J. Math. Anal. Appl. 491 (2020) 124370

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the existence of weak solutions to a class of nonlinear elliptic systems with drift term

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ARTICLE INFO

Article history:
Received 23 April 2020
Available online 10 July 2020
Submitted by G. Mingione

Keywords:
Existence
Weak solutions
Nonlinear elliptic systems
Drift term

ABSTRACT

We study the existence of a weak solution u of the following nonlinear vectorial Dirichlet problem

$$\begin{cases} u \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ -\sum_{i=1}^{n} \mathrm{D}_i A_i^\nu(x, u, \mathrm{D}u) = -\sum_{i=1}^{n} \mathrm{D}_i\big(\sum_{j=1}^{N} E_i^{\nu j}(x) u^j\big) + f^\nu(x) \quad x \in \Omega \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^n$, $n \geq 3$, u^ν and f^ν, for any $\nu = 1, 2, \ldots, N$, are the $\nu-$th components of the vectors u and f, respectively, and the tensors $A(x, s, \xi)$ and $E(x)$ satisfy suitable structural assumptions.

1. Introduction

This paper is devoted to the study of the existence of a weak solution u of the following nonlinear vectorial Dirichlet problem

$$\begin{cases} u \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ -\sum_{i=1}^{n} \mathrm{D}_i A_i^\nu(x, u, \mathrm{D}u) = -\sum_{i=1}^{n} \mathrm{D}_i\big(\sum_{j=1}^{N} E_i^{\nu j}(x) u^j\big) + f^\nu(x) \quad x \in \Omega \end{cases} \tag{1}$$

where, $N \geq 2$, Ω is a bounded subset of $\mathbb{R}^n$, $n \geq 3$, u^ν and f^ν, for any $\nu = 1, 2, \ldots, N$, are the $\nu-$th components of the vectors u and f respectively and the tensors

$$A(x, s, \xi) = (A_i^\nu(x, s, \xi))_{i=1,\ldots,n;\ \nu=1,\ldots,N}\,,$$

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https://doi.org/10.1016/j.jmaa.2020.124370
0022-247X/

$$E(x) = \left(E_i^{\nu j}\right)_{i=1,\ldots,n;\ j,\nu=1,\ldots,N}$$

satisfy suitable structural assumptions that will be precised in the next section.

The study of the above problem, in the linear scalar case $N = 1$, goes back to the papers [51,52] by G. Stampacchia and it presents a difficulty due to the noncoercivity of the differential operator $u \to -\text{div}[A(x)\nabla u - E(x)u]$. In the quoted papers the existence, uniqueness and regularity of a weak solution $u \in W_0^{1,2}(\Omega)$ have been proved assuming that

$$|E| \in L^n(\Omega), \text{ with } \|E\|_{L^n(\Omega)} \text{ sufficiently small,} \tag{2}$$

and

$$f \in L^m(\Omega), \quad \text{with } m \geq \frac{2n}{n+2}. \tag{3}$$

Later, using a nonlinear approach and exploiting techniques issued from those of G. Stampacchia, L. Boccardo in [2] retrieved the previous results without the smallness condition on $||E||_{L^n(\Omega)}$.

In previous two papers by the authors [10,11], again in the case of one single equation, the question of the local regularity of a weak solution had been taken into consideration and solved.

In this paper we want to take into account the case of a system of non coercive equations, i.e. $N \geq 2$, to prove the existence of a weak solution.

It is worth to point out that, in the case of vectorial problems, many consolidated techniques working for one single equation do not apply anymore as well as the choice of some usual test functions.

To overcome this issue, it has been necessary to introduce an "ad hoc" structural condition for first order term (drift term). This condition recalls back the "Landes condition" used for the principal part of the operator (see below) and still keeps the operator non coercive. This new condition allows us to adapt to our purposes the ideas contained in papers [30,1,2] and enables us to prove the existence of at least a weak solution for the problem.

There remain open the problems weather this existence result could be extended to measure right-hand sides or if the G. Mingione's Calderon–Zygmund theory on fractional differentiability could be applied to this operator (see for instance [48–50,24–28]).

In the framework of regularity theory of weak solutions the reader can also refer to the following papers [6–9,16,35,37,21,23,40,41,3,4,12–15,17–20,22,29,31–34,36,38,39,42–46].

2. Main notations, functions spaces and auxiliary lemmas

Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 3$ and $A(x, s, \xi)$ be a matrix-valued function whose entries are the functions

$$A_i^\nu : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$$

for $i = 1, ..., n$ and $\nu = 1, ..., N$ with $N \geq 2$.

We assume that each entry is a Carathéodory function (i.e. continuous in (s, ξ) for a.e. $x \in \Omega$ and measurable in x for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$) satisfying the following conditions for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^N$, $\xi, \eta \in \mathbb{R}^{nN}$ such that $\xi \neq \eta$ ([1]):

$$\exists \Lambda_1 > 0 : (A_i^\nu(x, s, \xi) - A_i^\nu(x, s, \eta))(\xi_i^\nu - \eta_i^\nu) \geq \Lambda_1 |\xi - \eta|^2, \tag{4}$$

[1] We assume the use of Einstein's convention over repeated indices throughout the paper.

$$\exists \Lambda_2 > 0 : |A(x,s,\xi)| \le \Lambda_2 \left[|s|^q + |\xi|\right], \quad \text{with } q \le \frac{n}{n-2} \tag{5}$$

$$A_i^\nu(x,s,0) = 0, \tag{6}$$

$$A_i^\nu(x,s,\xi)\left[\xi_i^\nu|\gamma|^2 - \gamma^\nu\gamma^\mu\xi_i^\mu\right] \ge 0, \tag{7}$$

for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and $\forall \gamma \in \mathbb{R}^N$.

Remark 2.1.

- The previous assumption (7) has been originally used by R. Landes in [30] and it is known as "Landes condition". It is automatically satisfied if $N = 1$.
- Whenever $N \ge 2$, examples of operators satisfying condition (7) can be found in the papers [30] and [1].

We denote by $E(x)$ a tensor-valued function in $\mathbb{R}^{nN^2}$ whose entries are the measurable functions

$$E_i^{\nu j} : \Omega \to \mathbb{R}$$

such that, for $i = 1, ..., n$ and $\nu, j = 1, ..., N$, the following condition holds:

$$E_i^{\nu j}(x)\gamma^j\left[\xi_i^\nu|\gamma|^2 - \gamma^\nu\gamma^\mu\xi_i^\mu\right] \le 0, \tag{8}$$

$\forall \gamma \in \mathbb{R}^N$, for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN}$.

Throughout the paper we will briefly denote by the symbol $||E||$ the norm in $\mathbb{R}^{nN^2}$ of the tensor E.

Remark 2.2.

- Condition (8) is automatically satisfied if $N = 1$.
- The tensor $E_i^{\nu j}(x) = E_i(x)\,\delta^{\nu j}$, where $\delta^{\nu j}$ is the Kronecker delta and $E_i : \Omega \to \mathbb{R}$ is a measurable function, is an example of tensor readily verifying condition (8) for any $N \ge 1$.
- The introduction of the structural condition (8), coupled with the classical Landes condition (7), allows us to use an appropriate truncation for vector–valued functions as test function in the weak formulation of our problem.

Given a vector–valued function $u = (u^\nu)_{\nu=1,2,...,N}$, $\mathrm{D}u$ denotes its gradient, that is

$$\mathrm{D}u \equiv \left(\frac{\partial u^\nu}{\partial x_i}\right)_{\nu=1,2,...,N;\, i=1,2,...,n} \equiv \left(\mathrm{D}_i u^\nu\right)_{\nu=1,2,...,N;\, i=1,2,...,n}.$$

We will define a truncation for vector–valued functions (see [30]) as follows.

Given a real number $\theta > 0$, let us denote by $T_\theta(s)$ the vector–valued function whose ν–th components are defined by

$$[T_\theta(s)]^\nu = \begin{cases} s^\nu & \text{if } |s| \le \theta \\ \theta\dfrac{s^\nu}{|s|} & \text{if } |s| > \theta \end{cases} \tag{9}$$

for $\nu = 1, ..., N$. Obviously

$$|T_\theta(s)| \le \theta, \quad |T_\theta(s)| \le |s| \quad \forall s \in \mathbb{R}^N, \ \forall \theta > 0.$$

Moreover, see [30], if $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ then $T_\theta(u) \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ and for any $i = 1, ..., n$ and $\nu = 1, ..., N$ we have

$$D_i[T_\theta(u)]^\nu = \begin{cases} D_i u^\nu & \text{if } |u| \le \theta \\ \dfrac{\theta}{|u|}\left[D_i u^\nu - \dfrac{1}{|u|^2} u^\nu u^\mu D_i u^\mu\right] & \text{if } |u| > \theta. \end{cases} \tag{10}$$

Bearing in mind the Einstein's convention over the repeated indices, we shall consider the following system

$$\begin{cases} u \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ -\mathrm{D}_i A_i^\nu(x, u, \mathrm{D}u) = -\mathrm{D}_i\big(E_i^{\nu j}(x) u^j\big) + f^\nu(x) \quad x \in \Omega \end{cases} \tag{11}$$

where, for any $\nu = 1, 2, \ldots, N$, f^ν is the ν–th component of the vector f with

$$f \in L^{\frac{2n}{n+2}}(\Omega, \mathbb{R}^N) \tag{12}$$

and

$$E \in L^n(\Omega, \mathbb{R}^{nN^2}). \tag{13}$$

Definition 2.3. A weak solution of the system (11) is a vector–function $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that the following integral identity

$$\int_\Omega A_i^\nu(x, u, \mathrm{D}u)\mathrm{D}_i\varphi^\nu \,\mathrm{d}\,x = \int_\Omega E_i^{\nu j}(x)\, u^j(x)\, \mathrm{D}_i\varphi^\nu \,\mathrm{d}\,x + \int_\Omega f^\nu(x)\,\varphi^\nu \,\mathrm{d}\,x \tag{14}$$

holds for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Remark 2.4. Observe that for some constants $c_0, c_1 > 0$, one has

$$|Eu| \le c_0|E|^{\frac{n}{2}} + c_1|u|^{\frac{2^*}{2}}$$

so that our lower order term lies in the framework of "controlli limite" as described in Campanato's book [5] pages 122 and 125.

We can now state our existence result whose proof is postponed in the next section.

Theorem 2.5. *Assume that hypotheses (4)–(8), (12) and (13) hold.*
Then there exists a weak solution $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ of the system (11).

In order to obtain the claimed result at first we need to consider, for every $k \in \mathbb{N}$, the following approximating problems

$$\begin{cases} u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N) \\ -\mathrm{D}_i A_i^\nu(x, u_k, \mathrm{D}u_k) = -\mathrm{D}_i\left(\dfrac{E_i^{\nu j}(x)}{1 + \frac{1}{k}||E||} \cdot \dfrac{u_k^j}{1 + \frac{1}{k}|u_k|}\right) + f^\nu(x) \quad x \in \Omega \end{cases} \tag{15}$$

for any $\nu, j = 1, 2, \ldots, N$.

Definition 2.6. A weak solution of problem (15) is a vector–function $u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that the following integral identity

$$\int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\mathrm{D}_i\varphi^\nu \,\mathrm{d}\,x = \int_\Omega \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\,\mathrm{D}_i\varphi^\nu \,\mathrm{d}\,x + \int_\Omega f^\nu\,\varphi^\nu\,\mathrm{d}\,x \tag{16}$$

holds for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Note that for every fixed $k \in \mathbb{N}$, it holds

$$\left|\frac{E_i^{\nu j}(x)}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j(x)}{1+\frac{1}{k}|u_k|}\right| \le k^2 \quad \text{for a.a. } x \in \Omega,$$

therefore a weak solution u_k of (15) exists thanks to Schauder fixed point theorem.

Given $\theta > 0$, let us denote by $A_k(\theta)$ the level sets of $|u_k|$, that is

$$A_k(\theta) = \big\{x \in \Omega \,:\, |u_k(x)| > \theta\big\},$$

and by $|A_k(\theta)|$ its n–dimensional Lebesgue measure.

In the next Lemma we will prove that, for every $k \in \mathbb{N}$, the measure of $A_k(\theta)$ is small enough for sufficiently large θ.

We make the reader aware that in the sequel we will denote by c various positive constants depending only on the known data and whose values may vary from line to line.

Lemma 2.7. *Assume (4), (6)–(8), $E \in L^2(\Omega, \mathbb{R}^{nN^2})$ and $f \in L^1(\Omega, \mathbb{R}^N)$. Let u_k be a weak solution of the problem (15). Then, for any $\varepsilon > 0$ there exists $\theta_\varepsilon > 0$, independent of k, such that*

$$\big|A_k(\theta_\varepsilon)\big|^{\frac{2}{2^*}} \le \varepsilon. \tag{17}$$

Proof. For every fixed $k \in \mathbb{N}$, let us choose as test function in (15) the vector–valued function φ_k whose entries are

$$\varphi_k^\nu = \frac{u_k^\nu}{1+|u_k|}, \quad \nu = 1, 2, \ldots, N,$$

and whose derivatives are

$$\mathrm{D}_i\varphi_k^\nu = \frac{\mathrm{D}_i u_k^\nu}{(1+|u_k|)^2} + \frac{1}{|u_k|}\left[\frac{\mathrm{D}_i u_k^\nu\,|u_k|^2 - u_k^\nu\,u_k^\mu\,\mathrm{D}_i u_k^\mu}{(1+|u_k|)^2}\right], \quad i = 1, \ldots, n;\ \nu = 1, \ldots, N.$$

Thanks to (4) and (7), we have

$$\begin{aligned}\Lambda_1\int_\Omega \frac{|\mathrm{D}u_k|^2}{(1+|u_k|)^2}\,\mathrm{d}\,x \quad &\le \int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\frac{\mathrm{D}_i u_k^\nu}{(1+|u_k|)^2}\,\mathrm{d}\,x \le \int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\mathrm{D}_i\varphi_k^\nu\,\mathrm{d}\,x \\ &= \int_\Omega \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\mathrm{D}_i\varphi_k^\nu\,\mathrm{d}\,x + \int_\Omega f^\nu\varphi_k^\nu\,\mathrm{d}\,x\,.\end{aligned} \tag{18}$$

By virtue of (8) and taking into account that $f^\nu \le |f|$ and $|\varphi_k^\nu| \le 1$ for every $\nu = 1, 2, \ldots, N$, we can estimate the right–hand side of (18) as follows

$$\int_\Omega \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\mathrm{D}_i\varphi_k^\nu\,\mathrm{d}\,x+\int_\Omega f^\nu\varphi_k^\nu\,\mathrm{d}\,x$$
$$\le \int_\Omega ||E||\frac{|u_k|}{1+|u_k|}\frac{|Du_k|}{1+|u_k|}\,\mathrm{d}\,x$$
$$+\int_\Omega \frac{1}{|u_k|}E_i^{\nu j}\,u_k^j\left[\frac{\mathrm{D}_i u_k^\nu\,|u_k|^2-u_k^\nu\,u_k^\mu\,\mathrm{D}_i u_k^\mu}{(1+|u_k|)^2}\right]\mathrm{d}\,x+\int_\Omega |f|\,\mathrm{d}\,x$$
$$\le \int_\Omega ||E||\,\frac{|Du_k|}{1+|u_k|}\,\mathrm{d}\,x+\int_\Omega |f|\,\mathrm{d}\,x\,. \tag{19}$$

By using Young's inequality in the right–hand side of (19), from (18), for any $\varepsilon>0$, it follows

$$\Lambda_1\int_\Omega \frac{|\mathrm{D}u_k|^2}{(1+|u_k|)^2}\,\mathrm{d}\,x\le\varepsilon\int_\Omega\frac{|\mathrm{D}u_k|^2}{(1+|u_k|)^2}\,\mathrm{d}\,x+\frac{1}{4\varepsilon}\int_\Omega ||E||^2\,\mathrm{d}\,x+\int_\Omega |f|\,\mathrm{d}\,x\,. \tag{20}$$

Applying Sobolev's embedding inequality and choosing a suitable $0<\varepsilon<\Lambda_1$, (20) implies[2]

$$\begin{aligned}\Big(\int_\Omega \big|\log(1+|u_k|)\big|^{2^*}\,\mathrm{d}\,x\Big)^{\frac{2}{2^*}}&\le\mathcal{S}\int_\Omega\big|\mathrm{D}\log(1+|u_k|)\big|^2\,\mathrm{d}\,x\\ \le\ \ \mathcal{S}\int_\Omega\frac{|\mathrm{D}u_k|^2}{(1+|u_k|)^2}\,\mathrm{d}\,x&\le c\left[\int_\Omega ||E||^2\,\mathrm{d}\,x+\int_\Omega|f|\,\mathrm{d}\,x\right].\end{aligned} \tag{21}$$

Therefore, given $\theta>0$ it holds

$$|\{x\in\Omega\,:\,|u_k(x)|>\theta\}|^{\frac{2}{2^*}}\le\frac{c}{\log^2(1+\theta)}\int_\Omega\left[||E||^2+|f|\right]\mathrm{d}\,x \tag{22}$$

which implies (17). □

Lemma 2.8. *Assume* (4), (6)–(8), (12) *and* (13).

Then, there exist two positive constants c and θ_0, independent of k, such that

$$\int_\Omega |\mathrm{D}u_k|^2\,\mathrm{d}\,x\le c\Big[\theta_0^2\int_\Omega\left[||E||^n+1\right]\mathrm{d}\,x+\Big(\int_\Omega|f|^{\frac{2n}{n+2}}\,\mathrm{d}\,x\Big)^{\frac{n+2}{n}}\Big] \tag{23}$$

for every $k\in\mathbb{N}$.

Proof. For a fixed $k\in\mathbb{N}$, we pick u_k as test function in (16). Thanks to the ellipticity condition we have

$$\Lambda_1\int_\Omega|\mathrm{D}u_k|^2\,\mathrm{d}\,x\le\int_\Omega\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\mathrm{D}_i u_k^\nu\,\mathrm{d}\,x+\int_\Omega f_k^\nu\,u_k^\nu\,\mathrm{d}\,x\,. \tag{24}$$

[2] $\mathcal{S}$ represents the Sobolev embedding constant.

By using Hölder's, Sobolev's and Young's inequalities, for any $\varepsilon > 0$, we have

$$\int_\Omega f_k^\nu\, u_k^\nu \,\mathrm{d}\,x \le \mathcal{S}\Big(\int_\Omega |f|^{\frac{2n}{n+2}} \,\mathrm{d}\,x\Big)^{\frac{n+2}{2n}} \Big(\int_\Omega |\mathrm{D}u_k|^2 \,\mathrm{d}\,x\Big)^{\frac{1}{2}}$$

$$\le \varepsilon \int_\Omega |\mathrm{D}u_k|^2 \,\mathrm{d}\,x + \mathcal{S}^2 \frac{1}{4\varepsilon}\Big(\int_\Omega |f|^{\frac{2n}{n+2}} \,\mathrm{d}\,x\Big)^{\frac{n+2}{n}}. \tag{25}$$

Applying again Young's inequality, for any $\varepsilon > 0$ we deduce that

$$\int_\Omega \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||} \cdot \frac{u_k^j}{1+\frac{1}{k}|u_k|} \mathrm{D}_i u_k^\nu \,\mathrm{d}\,x \le \varepsilon \int_\Omega |Du_k|^2 \,\mathrm{d}\,x + \frac{1}{4\varepsilon} \int_\Omega ||E||^2\, |u_k|^2 \,\mathrm{d}\,x\,. \tag{26}$$

On the other hand, for any $\theta > 0$ fixed, by using Hölder's and Sobolev's inequality we have

$$\int_\Omega ||E||^2\, |u_k|^2 \,\mathrm{d}\,x = \int_{\Omega\cap\{|u_k|>\theta\}} ||E||^2\, |u_k|^2 \,\mathrm{d}\,x + \int_{\Omega\cap\{|u_k|\le\theta\}} ||E||^2\, |u_k|^2 \,\mathrm{d}\,x \le$$

$$\le \theta^2 \int_\Omega ||E||^2 \,\mathrm{d}\,x + \mathcal{S}^2 \Big(\int_{\Omega\cap\{|u_k|>\theta\}} ||E||^n \,\mathrm{d}\,x\Big)^{\frac{2}{n}} \int_\Omega |\mathrm{D}u_k|^2 \,\mathrm{d}\,x\,. \tag{27}$$

Gathering together (25)–(27), from (24), we obtain

$$\Big[\Lambda_1 - 2\varepsilon - \frac{\mathcal{S}^2}{4\varepsilon}\Big(\int_{\Omega\cap\{|u_k|>\theta\}} ||E||^n \,\mathrm{d}\,x\Big)^{\frac{2}{n}}\Big] \int_\Omega |\mathrm{D}u_k|^2 \,\mathrm{d}\,x \le$$

$$\le \frac{\theta^2}{4\varepsilon} \int_\Omega ||E||^2 \,\mathrm{d}\,x + \frac{\mathcal{S}^2}{4\varepsilon}\Big(\int_\Omega |f|^{\frac{2n}{n+2}} \,\mathrm{d}\,x\Big)^{\frac{n+2}{n}} \quad \text{for any } \theta > 0 \text{ fixed.} \tag{28}$$

Let us choose in (28) $\varepsilon = \frac{\Lambda_1}{8}$ and $\theta = \theta_0$ with $\theta_0 > 0$, independent of k, such that

$$\frac{\mathcal{S}^2}{4\varepsilon}\Big(\int_{\Omega\cap\{|u_k|>\theta_0\}} ||E||^n \,\mathrm{d}\,x\Big)^{\frac{2}{n}} \le \frac{\Lambda_1}{4}.$$

We point out that the existence of θ_0 follows from Lemma 2.7. Then, we deduce

$$\int_\Omega |\mathrm{D}u_k|^2 \,\mathrm{d}\,x \le c\left[\theta_0^2 \int_\Omega \big[||E||^n + 1\big] \,\mathrm{d}\,x + \Big(\int_\Omega |f|^{\frac{2n}{n+2}} \,\mathrm{d}\,x\Big)^{\frac{n+2}{n}}\right] \tag{29}$$

with the constant c independent of k. □

3. Proof of the Theorem 2.5

From Lemma 2.8 there exists a positive constant c, independent of k, such that

$$||u_k||_{W_0^{1,2}(\Omega)} \le c \quad \text{for any } k \in \mathbb{N}, \tag{30}$$

hence we can pick a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } W_0^{1,2}(\Omega, \mathbb{R}^N), \\ \\ u_k \to u & \text{strongly in } L^2(\Omega, \mathbb{R}^N) \text{ and a.e. in } \Omega. \end{cases} \tag{31}$$

Moreover, estimate (30) implies that

$$||T_\theta(u_k)||_{W_0^{1,2}(\Omega)} \le c \quad \forall k \in \mathbb{N} \tag{32}$$

for any $\theta > 0$ fixed and by (31) we also deduce

$$\begin{cases} T_\theta(u_k) \rightharpoonup T_\theta(u) & \text{weakly in } W_0^{1,2}(\Omega, \mathbb{R}^N), \\ \\ T_\theta(u_k) \to T_\theta(u) & \text{strongly in } L^2(\Omega, \mathbb{R}^N) \text{ and a.e. in } \Omega. \end{cases}$$

Now, we will prove that u is a solution of the system (11) in the sense of Definition 2.3.

To this aim, we have to pass to the limit in the approximating problems (15) and, therefore, we need to prove that

$$\mathrm{D}u_k \to \mathrm{D}u \quad \text{a.e. in } \Omega \text{ as } k \to +\infty.$$

The above convergence will be a consequence of the following Lemma whose proof can be found in the paper [1]. For the reader's convenience we will give its proof in the Appendix.

Lemma 3.1. *Let the assumptions (4)–(7) be satisfied and let $\{u_k\} \subset W_0^{1,2}(\Omega, \mathbb{R}^N)$ be a sequence such that*

$$u_k \to u$$

weakly in $W_0^{1,2}(\Omega, \mathbb{R}^N)$, strongly in $L^2(\Omega, \mathbb{R}^N)$ and a.e. in Ω.

Assume that the following estimate

$$\begin{aligned} &\int_\Omega \left[A_i^\nu(x, u_k, \mathrm{D}u_k) - A_i^\nu(x, u_k, \mathrm{D}u)\right]\left[\mathrm{D}_i u_k^\nu - \mathrm{D}_i u^\nu\right] \chi_{\{|u_k - u| \le \theta\}} \, \mathrm{d}\, x \\ \le \quad & c\,\theta + \omega(k, \theta) \end{aligned} \tag{33}$$

holds[3]*, where c is a positive constant and $\omega(k, \theta)$ is a function such that*

$$\lim_{k \to +\infty} \omega(k, \theta) = 0 \qquad \textit{for any } \theta > 0 \textit{ fixed.}$$

Then $u_k \to u$ strongly in $W_0^{1,2}(\Omega, \mathbb{R}^N)$.

In view of exploiting Lemma 3.1, for any fixed $\theta > 0$ we set

$$J_{k,\theta} = \int_\Omega \left[A_i^\nu(x, u_k, \mathrm{D}u_k) - A_i^\nu(x, u_k, \mathrm{D}u)\right]\left[\mathrm{D}_i (u_k - u)^\nu\right] \chi_{\{|u_k - u| \le \theta\}} \, \mathrm{d}\, x\,. \tag{34}$$

[3] We denote by $\chi_B(x)$ the characteristic function of the set $B \subset \mathbb{R}^n$.

It's easy to prove that, for fixed $\theta > 0$,

$$\int_\Omega A_i^\nu(x, u_k, \mathrm{D}u)\left[\mathrm{D}_i u_k^\nu - \mathrm{D}_i u^\nu\right]\chi_{\{|u_k-u|\le\theta\}}\,\mathrm{d}\,x \to 0 \text{ as } k \to +\infty. \tag{35}$$

Indeed,

$$A_i^\nu(x, u_k, \mathrm{D}u)\chi_{\{|u_k-u|\le\theta\}} \to A_i^\nu(x, u, \mathrm{D}u) \text{ a.e. in } \Omega.$$

Moreover, if B is a measurable subset of Ω, applying the growth condition (5), we have

$$\begin{aligned}
&\left(\int_B |A_i^\nu(x, u_k, \mathrm{D}u)|^2\chi_{\{|u_k-u|\le\theta\}}\,\mathrm{d}\,x\right)^{\frac{1}{2}}\\
&\le \Lambda_2\left(\int_{B\cap\{|u_k-u|\le\theta\}}\left[|u_k-u|^q + |u|^q + |\mathrm{D}u|\right]^2\mathrm{d}\,x\right)^{\frac{1}{2}}\\
&\le \Lambda_2\left[\theta|B|^{\frac{1}{2q}} + \left(\int_B |u|^{2q}\,\mathrm{d}\,x\right)^{\frac{1}{2q}}\right]^q + \Lambda_2\left(\int_B |\mathrm{D}u|^2\,\mathrm{d}\,x\right)^{\frac{1}{2}}.
\end{aligned} \tag{36}$$

consequently the functions $A_i^\nu(x, u_k, \mathrm{D}u)\chi_{\{|u_k-u|\le\theta\}}$ are equi–integrable in $L^2(\Omega)$ and therefore

$$A_i^\nu(x, u_k, \mathrm{D}u)\chi_{\{|u_k-u|\le\theta\}} \to A_i^\nu(x, u, \mathrm{D}u) \quad \text{strongly in } L^2(\Omega)$$

and assertion (35) is proved.

Now we observe that we can rewrite

$$\begin{aligned}
&\int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\,\mathrm{D}_i\left(u_k - u\right)^\nu\chi_{\{|u_k-u|\le\theta\}}\,\mathrm{d}\,x\\
&= \int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\,\mathrm{D}_i T_\theta\left(u_k - u\right)^\nu\,\mathrm{d}\,x\\
&\quad - \int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\,\mathrm{D}_i T_\theta(u_k - u)^\nu\chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x.
\end{aligned} \tag{37}$$

Testing (16) with the function $T_\theta(u_k - u)$ we obtain

$$\begin{aligned}
&\int_\Omega A_i^\nu(x, u_k, \mathrm{D}u_k)\mathrm{D}_i\left[T_\theta(u_k-u)\right]^\nu\,\mathrm{d}\,x\\
&\le \int_{\Omega\cap\{|u_k-u|\le\theta\}} \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\mathrm{D}_i(u_k-u)^\nu\,\mathrm{d}\,x\\
&\quad + \int_{\Omega\cap\{|u_k-u|>\theta\}} \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\Big[\mathrm{D}_i(u_k-u)^\nu\\
&\quad - \frac{(u_k-u)^\nu(u_k-u)^\mu\mathrm{D}_i(u_k-u)^\mu}{|u_k-u|^2}\Big]\,\mathrm{d}\,x + \theta||f||_{L^{\frac{2n}{n+2}}(\Omega)}\\
&\equiv I_k^1 + I_k^2 + \theta||f||_{L^{\frac{2n}{n+2}}(\Omega)}.
\end{aligned} \tag{38}$$

We claim that

$$\lim_{k\to+\infty}(I_k^1+I_k^2)=0 \quad \text{for any } \theta>0 \text{ fixed.} \tag{39}$$

Thanks to the following inequality

$$\left|\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\chi_{\{|u_k-u|\le\theta\}}\right|\le ||E||(\theta+|u|) \quad \text{for any } k\in\mathbb{N},$$

by the Lebesgue's dominated convergence Theorem and (31), it follows that

$$\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\chi_{\{|u_k-u|\le\theta\}}\to E_i^{\nu j}\,u^j \quad \text{strongly in } L^2(\Omega)$$

which, in turn, implies that $I_k^1\to 0$ as $k\to+\infty$.

On the other hand, using condition (8), we deduce

$$\begin{aligned}
I_k^2 = &\int\limits_{\Omega\cap\{|u_k-u|>\theta\}} \frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{(u_k-u)^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\Big[\mathrm{D}_i(u_k-u)^\nu \\
&-\frac{(u_k-u)^\nu(u_k-u)^\mu\mathrm{D}_i(u_k-u)^\mu}{|u_k-u|^2}\Big]\,\mathrm{d}\,x \\
&+\int\limits_{\Omega}\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\Big[\mathrm{D}_i(u_k-u)^\nu \\
&-\frac{(u_k-u)^\nu(u_k-u)^\mu\mathrm{D}_i(u_k-u)^\mu}{|u_k-u|^2}\Big]\chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x \\
\le &\int\limits_{\Omega}\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\Big[\mathrm{D}_i(u_k-u)^\nu \\
&-\frac{(u_k-u)^\nu(u_k-u)^\mu\mathrm{D}_i(u_k-u)^\mu}{|u_k-u|^2}\Big]\chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x\,.
\end{aligned} \tag{40}$$

Since $u_k\to u$ a.e. in Ω, it follows

$$\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\chi_{\{|u_k-u|>\theta\}}\to 0 \quad \text{a.e. in } \Omega \text{ as } k\to+\infty.$$

Moreover, the following inequality holds

$$\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\chi_{\{|u_k-u|>\theta\}}\le |E_i^{\nu j}\,u^j| \quad \text{for any } k\in\mathbb{N},$$

so by the Lebesgue's dominated convergence Theorem

$$\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u^j}{1+\frac{1}{k}|u_k|}\frac{\theta}{|u_k-u|}\chi_{\{|u_k-u|>\theta\}}\to 0 \quad \text{strongly in } L^2(\Omega)$$

hence, thanks to (31) the right-hand side of (40) goes to 0, for any $\theta>0$ fixed, and, at least we conclude that $I_k^2\to 0$, for any $\theta>0$ fixed and (39) holds.

On the other hand, using formula (10) and taking into account condition (7) and (30), one deduces that

$$\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u_k)\,\mathrm{D}_i\left[T_\theta(u_k-u)\right]^\nu\ \chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x \to 0 \quad \text{as } k\to+\infty, \tag{41}$$

for any $\theta>0$ fixed.

Indeed, by (7) we have

$$\begin{aligned}
&\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u_k)\,\mathrm{D}_i\left[T_\theta(u_k-u)\right]^\nu\ \chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x\\
&\quad=\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u_k)\,\frac{\theta}{|u_k-u|}\left[D_iu_k^\nu-\frac{(u_k-u)^\nu(u_k-u)^\mu}{|u_k-u|^2}D_iu_k^\mu\right]\chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x\\
&\qquad-\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u_k)\,\frac{\theta}{|u_k-u|}\left[D_iu^\nu-\frac{(u_k-u)^\nu(u_k-u)^\mu}{|u_k-u|^2}D_iu^\mu\right]\chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x\\
&\quad\le-\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u_k)\,\frac{\theta}{|u_k-u|}\left[D_iu^\nu-\frac{(u_k-u)^\nu(u_k-u)^\mu}{|u_k-u|^2}D_iu^\mu\right]\chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x.
\end{aligned} \tag{42}$$

We note that by (5) and (30), for any fixed $i=1,\dots,n$ the sequence $\{A_i^\nu(x,u_k,\mathrm{D}u_k)\}$ converges weakly in $L^2(\Omega)$ while

$$\frac{\theta}{|u_k-u|}\left[D_iu^\nu-\frac{(u_k-u)^\nu(u_k-u)^\mu}{|u_k-u|^2}D_iu^\mu\right]\chi_{\{|u_k-u|>\theta\}}\to 0 \quad \text{strongly in } L^2(\Omega)$$

so the right–hand side of (42) goes to zero.

Finally, from (34), (37) and (38) we obtain the following basic inequality

$$J_{k,\theta}\le c\,\theta+\omega(k,\theta) \tag{43}$$

where ω is the function defined by

$$\begin{aligned}
\omega(k,\theta)=&-\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u)\,\mathrm{D}_i(u_k-u)^\nu\ \chi_{\{|u_k-u|\le\theta\}}\,\mathrm{d}\,x+I_k^1+I_k^2\\
&+\int_\Omega A_i^\nu(x,u_k,\mathrm{D}u_k)\,\mathrm{D}_i\left[T_\theta(u_k-u)\right]^\nu\ \chi_{\{|u_k-u|>\theta\}}\,\mathrm{d}\,x
\end{aligned}$$

which converges to 0, for any $\theta>0$ fixed, by virtue of (35), (39) and (41).

By Lemma 3.1, up to a subsequence still relabeled u_k, it follows that

$$\mathrm{D}u_k\to\mathrm{D}u \quad \text{strongly in } L^2(\Omega,\mathbb{R}^N), \tag{44}$$

$$\mathrm{D}u_k(x)\to\mathrm{D}u(x) \quad \text{a.e in } \Omega. \tag{45}$$

Moreover it's easy to see

$$\frac{E_i^{\nu j}}{1+\frac{1}{k}||E||}\cdot\frac{u_k^j}{1+\frac{1}{k}|u_k|}\to E_i^{\nu j}\,u \quad \text{strongly in } L^2(\Omega). \tag{46}$$

Therefore, for any $\varphi\in W_0^{1,2}(\Omega,\mathbb{R}^N)$, we can pass to the limit in (16) for the above subsequence (for which (44)–(46) hold) and therefore for the limit u the integral identity (14) is satisfied.

4. Appendix

In this section we will reproduce the proof of Lemma 3.1, already contained in [1], pages 151–155.
For any $k \in \mathbb{N}$ and $x \in \Omega$ we set

$$\Delta^k(x) = \left[A_i^\nu(x, u_k, \mathrm{D}u_k) - A_i^\nu(x, u_k, \mathrm{D}u)\right]\left[\mathrm{D}_i u_k^\nu - \mathrm{D}_i u^\nu\right]$$

and we claim that there exists a subsequence, still denoted by $\{\Delta^k(x)\}$, such that

$$\Delta^k(x) \to 0 \quad \text{for a.e. } x \in \Omega. \tag{47}$$

Indeed, let $E \subseteq \Omega$ with $|E| = 0$ such that

$$u_k(x) \to u(x) \quad \forall x \in \Omega \setminus E.$$

For any fixed $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that the set

$$\Omega_{l,m} = \left\{x \in \Omega \setminus E \quad : |u_k(x) - u(x)| < \frac{1}{m}, \quad k \geq l\right\}$$

is nonempty. Moreover

$$\Omega_{l,m} \subseteq \Omega_{l+1,m}$$

and

$$\bigcup_{l=1}^{+\infty} \Omega_{l,m} = \Omega \setminus E$$

which implies

$$\lim_{l \to +\infty} |\Omega - \Omega_{l,m}| = 0.$$

For any $p \in \mathbb{N}$ let

$$l(m,p) = \min\left\{l \in \mathbb{N} \quad : |\Omega - \Omega_{l,m}| \leq \frac{1}{p2^m}\right\}.$$

Then

$$|\Omega - \Omega_{l(m,p),m}| \leq \frac{1}{p2^m}$$

and

$$l(m,p) \leq l(m,p+1).$$

Moreover, if $x \in \Omega_{l(m,p),m}$ then

$$|u_k(x) - u(x)| < \frac{1}{m} \quad \forall k \geq l(m,p).$$

We define

$$\Omega^p = \bigcap_{m=1}^{+\infty} \Omega_{l(m,p),m}.$$

Then

$$\Omega^p \subseteq \Omega^{p+1}$$

and

$$|\Omega \setminus \Omega^p| = |\bigcup_{m=1}^{+\infty} |\Omega \setminus \Omega_{l(m,p),m}| \leq \sum_{m=1}^{+\infty} \frac{1}{p2^m} = \frac{1}{p}.$$

Moreover,

$$\sup_{x\in\Omega^p} |u_k(x) - u(x)| \leq \frac{1}{m} \quad \forall k \geq l(m,p).$$

By virtue of the hypothesis (4), $\Delta^k(x) \geq 0$ a.e. $x \in \Omega$ and using the assumption (33) we obtain

$$\int_{\Omega^p} \Delta^k(x)\,\mathrm{d}\,x \leq \int_{\Omega_{l(m,p),m}} \Delta^k(x)\,\mathrm{d}\,x$$
$$= \int_{\Omega} \Delta^k(x)\chi_{\Omega_{l(m,p),m}}\,\mathrm{d}\,x \leq \frac{c}{m} + \omega\left(k, \frac{1}{m}\right), \quad \forall k \geq l(m,p)$$

which, in turn, implies

$$\lim_{k\to+\infty} \int_{\Omega^p} \Delta^k(x)\,\mathrm{d}\,x = 0 \quad \forall p \in \mathbb{N}. \tag{48}$$

Since $\Delta^k(x) \to 0$ in $L^1(\Omega^1)$, there exist a subsequence $\{n_k^1\}_k \subseteq \{k\}$ and a subset $E^1 \subseteq \Omega^1$ with $|E^1| = 0$ such that

$$\Delta^{n_k^1}(x) \to 0 \quad \forall x \in \Omega^1 \setminus E^1.$$

Since

$$\lim_{k\to+\infty} \int_{\Omega^2} \Delta^{n_k^1}(x)\,\mathrm{d}\,x = 0$$

then $\Delta^{n_k^1}(x) \to 0$ in $L^1(\Omega^2)$ and, therefore, there exists a subsequence $\{n_k^2\} \subseteq \{n_k^1\}$, with $n_k^2 \geq n_k^1$ $\forall k \in \mathbb{N}$, and a subset $E^2 \subseteq \Omega^2$ with $|E^2| = 0$ such that

$$\Delta^{n_k^2}(x) \to 0 \quad \forall x \in \Omega^2 \setminus E^2.$$

Moreover,

$$\Delta^{n_k^2}(x) \to 0 \quad \forall x \in \Omega^1 \setminus E^1$$

because of $\left\{\Delta^{n_k^2}(x)\right\}$ is a subsequence of $\left\{\Delta^{n_k^1}(x)\right\}$. Iterating this procedure, for any $p \in \mathbb{N}$ we construct a subsequence $\{n_k^{p+1}\} \subseteq \{n_k^p\}$, with $n_k^{p+1} \geq n_k^p$ $\forall k \in \mathbb{N}$, and a subset $E^p \subset \Omega^p$ with $|E^p| = 0$ such that

$$\lim_{k\to+\infty} \Delta^{n_k^p}(x) \to 0 \quad \forall x \in \bigcup_{j=1}^{p} \left[\Omega^j \setminus E^j\right]$$

We define

$$\tilde{\Omega}^p = \bigcup_{j=1}^{p} \left[\Omega^j \setminus E^j\right] \quad \forall p \in \mathbb{N}$$

then

$$\lim_{k\to+\infty} \Delta^{n_k^p}(x) \to 0 \quad \forall x \in \tilde{\Omega}^p. \tag{49}$$

Moreover

$$\Omega \setminus \tilde{\Omega}^p \subseteq [\Omega \setminus \Omega^p] \cup E^p \quad \forall p \in \mathbb{N}$$

and

$$|\Omega \setminus \tilde{\Omega}^p| \leq \frac{1}{p}, \quad \forall p \in \mathbb{N}.$$

Therefore, if we set

$$\tilde{\Omega} = \bigcup_{p=1}^{+\infty} \tilde{\Omega}^p$$

then

$$|\Omega \setminus \tilde{\Omega}| = \left| \bigcap_{p=1}^{+\infty} \left[\Omega \setminus \tilde{\Omega}^p\right] \right| = \lim_{p\to+\infty} |\Omega \setminus \tilde{\Omega}^p| = 0$$

and, up to a subsequence still denoted by $\left\{\Delta^k(x)\right\}$, we have

$$\lim_{k\to+\infty} \Delta^k(x) = 0 \quad \forall x \in \tilde{\Omega}. \tag{50}$$

Indeed, given $x \in \tilde{\Omega}$, let p_x be such that $x \in \tilde{\Omega}^{p_x}$. Using (49) we deduce that

$$\lim_{k\to+\infty} \Delta^{n_k^{p_x}}(x) = 0$$

hence, for any $\varepsilon > 0$ there exists $k(x, \varepsilon)$ such that

$$0 \leq \Delta^{n_k^{p_x}}(x) \leq \varepsilon \quad \forall k \geq k(x, \varepsilon). \tag{51}$$

Let

$$\tilde{k} = \max\{k(x, \varepsilon), p_x\}$$

if $k \geq \tilde{k}$ then $k \geq p_x$ and $\{n_k^k\} \subseteq \{n_k^{p_x}\}$. Consequently $\left\{\Delta^{n_k^k}(x)\right\}$ is a subsequence of $\left\{\Delta^{n_k^{p_x}}(x)\right\}$ that satisfies (51) since $k \geq k(x, \varepsilon)$. At least

$$0 \leq \Delta^{n_k^k}(x) \leq \varepsilon \quad \forall k \geq \tilde{k}$$

and if we set $\Delta^k(x) = \Delta^{n_k^k}(x)$ we get (50) and conclude that (47) holds. Once (47) is achieved the strong convergence of $\{Du_k\}$ to Du will be proved following a standard argument due to Leray and Lions [47].

Acknowledgments

This work has been supported by “Programma Ricerca di Ateneo UNICT 2020-22 linea di intervento 2”.

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