p-1}\Big(1-\dfrac{\lambda}{R^s}\Big) & \text{in } D_R \\ w_\lambda = M(1-\lambda)\mathrm{d}_\Omega^s & \text{in } D_{R/2} \end{cases}$$

($C_5 > 0$ as in Lemma 3.4). Now set for all $x \in \mathbb{R}^N$

$$v_\lambda(x) = \begin{cases} w_\lambda(x) & \text{if } x \in \tilde{B}_R^c \\ u(x) & \text{if } x \in \tilde{B}_R, \end{cases}$$

where $\tilde{B}_R$ is defined as in (2.2). By Proposition 2.6, we have for all $x \in D_R$

$$(-\Delta)_p^s\, v_\lambda(x) = (-\Delta)_p^s\, w_\lambda(x) + 2\int_{\tilde{B}_R}\frac{(w_\lambda(x)-u(y))^{p-1}-(w_\lambda(x)-w_\lambda(y))^{p-1}}{|x-y|^{N+ps}}\,dy,$$

and estimating the integral term as in the proof of Lemma 3.6, we obtain

$$(-\Delta)_p^s\, v_\lambda \geqslant -CM^{p-1} - \frac{M^{p-2}}{R^s}\Big(CM\lambda - \frac{\mathrm{Ex}(u,M,R)}{C}\Big), \tag{4.25}$$

for some $C > 1$ (depending on N, p, s, and Ω). Now we fix $\theta \geqslant 1$ and set

$$\sigma_\theta' = \frac{\lambda_1}{2\theta C^2}, \quad C_\theta' = \sigma_\theta' \max\big\{4C,\, (4C^2\theta^{p-2})^{\frac{1}{p-1}}\big\}, \quad \lambda = \frac{\sigma_\theta'\mathrm{Ex}(u,M,R)}{M}.$$

Note that $\sigma_\theta' \leqslant 1$. We also assume

$$\mathrm{Ex}(u,M,R) \leqslant M\theta. \tag{4.26}$$

Then, by the choice of constants we have

$$\lambda < \lambda_1, \quad CM\lambda \leqslant \frac{\mathrm{Ex}(u,M,R)}{2C}.$$

These inequalities and (4.25) give

$$(-\Delta)_p^s\, v_\lambda \geqslant -CM^{p-1} + \frac{M^{p-2}}{R^s}\frac{\mathrm{Ex}(u,M,R)}{2C} \quad \text{in } D_R.$$

Assuming also

$$\sigma'_\theta \mathrm{Ex}(u,M,R) \geqslant C'_\theta (M^{p-1}+K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} + C'_\theta H R^s$$

(otherwise the thesis is trivial), the choice of the parameters and (4.26) imply

$$M^{p-2}\mathrm{Ex}(u,M,R) \geqslant 2C(CM^{p-1}+K+M^{p-2}H)R^s,$$

exactly as in the proof of Lemma 3.6, and therefore

$$(-\Delta)_p^s v_\lambda \geqslant K + M^{p-2}H \quad \text{in } D_R.$$

Moreover in D_R^c we have by construction either $v_\lambda = u$ in $\tilde{B}_R$, or $v_\lambda = w_\lambda = M\mathrm{d}_\Omega^s \geqslant u$. Thus

$$\begin{cases} (-\Delta)_p^s u \leqslant (-\Delta)_p^s v_\lambda & \text{in } D_R \\ u \leqslant v_\lambda & \text{in } D_R^c. \end{cases}$$

Proposition 2.1 ensures $u \leqslant v_\lambda$ in all of $\mathbb{R}^N$. In particular $u \leqslant w_\lambda = M(1-\lambda)\mathrm{d}_\Omega^s$ in $D_{R/2}$. So,

$$\inf_{D_{R/2}} \Big(M - \frac{u}{\mathrm{d}_\Omega^s}\Big) \geqslant \inf_{D_{R/2}} \Big(M - \frac{w_\lambda}{\mathrm{d}_\Omega^s}\Big) \geqslant M\lambda = \sigma'_\theta \mathrm{Ex}(u,M,R)$$

and the conclusion follows. □

Now we present the analog of Proposition 3.7, dealing with the problem

$$\begin{cases} (-\Delta)_p^s u \leqslant \tilde{K} & \text{in } D_R \\ u \leqslant M\mathrm{d}_\Omega^s & \text{in } D_{2R}, \end{cases} \tag{4.27}$$

with $\tilde{K}, M \geqslant 0$.

Proposition 4.5. (Upper bound) *Let $\partial\Omega$ be $C^{1,1}$, $p \geqslant 2$, $u \in \widetilde{W}_0^{s,p}(D_R)$ solve (4.27) and $R \in \,]0, \rho/4[$. Then, for all $\varepsilon > 0$ there exist $\tilde{C}'_\varepsilon = \tilde{C}'_\varepsilon(N,p,s,\Omega,\varepsilon) > 0$ and two more constants $\sigma'_2 = \sigma'_2(N,p,s,\Omega) \in]0,1]$, $C'_6 = C'_6(N,p,s,\Omega) > 1$ s.t.*

$$\begin{aligned} \inf_{D_{R/4}} \Big(M - \frac{u}{\mathrm{d}_\Omega^s}\Big) \geqslant \sigma'_2 \mathrm{Ex}(u,M,R) - \varepsilon \Big\| M - \frac{u}{\mathrm{d}_\Omega^s}\Big\|_{L^\infty(D_R)} - C'_6 \mathrm{tail}_1\Big(\Big(\frac{u}{\mathrm{d}_\Omega^s} - M\Big)_+, 2R\Big) R^s \\ - \tilde{C}'_\varepsilon \Big[M + \tilde{K}^{\frac{1}{p-1}} + \mathrm{tail}_{p-1}\Big(\Big(\frac{u}{\mathrm{d}_\Omega^s} - M\Big)_+, 2R\Big)\Big] R^{\frac{s}{p-1}}. \end{aligned}$$

Proof. The proof is identical to the one of Proposition 3.7, so we only sketch it. Consider $v = u \wedge M\mathrm{d}_\Omega^s$ and fix $\varepsilon > 0$. By Proposition 2.7 *(ii)*

$$\begin{cases} (-\Delta)_p^s v \leqslant K + M^{p-2}H & \text{in } D_R \\ v \leqslant M\mathrm{d}_\Omega^s & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$K := \tilde{K} + \frac{\varepsilon^{p-1}}{R^s}\Big\| M - \frac{u}{\mathrm{d}_\Omega^s}\Big\|_{L^\infty(D_R)}^{p-1} + C'_\varepsilon \mathrm{tail}_{p-1}\Big(\Big(\frac{u}{\mathrm{d}_\Omega^s} - M\Big)_+, 2R\Big)^{p-1},$$

$$H := C'_2 \mathrm{tail}_1\Big(\Big(\frac{u}{\mathrm{d}_\Omega^s} - M\Big)_+, 2R\Big).$$

Let $0 < \sigma'_1 \leqslant 1 \leqslant \theta'_1, C'_5$ given in Lemma 4.3 and choose $\theta = \theta'_1$ in Lemma 4.4, with corresponding $0 < \sigma'_{\theta'_1} \leqslant 1 \leqslant C'_{\theta'_1}$ given therein. Define

$$\sigma'_2 = \min\{\sigma'_1, \sigma'_{\theta'_1}\}, \quad C = \max\{C'_5, C'_{\theta'_1}\}.$$

Considering separately the cases $\mathrm{Ex}(u,M,R) \geqslant M\theta_1$ and $\mathrm{Ex}(u,M,R) < M\theta_1$ we obtain that

$$\inf_{D_{R/4}} \Big(M - \frac{v}{\mathrm{d}_\Omega^s}\Big) \geqslant \sigma'_2 \mathrm{Ex}(v,M,R) - C(M^{p-1}+K)^{\frac{1}{p-1}} R^{\frac{s}{p-1}} - CHR^s.$$

Since $u = v$ in D_{2R}, after standard estimates we conclude. □

5. Weighted Hölder regularity

This final section is devoted to the proof of Theorem 1.1, i.e., of weighted Hölder regularity for the solutions of problem (1.1). We follow a standard approach, starting with an estimate of the oscillation near the boundary of u/d_Ω^s, where u satisfy

$$\begin{cases} |(-\Delta)_p^s\, u| \leqslant K & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{5.1}$$

with some $K > 0$. Our estimate reads as follows:

Theorem 5.1. *Let $\partial\Omega$ be $C^{1,1}$, $p \geqslant 2$, $x_1 \in \partial\Omega$ and $u \in W_0^{s,p}(\Omega)$ solve (5.1). Then there exist $\alpha_1 \in]0, s]$, $C_7 > 1$, $R_0 \in]0, \rho/4[$ all depending on N, p, s and Ω s.t. for all $r \in]0, R_0[$*

$$\operatorname*{osc}_{D_r(x_1)} \frac{u}{\mathrm{d}_\Omega^s} \leqslant C_7 K^{\frac{1}{p-1}} r^{\alpha_1}.$$

Proof. First we assume $x_1 = 0$ and $K = 1$ in (5.1). We set $v = u/\mathrm{d}_\Omega^s \in \widetilde{W}_0^{s,p}(\Omega)$, $R_0 = \min\{1, \rho/4\}$, and for all $n \in \mathbb{N}$ we define $R_n = R_0/8^n$, $D_n = D_{R_n}$, and $\tilde{B}_n = \tilde{B}_{R_n/2}$ (see (2.2)). We claim that there exist $\alpha_1 \in]0, s]$, $\mu \geqslant 1$, a nondecreasing sequence $\{m_n\}$, and a nonincreasing sequence $\{M_n\}$ in $\mathbb{R}$ (all depending on N, p, s, and Ω) s.t. for all $n \in \mathbb{N}$

$$m_n \leqslant \inf_{D_n} v \leqslant \sup_{D_n} v \leqslant M_n, \quad M_n - m_n = \mu R_n^{\alpha_1}. \tag{5.2}$$

Pick $\alpha_1 \in]0, s]$ (to be determined later). We argue by (strong) induction on $n \in \mathbb{N}$. The first step $n = 0$ follows from [14, Theorem 4.4], which (slightly rephrased) ensures existence of $\tilde{C}_\Omega > 1$ (depending on N, p, s, and Ω) s.t.

$$|v| \leqslant \tilde{C}_\Omega \quad \text{in } \Omega.$$

So we set $M_0 = \tilde{C}_\Omega$, $m_n = -\tilde{C}_\Omega$, $\mu = 2\tilde{C}_\Omega/R_0^{\alpha_1}$, and (5.2) holds. Now let $n \in \mathbb{N}$ and

$$m_0 \leqslant \ldots \leqslant m_n < M_n \leqslant \ldots \leqslant M_0$$

be defined and satisfy (5.2). We set $R = R_n/2$, so $D_{n+1} = D_{R/4}$ and $\tilde{B}_n = \tilde{B}_R$, and aim at applying our lower and upper bounds on v, by distinguishing three cases:

(*a*) If $0 \leqslant m_n < M_n$, then u satisfies both (3.25) and (4.27) with $\tilde{K} = 1$ and non-negative multipliers of d_Ω^s, namely

$$\begin{cases} -1 \leqslant (-\Delta)_p^s\, u \leqslant 1 & \text{in } D_{R_n/2} \\ m_n \mathrm{d}_\Omega^s \leqslant u \leqslant M_n \mathrm{d}_\Omega^s & \text{in } D_n. \end{cases}$$

Thus, Propositions 3.7 and 4.5 apply, yielding constants $0 < \sigma \leqslant 1 < C_6, C_\varepsilon$ (we take here the smaller of σ's and the bigger of C_6's and of C_ε's, all depending on N, p, s, Ω with C_ε also depending on ε) s.t. the following bounds hold:

$$\begin{aligned} \inf_{D_{n+1}} (v - m_n) \geqslant \sigma \fint_{\tilde{B}_n} (v - m_n)\, dx - C_\varepsilon \big[m_n + 1 + \mathrm{tail}_{p-1}((m_n - v)_+, R_n)\big] R_n^{\frac{s}{p-1}} \\ - C_6 \Big[\varepsilon \sup_{D_{R_n/2}} (v - m_n) + \mathrm{tail}_1((m_n - v)_+, R_n) R_n^s\Big], \end{aligned} \tag{5.3}$$

$$\begin{aligned} \inf_{D_{n+1}} (M_n - v) \geqslant \sigma \fint_{\tilde{B}_n} (M_n - v)\, dx - C_\varepsilon \big[M_n + 1 + \mathrm{tail}_{p-1}((v - M_n)_+, R_n)\big] R_n^{\frac{s}{p-1}} \\ - C_6 \Big[\varepsilon \sup_{D_{R_n/2}} (M_n - v) + \mathrm{tail}_1((v - M_n)_+, R_n) R_n^s\Big]. \end{aligned} \tag{5.4}$$

(*b*) If $m_n < 0 < M_n$, then we can similarly apply Proposition 4.5 to u with upper bound $M_n \mathrm{d}_\Omega^s$ and to $-u$ with upper bound $-m_n \mathrm{d}_\Omega^s$. After substitution, this provides (5.4) and (5.3) respectively.

(c) If $m_n < M_n \leqslant 0$, then we apply Proposition 3.7 to $-u$ with lower bound $-M_n \mathrm{d}_\Omega^s$ and Proposition 4.5 to $-u$ with upper bound $-m_n \mathrm{d}_\Omega^s$, getting again (5.4) and (5.3) respectively.

All in all, by taking convenient constants and replacing ε with ε/C_6, we have

$$\begin{aligned}\sigma(M_n - m_n) &= \sigma \fint_{\tilde{B}_n} (M_n - v)\,dx + \sigma \fint_{\tilde{B}_n} (v - m_n)\,dx \\ &\leqslant \inf_{D_{n+1}} (M_n - v) + \inf_{D_{n+1}} (v - m_n) + \varepsilon \sup_{D_n}(M_n - v) + \varepsilon \sup_{D_n}(v - m_n) \\ &\quad + C_\varepsilon \big[1 + |M_n| + |m_n| + \mathrm{tail}_{p-1}((v - M_n)_+, R_n) + \mathrm{tail}_{p-1}((m_n - v)_+, R_n)\big] R_n^{\frac{s}{p-1}} \\ &\quad + C_6 \big[\mathrm{tail}_1((v - M_n)_+, R_n) + \mathrm{tail}_1((m_n - v)_+, R_n)\big] R_n^s.\end{aligned}$$

Notice that

$$\inf_{D_{n+1}} (M_n - v) + \inf_{D_{n+1}} (v - m_n) = (M_n - m_n) - \operatorname*{osc}_{D_{n+1}} v$$

and by the inductive hypothesis (5.2),

$$\sup_{D_n}(M_n - v) + \sup_{D_n}(v - m_n) \leqslant 2(M_n - m_n).$$

Now fix $\varepsilon = \sigma/4$ and, recalling that $|m_n|, |M_n| \leqslant \tilde{C}_\Omega$, we get

$$\begin{aligned}\sigma(M_n - m_n) \leqslant \Big(1 + \frac{\sigma}{2}\Big)(M_n - m_n) - \operatorname*{osc}_{D_{n+1}} v& \\ + C\big[1 + \mathrm{tail}_{p-1}((v - M_n)_+, R_n) + \mathrm{tail}_{p-1}((m_n - v)_+, R_n)\big] R_n^{\frac{s}{p-1}}& \\ + C\big[\mathrm{tail}_1((v - M_n)_+, R_n) + \mathrm{tail}_1((m_n - v)_+, R_n)\big] R_n^s&,\end{aligned}$$

for some $C > 1$ depending on N, p, s and Ω. Rearranging and using (5.2), we get

$$\begin{aligned}\operatorname*{osc}_{D_{n+1}} v \leqslant \Big(1 - \frac{\sigma}{2}\Big)\mu R_n^{\alpha_1} + C\big[1 + \mathrm{tail}_{p-1}((v - M_n)_+, R_n) + \mathrm{tail}_{p-1}((m_n - v)_+, R_n)\big] R_n^{\frac{s}{p-1}}& \\ + C\big[\mathrm{tail}_1((v - M_n)_+, R_n) + \mathrm{tail}_1((m_n - v)_+, R_n)\big] R_n^s&.\end{aligned} \tag{5.5}$$

Now we need to estimate the tail terms. We note that for all $x \in D_i \setminus D_{i+1}$, $i \in \{0, \dots n-1\}$, by (5.2) and monotonicity of the sequences $\{m_n\}$, $\{M_n\}$ we have

$$m_n - v(x) \leqslant m_n - m_i \leqslant (m_n - M_n) + (M_i - m_i) \leqslant \mu(R_i^{\alpha_1} - R_n^{\alpha_1}).$$

Using $|m_n|, |M_n|, \|v\|_{L^\infty(\Omega)} \leqslant \tilde{C}_\Omega$, for all $q \geqslant 1$ we have

$$\begin{aligned}\int_{\Omega \cap B_n^c} \frac{(m_n - v(x))_+^q}{|x|^{N+s}}\,dx &\leqslant \int_{\Omega \cap B_0^c} \frac{(m_n - v(x))_+^q}{|x|^{N+s}}\,dx + \sum_{i=0}^{n-1} \int_{D_i \setminus D_{i+1}} \frac{(m_n - v(x))_+^q}{|x|^{N+s}}\,dx \\ &\leqslant C + \mu^q \sum_{i=0}^{n-1} \int_{D_i \setminus D_{i+1}} \frac{(R_i^{\alpha_1} - R_n^{\alpha_1})^q}{|x|^{N+s}}\,dx \\ &\leqslant C + C\mu^q \sum_{i=0}^{n-1} \frac{(R_i^{\alpha_1} - R_n^{\alpha_1})^q}{R_i^s} \leqslant C + C\mu^q S_q(\alpha_1) R_n^{q\alpha_1 - s},\end{aligned}$$

where we have set

$$S_q(\alpha_1) = \sum_{j=1}^{\infty} \frac{(8^{\alpha_1 j} - 1)^q}{8^{sj}}.$$

Recalling the definition (2.3) and setting $q = p - 1$, we get (by convexity)

$$\begin{aligned}\mathrm{tail}_{p-1}((m_n - v)_+, R_n)R_n^{\frac{s}{p-1}} &\leqslant C\big(1 + \mu^{p-1}S_{p-1}(\alpha_1)R_n^{(p-1)\alpha_1 - s}\big)^{\frac{1}{p-1}}R_n^{\frac{s}{p-1}} \\ &\leqslant CR_n^{\frac{s}{p-1}} + C\mu S_{p-1}^{\frac{1}{p-1}}(\alpha_1)R_n^{\alpha_1},\end{aligned}$$

while for $q = 1$ we immediately get

$$\mathrm{tail}_1((m_n - v)_+, R_n)R_n^s \leqslant CR_n^s + C\mu S_1(\alpha_1)R_n^{\alpha_1}.$$

Similarly we prove the estimates

$$\begin{aligned}\mathrm{tail}_{p-1}((v - M_n)_+, R_n)R_n^{\frac{s}{p-1}} &\leqslant CR_n^{\frac{s}{p-1}} + C\mu S_{p-1}^{\frac{1}{p-1}}(\alpha_1)R_n^{\alpha_1}, \\ \mathrm{tail}_1((v - M_n)_+, R_n)R_n^s &\leqslant CR_n^s + C\mu S_1(\alpha_1)R_n^{\alpha_1}.\end{aligned}$$

Plugging these estimates into (5.5), and recalling that $R_0 < 1$, we get

$$\begin{aligned}\operatorname*{osc}_{D_{n+1}} v &\leqslant \Big(1 - \frac{\sigma}{2}\Big)\mu R_n^{\alpha_1} + C\big(S_1(\alpha_1) + S_{p-1}^{\frac{1}{p-1}}(\alpha_1)\big)\mu R_n^{\alpha_1} + C\big(R_n^{\frac{s}{p-1}} + R_n^s\big) \\ &\leqslant \Big(1 - \frac{\sigma}{2} + CS_1(\alpha_1) + CS_{p-1}^{\frac{1}{p-1}}(\alpha_1)\Big)8^{\alpha_1}\mu R_{n+1}^{\alpha_1} + C\big(R_0^{\frac{s}{p-1}-\alpha_1} + R_0^{s-\alpha_1}\big)8^{\alpha_1}R_{n+1}^{\alpha_1}.\end{aligned} \tag{5.6}$$

We claim that for all $q \geqslant 1$

$$\lim_{\alpha_1 \to 0^+} S_q(\alpha_1) = 0. \tag{5.7}$$

Indeed, for all $\alpha_1 \in]0, s/q[$ we have

$$S_q(\alpha_1) \leqslant \sum_{j=1}^{\infty} \frac{1}{8^{(s-\alpha_1 q)j}} < \infty,$$

while clearly $(8^{\alpha_1 j} - 1)^q/8^{sj} \to 0$ as $\alpha_1 \to 0^+$, for all $j \in \mathbb{N}$, so $S_q(\alpha_1) \to 0$ as well. Applying (5.7) with $q = 1, p - 1$ respectively, for all $\alpha_1 > 0$ small enough we have

$$\Big(1 - \frac{\sigma}{2} + CS_1(\alpha_1) + CS_{p-1}^{\frac{1}{p-1}}(\alpha_1)\Big)8^{\alpha_1} < 1 - \frac{\sigma}{4},$$

while we may choose $\mu > 1$ big enough to have

$$\Big(1 - \frac{\sigma}{4}\Big)\mu + C\big(R_0^{\frac{s}{p-1}-\alpha_1} + R_0^{s-\alpha_1}\big)8^{\alpha_1} \leqslant \mu,$$

so from (5.6) we have

$$\operatorname*{osc}_{D_{n+1}} v \leqslant \mu R_{n+1}^{\alpha_1}.$$

Thus, we can find $m_{n+1}, M_{n+1} \in [m_n, M_n]$ s.t.

$$m_{n+1} \leqslant \inf_{D_{n+1}} v \leqslant \sup_{D_{n+1}} v \leqslant M_{n+1}, \quad M_{n+1} - m_{n+1} = \mu R_{n+1}^{\alpha_1},$$

hence (5.2) holds at step $n + 1$, which concludes the induction step. For any $r \in]0, R_0[$ there exists $n \in \mathbb{N}$ s.t. $R_{n+1} < r \leqslant R_n$, so we have

$$\operatorname*{osc}_{D_r} v \leqslant \operatorname*{osc}_{D_{n+1}} v \leqslant \mu 8^{\alpha_1} r^{\alpha_1}.$$

Setting $C_7 = \mu 8^{\alpha_1}$, we have

$$\operatorname*{osc}_{D_r} \frac{u}{\mathrm{d}_\Omega^s} \leqslant C_7 r^{\alpha_1}.$$

Finally, for any $x_1 \in \partial\Omega$ and an arbitrary $K > 0$ in (5.1), translation invariance and homogeneity of $(-\Delta)_p^s$ yield the conclusion. □

Our final steps require a technical lemma, which is contained in the proof of [28, Theorem 1.2]:

Lemma 5.2. *Let $\partial\Omega$ be $C^{1,1}$. If $v \in L^\infty(\Omega)$ satisfies the following conditions:*

(i) $\|v\|_{L^\infty(\Omega)} \leqslant C_8$;

(ii) for all $x_1 \in \partial\Omega$, $r > 0$ we have $\operatorname*{osc}_{D_r(x_1)} v \leqslant C_8 r^{\beta_1}$;

(iii) for all $x_0 \in \Omega$ with $\mathrm{d}_\Omega(x_0) = R$, $v \in C^{\beta_2}(B_{R/2}(x_0))$ with $[v]_{C^{\beta_2}(B_{R/2}(x_0))} \leqslant C_8(1 + R^{-\mu})$,

for some $C_8, \mu > 0$ and $\beta_1, \beta_2 \in]0,1[$, then there exist $\alpha \in]0,1[$, $C_9 > 0$ depending on the parameters and Ω s.t. $v \in C^\alpha(\overline{\Omega})$ and $[v]_{C^\alpha(\overline{\Omega})} \leqslant C_9$.

Now we can prove our main result.

Proof of Theorem 1.1. Let $u \in W^{s,p}_0(\Omega)$, $f \in L^\infty(\Omega)$ satisfy (1.1), and set $K = \|f\|_{L^\infty(\Omega)}$, so u satisfies (5.1). By homogeneity we can assume $K = 1$. Let us collect some known facts about u. From [14, Theorem 1.1] we know that there exist $\alpha_2 \in]0,s]$, $C > 0$ s.t. $u \in C^{\alpha_2}(\overline{\Omega})$ and

$$\|u\|_{C^{\alpha_2}(\overline{\Omega})} \leqslant C \tag{5.8}$$

(in what follows, all constants depend on N, p, s, and Ω), in particular $\|u\|_{L^\infty(\Omega)} \leqslant C$. Besides, from [14, Corollary 5.5] we know that for all $x_0 \in \Omega$ with $R = \mathrm{d}_\Omega(x_0)$

$$\begin{aligned} [u]_{C^{\alpha_2}(B_{R/2}(x_0))} &\leqslant \frac{C}{R^{\alpha_2}}\Big[R^{p's} + \|u\|_{L^\infty(\Omega)} + R^{p's}\Big(\int_{B^c_R(x_0)} \frac{|u(y)|^{p-1}}{|x_0 - y|^{N+ps}}\,dy\Big)^{\frac{1}{p-1}}\Big] \\ &\leqslant \frac{C}{R^{\alpha_2}}\Big[R^{p's} + 1 + R^{p's}\Big(\int_{B^c_R(x_0)} \frac{1}{|x_0 - y|^{N+ps}}\,dy\Big)^{\frac{1}{p-1}}\Big] \leqslant \frac{C}{R^{\alpha_2}}, \end{aligned} \tag{5.9}$$

since $R \leqslant \mathrm{diam}(\Omega)$. Finally, from [28, p. 292] we know that, with the same choice of x_0 and R as above, the following estimate can be obtained by interpolation:

$$[\mathrm{d}_\Omega^{-s}]_{C^{\alpha_2}(B_{R/2}(x_0))} \leqslant \frac{C}{R^{s+\alpha_2}}. \tag{5.10}$$

Now we set $v = u/\mathrm{d}^s_\Omega$, and aim at applying Lemma 5.2 to this function. First, from [14, Theorem 4.4] we know that $v \in L^\infty(\Omega)$ with

$$\|v\|_{L^\infty(\Omega)} \leqslant C. \tag{5.11}$$

Further, chosen $x_0 \in \Omega$, $R = \mathrm{d}_\Omega(x_0)$, we have for all $x, y \in B_{R/2}(x_0)$

$$\begin{aligned} \frac{|v(x) - v(y)|}{|x - y|^{\alpha_2}} &\leqslant \frac{|u(x)\mathrm{d}_\Omega^{-s}(x) - u(y)\mathrm{d}_\Omega^{-s}(x)|}{|x - y|^{\alpha_2}} + \frac{|u(y)\mathrm{d}_\Omega^{-s}(x) - u(y)\mathrm{d}_\Omega^{-s}(y)|}{|x - y|^{\alpha_2}} \\ &\leqslant [u]_{C^{\alpha_2}(B_{R/2}(x))}\|\mathrm{d}_\Omega^{-s}\|_{L^\infty(B_{R/2}(x_0))} + \|u\|_{L^\infty(\Omega)}[\mathrm{d}_\Omega^{-s}]_{C^{\alpha_2}(B_{R/2}(x_0))} \\ &\leqslant \frac{C}{R^{\alpha_2}}\Big(\frac{2}{R}\Big)^s + \frac{C}{R^{s+\alpha_2}} \leqslant \frac{C}{R^{s+\alpha_2}}, \end{aligned}$$

for some $C > 0$. Here we have used (5.8), (5.9), and (5.10). Finally, let $x_1 \in \partial\Omega$ and $r > 0$, and $\alpha_1 \in]0,s]$, $C_7 > 0$, and $R_0 \in]0,\rho/4]$ be as in Theorem 5.1. We distinguish two cases:

(a) If $r \in]0,R_0[$, then by Theorem 5.1 we have

$$\operatorname*{osc}_{D_r(x_1)} v \leqslant C_7 r^{\alpha_1}.$$

(b) If $r \geqslant R_0$, then by (5.11) we have

$$\operatorname*{osc}_{D_r(x_1)} v \leqslant 2\|v\|_{L^\infty(\Omega)} \leqslant \frac{C}{R_0^{\alpha_1}} r^{\alpha_1}.$$

In both cases, we can find $C > 0$ s.t.

$$\operatorname*{osc}_{D_r(x_1)} v \leqslant C r^{\alpha_1} \quad \text{for all } r > 0.$$

Then, hypotheses (i), (iii), (ii) of Lemma 5.2 hold with $C_8 = C$, $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, and $\mu = \alpha_2 + s$. Thus, we conclude that $v \in C^\alpha(\overline{\Omega})$ and $[v]_{C^\alpha(\overline{\Omega})} \leqslant C$, which by (5.11) implies $\|v\|_{C^\alpha(\overline{\Omega})} \leqslant C$, for $\alpha \in]0, s]$ and $C > 0$ only depending on N, p, s, and Ω. □

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