Two positive solutions for a nonlinear Neumann problem involving the discrete $p-$Laplacian

G. Bonanno, P. Candito and G. D'Aguì

Abstract This paper is devoted to study of existence of at least two positive solutions for a nonlinear Neumann boundary value problem involving the discrete p-Laplacian.

Key words: Multiple solutions, Difference equations, Neumann problem [1]

1 Introduction

In this paper, we investigate the existence of two positive solutions for the following nonlinear discrete Neumann boundary value problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) = \lambda f(k, u(k)), \quad k \in [1, N], \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \qquad (N_{\lambda,\underline{f}})$$

where λ is a positive parameter, N is a fixed positive integer, $[0, N+1]$ is the discrete interval $\{0, ..., N+1\}$, $\phi_p(s) := |s|^{p-2}s$, $1 < p < +\infty$ and for all $k \in [0, N+1]$, $q(k) > 0$, $\Delta u(k) := u(k+1) - u(k)$ denotes the forward difference operator and

G. Bonanno
Department of Engineering, University of Messina, Contrada Di Dio, (S. Agata), 98166 Messina, Italy, e-mail: bonanno@unime.it

P. Candito
Department DICEAM, University of Reggio Calabria, Via Graziella (Feo Di Vito), 89122 Reggio Calabria, Italy e-mail: pasquale.candito@unirc.it

G. D'Aguì
Department of Engineering, University of Messina, Contrada Di Dio, (S. Agata), 98166 Messina, Italy, e-mail: dagui@unime.it

[1] *2010 Mathematics Subject Classification:* 39A10; 39A12; 34B15.

$f : [0, N+1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The theory of difference equations employs numerical analysis, fixed point methods, upper an lower solutions methods (see, for instance, [4, 6, 8, 25]). The variational approach represents an important advance as it allows to prove multiplicity results, usually, under a suitable condition on the nonlinearities, see [2, 3, 8, 9, 10, 11, 12, 15, 16, 17, 18, 21, 22, 23, 26, 28].

In the present paper, we study the problem $(N_{\lambda,f})$ following a variational approach, based on a recent result of Bonanno and D'Aguì (see [7]), that assures the existence of at least two non trivial critical points for a certain class of functionals defined on infinite-dimensional Banach space. This theorem is obtained by combining a local minimum result given in [14], together with the Ambrosetti-Rabinowitz theorem (see [5]). In the application of the mountain pass theorem, to prove the Palais-Smale condition of the energy functional associated to the nonlinear differential problems, the Ambrosetti-Rabinowitz condition is requested on the nonlinear term, in particular this means that the nonlinear term has to be more than $p-$superlinear at infinity.
In this paper, exploiting that the variational framework of the problem $(N_{\lambda,f})$ is defined in a finite-dimensional space, we prove that the $p-$superlinearity at infinity of the primitive on the nonlinearity is enough to prove the Palais-Smale condition. For a complete overview on variational methods on finite Banach spaces and discrete problems, see [13]. We obtain, here, Theorem (2), which gived the existence of two positive solutions, by requirin an algebraic condition on the nonlinearity (we mean (6) in 2).
The paper is so organized: Section 2, contains basic definitions and main results on difference equations and some critical point tools, in addition, Lemma 2 is given in order to prove the Palais-Smale condition of the functional associated to problem $(N_{\lambda,f})$. Section 3 is devoted to our main result. In particular, our main theorem allows us to obtain two positive solutions with only one hypothesis on the primitive of the nonlinear term f without any asymptotic behaviour at zero. moreover, a consequence (Corollary 1) (requiring the $p-$superlinearity at infinity and the $p-$sublinearity at zero on the primitive of f) of our main result is presented in order to show the applicability of our results.

2 Mathematical Background

In the $N+2$-dimensional Banach space

$$X = \{u : [0, N+1] \to \mathbb{R} : \Delta u(0) = \Delta u(N) = 0\},$$

we consider the norm

$$\|u\| := \left(\sum_{k=1}^{N+1} |\Delta u(k-1)|^p + \sum_{k=1}^{N} q(k)|u(k)|^p\right)^{1/p} \quad \forall u \in X.$$

Moreover, we will use also the equivalent norm

$$\|u\|_\infty := \max_{k\in[0,N+1]} |u(k)|, \quad \forall u \in X.$$

For our purpose, it will be useful the following inequality

$$\|u\|_\infty \le \|u\| q^{-1/p}, \quad \forall u \in X, \quad \text{where} \quad q := \min_{k\in[1,N]} q_k. \tag{1}$$

Moreover, we mention the classical Hölder norm on X.

$$\|u\|_p = \left(\sum_{k=0}^{N+1} |u(k)|^p\right)^{\frac{1}{p}}.$$

We observe that being X a finite dimensional Banach space, all norms defined on it are equivalent and in particular, there exist two positive constants L_1 and L_2 such that

$$L_1\|u\|_p \le \|u\| \le L_2\|u\|_p. \tag{2}$$

To describe the variational framework of problem $(N_{\lambda,\underline{f}})$, we introduce the following two functions

$$\Phi(u) := \frac{\|u\|^p}{p} \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{N} F(k,u(k)), \quad \forall u \in X, \tag{3}$$

where $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t) \in [1,N] \times \mathbb{R}$. Clearly, Φ and Ψ are two functionals of class $C^1(X,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \sum_{k=1}^{N+1} \phi_p\left(\Delta u(k-1)\right)\Delta v(k-1) + q(k)\left|u(k)\right|^{p-2} u(k)\, v(k),$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{N} f\left(k,u(k)\right) v(k),$$

for all u, $v \in X$. Taking into account that

$$-\sum_{k=1}^{N} \Delta(\phi_p(\Delta u(k-1)))v(k) = \sum_{k=1}^{N+1} \phi_p(\Delta u(k-1))\Delta v(k-1), \quad \forall\ u\ v, \in X,$$

it is easy to verify, see also [28], that

Lemma 1. *A vector $u \in X$ is a solution of problem $(N_{\lambda,\underline{f}})$ if and only if u is a critical point of the function $I_\lambda = \Phi - \lambda\Psi$.*

Let $(X, \|\cdot\|)$ be a Banach space and let $I \in C^1(X, \mathbb{R})$. We say that I satisfies the Palais-Smale condition, (in short $(PS)-$condition), if any sequence $\{u_n\}_{n\in\mathbf{N}} \subseteq X$ such that

1. $\{I(u_n)\}_{n\in\mathbf{N}}$ is bounded,
2. $\{I'(u_n)\}_{n\in\mathbf{N}}$ converges to 0 in X^*,

admits a subsequence which is convergent in X.

Here, we recall the abstract result established in [7], on the existence of two non-zero critical points.

Theorem 1. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{4}$$

and, for each

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

Here and in the sequel we suppose $f(k,0) \geq 0$ for all $k \in [1,N]$. We assume that $f(k,x) = f(k,0)$ for all $x < 0$ and for all $k \in [1,N]$. Put

$$L_\infty(k) := \liminf_{s\to+\infty} \frac{F(k,s)}{s^p}, \quad L_\infty := \min_{k\in[1,N]} L_\infty(k).$$

We give the following lemma.

Lemma 2. *If $L_\infty > 0$ then I_λ satisfies $(PS)-$condition and it is unbounded from below for all $\lambda \in \left] \frac{L_2^p}{pL_\infty}, +\infty \right[$, where L_2 is given in (2).*

Proof. Since $L_\infty > 0$ we put $\lambda > \dfrac{L_2^p}{pL_\infty}$ and l such that $L_\infty > l > \dfrac{L_2^p}{p\lambda}$. Let $\{u_n\}$ be a sequence such that $\lim\limits_{n\to+\infty} I_\lambda(u_n) = c$ and $\lim\limits_{n\to+\infty} I'_\lambda(u_n) = 0$. Put $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$ for all $n \in \mathbb{N}$. We have that $\{u_n^-\}$ is bounded. In fact, one has

$$\left|\Delta u_n^-(k-1)\right|^p \leq -\phi_p\left(\Delta u_n(k-1)\right)\Delta u_n^-(k-1),$$

for all $k \in [1, N+1]$, and

$$q(k)\left|u_n^-(k)\right|^p = -q(k)\left|u_n(k)\right|^{p-2}u_n(k)u_n^-(k),$$

for all $k \in [1, N+1]$.
So we have,

$$\sum_{k=1}^{N+1}\left(\left|\Delta u_n^-(k-1)\right|^p + q(k)\left|u_n^-(k)\right|^p\right)$$
$$\leq -\sum_{k=1}^{N+1}\Big(\phi_p\left(\Delta u_n(k-1)\right)\Delta u_n^-(k-1) + q(k)\left|u_n(k)\right|^{p-2}u_n(k)u_n^-(k)\Big).$$

So,

$$\begin{aligned}
\|u_n^-\|^p &= \sum_{k=1}^{N+1}\left(\left|\Delta u_n^-(k-1)\right|^p + q(k)\left|u_n^-(k)\right|^p\right)\\
&\leq -\sum_{k=1}^{N+1}\Big(\phi_p\left(\Delta u_n(k-1)\right)\Delta u_n^-(k-1) + q(k)\left|u_n(k)\right|^{p-2}u_n(k)u_n^-(k)\Big)\\
&= -\Phi'(u_n)(u_n^-).
\end{aligned}$$

By definition of u_n^- and taking into account that $f(k,x) = f(k,0)$ for all $x < 0$ and for all $k \in [1,N]$, we have

$$\Psi'(u_n)(u_n^-) = \sum_{k=1}^{N} f\left(k, u_n(k)\right)u_n^-(k) \geq 0.$$

So, we get

$$\|u_n^-\|^p \leq -\Phi'(u_n)(u_n^-) \leq -\Phi'(u_n)(u_n^-) + \lambda\Psi'(u_n)(u_n^-),$$

that is

$$\|u_n^-\|^p \leq -I'_\lambda(u_n)(u_n^-), \tag{5}$$

for all $n \in \mathbb{N}$. Now, from $\lim\limits_{n\to+\infty} I'_\lambda(u_n) = 0$, one has $\lim\limits_{n\to+\infty} \dfrac{I'_\lambda(u_n)(u_n^-)}{\|u_n^-\|} = 0$, for which, taking (5) into account, gives $\lim\limits_{n\to+\infty} \|u_n^-\| = 0$. So, we obtain the clain. And, there is

$M > 0$ such that $\|u_n^-\| \leq M$, $\|u_n^-\|_p \leq \dfrac{M}{L_1} = L$, $0 \leq u_n^-(k) \leq L$ for all $k \in [1,N]$ for all $n \in \mathbb{N}$.

At this point, by contradiction argument, assume that $\{u_n\}$ is unbounded (that is, $\{u_n^+\}$ is unbounded).

From $\liminf\limits_{s\to+\infty} \dfrac{F(k,s)}{s^p} = L_\infty(k) \geq L_\infty > l$ there is $\delta_k > 0$ such that $F(k,s) > ls^p$ for all $s > \delta_k$. Moreover,

$$\begin{aligned} F(k,s) &\geq \min_{s\in[-L,\delta_k]} F(k,s) \geq ls^p - l\,(\max\{\delta_k, L\})^p + \min_{s\in[-L,\delta_k]} F(k,s) \\ &\geq ls^p - \max\{l\,(\max \delta_k, L)^p - \min_{s\in[-L,\delta_k]} F(k,s), 0\} = ls^p - Q(k) \end{aligned}$$

for all $s \in [-L, \delta_k]$. Hence, $F(k,s) \geq ls^p - Q(k)$ for all $s \geq -L$. It follows that $F(k, u_n(k)) \geq l\,(u_n(k))^p - Q(k)$ for all $n \in \mathbb{N}$ and for all $k \in [1,N]$, $\sum_{k=1}^{N} F(k, u_n(k)) \geq \sum_{k=1}^{N} [l\,(u_n(k))^p - Q(k)] = l\|u_n\|_p^p - \sum_{k=1}^{N} Q(k) = l\|u_n\|_p^p - \overline{Q}$, that is,

$$\Psi(u_n) \geq l\|u_n\|_p^p - \overline{Q},$$

for all $n \in \mathbb{N}$. Therefore, one has

$$I_\lambda(u_n) = \Phi(u_n) - \lambda\Psi(u_n) = \frac{1}{p}\|u_n\|^p - \lambda\Psi(u_n) \leq \frac{L_2^p}{p}\|u_n\|_p^p - \lambda l\|u_n\|_p^p + \lambda\overline{Q},$$

that is

$$I_\lambda(u_n) \leq \left(\frac{L_2^p}{p} - \lambda l\right)\|u_n\|_p^p + \lambda\overline{Q},$$

for all $n \in \mathbb{N}$. Since $\|u_n\|_p \to +\infty$ and $\dfrac{L_2^p}{p} - \lambda l < 0$, one has $\lim\limits_{n\to+\infty} I_\lambda(u_n) = -\infty$ and this is absurd. Hence, I_λ satisfies $(PS)-$condition.

Finally, we get that I_λ is unbounded from below. Let $\{u_n\}$ be such that $\{u_n^-\}$ is bounded and $\{u_n^+\}$ is unbounded. As before, we obtain $\Psi(u_n) \geq l\|u_n\|_p^p - \overline{Q}$, for all $n \in \mathbb{N}$ and, consequently, $I_\lambda(u_n) \leq \left(\dfrac{L_2^p}{p} - \lambda l\right)\|u_n\|_p^p + \lambda\overline{Q}$, for all $n \in \mathbb{N}$. Hence, $\lim\limits_{n\to+\infty} I_\lambda(u_n) = -\infty$ and the proof is complete.

3 Main Results

In this section, we present the main existence result of our paper. We start putting

$$Q = \sum_{k=1}^{N} q(k).$$

Theorem 2. *Let $f : [1,N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(k,0) \geq 0$ for all $k \in [1,N]$, and $f(k,0) \neq 0$ for some $k \in [1,N]$. Assume also that there exist two positive constants c and d with $d < c$ such that*

$$\frac{\sum_{k=1}^{N} \max_{|\xi| \leq c} F(k,\xi)}{c^p} < q \min \left\{ \frac{1}{Q} \frac{\sum_{k=1}^{N} F(k,d)}{d^p}, \frac{L_\infty}{L_2^p} \right\}. \tag{6}$$

Then, for each $\lambda \in \bar{\Lambda}$ with

$$\bar{\Lambda} = \left] \max \left\{ \frac{Q}{p} \frac{d^p}{\sum_{k=1}^{N} F(k,d)}, \frac{L_2^p}{pL_\infty} \right\}, \frac{q}{p} \frac{c^p}{\sum_{k=1}^{N} \max_{|\xi| \leq c} F(k,\xi)} \right[,$$

the problem $(N_{\lambda,\underline{f}})$ admits at least two positive solutions.

Proof.

We consider the functionals Φ and Ψ given in (3). Φ and Ψ satisfy all regularity assumptions requested in Theorem 1, moreover we have that any critical point in X of the functional I_λ is exactly a solution of problem $(N_{\lambda,\underline{f}})$. Furthermore, $\inf_S \Phi = \Phi(0) = \Psi(0) = 0$. In order to prove our result, we need to verify condition (4) of Theorem 1. Fix $\lambda \in \bar{\Lambda}$, from (6) one has that $L_\infty > 0$ and $\bar{\Lambda}$ is non-degenerate. From Lemma 2, the functional I_λ satisfies the $(PS)-$condition for each $\lambda > \dfrac{L_2^p}{pL_\infty}$, and it is unbounded from below. Now, put $r = \dfrac{qc^p}{p}$, an condier $u \in \Phi^{-1}(]-\infty, r])$; so such a u satisfies

$$\frac{1}{p}\|u\|^p \leq r,$$

so

$$\|u\| \leq (pr)^{\frac{1}{p}}.$$

One has

$$|u| \leq \frac{1}{q^{\frac{1}{p}}}\|u\| \leq \left(\frac{pr}{q}\right)^{\frac{1}{p}} = c.$$

So,

$$\Psi(u) = \sum_{k=1}^{N} F(k, u(k)) \leq \sum_{k=1}^{N} \max_{|\xi| \leq c} F(k, \xi),$$

for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r])$.
Hence,

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} \leq \frac{p}{q} \frac{\sum_{k=1}^{N} \max_{|\xi| \leq c} F(k, \xi)}{c^p}. \tag{7}$$

Now, let be $\tilde{u} \in \mathbb{R}^{N+2}$ be such that $\tilde{u}(k) = d$ for all $k \in [0, N+1]$. Clearly, $\tilde{u} \in X$ and it holds

$$\Phi(\tilde{u}) = \frac{Qd^p}{p}, \tag{8}$$

and so, we have

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \frac{p}{Q} \frac{\sum_{k=1}^{N} F(k, d)}{d^p}. \tag{9}$$

Therefore, from (7), (9) and assumption (6) one has

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Moreover, taking into account that $0 < d < c$ and again by (6), we have that

$$0 < d < \left(\frac{q}{Q}\right)^{\frac{1}{p}} c. \tag{10}$$

Indeed, by contradiction, if we suppose that $d \geq \left(\frac{q}{Q}\right)^{\frac{1}{p}} c$, we have

$$\frac{\sum_{k=1}^{N} \max_{|\xi| \leq c} F(k, \xi)}{c^p} \geq \frac{\sum_{k=1}^{N} F(k, d)}{c^p} \geq \frac{q}{Q} \frac{\sum_{k=1}^{N} F(k, d)}{d^p},$$

which contradicts (6). Hence by (8) and (10) we get $0 < \Phi(\tilde{u}) < r$.

So, finally we obtain tha that I_λ admits at least two non-zero critical points and then, for all $\lambda \in \bar{\Lambda} \subset \Lambda$, these are non zero solutions of $(N_{\lambda, \underline{f}})$.

Since we are interested to obtain a positive solution for problem $(N_{\lambda,\underline{f}})$, we adopt the following truncation on the functions $f(k,s)$,

$$f^{+}(k,s)=\begin{cases} f(k,s), & \text{if } s\geq 0;\\ f(k,0), & \text{if } s<0.\end{cases}$$

Fixed $\lambda\in\Lambda_c^{+}$. Working with the truncations $f^{+}(k,s)$, since we have that $f(k(0,s)\neq 0$ for some $k\in[1,N]$, let u a non trivial solution guaranteed in the first part of the proof, now, to prove the u is nonnegative, we exploit the u is a critical point of the energy functional $I_\lambda=\Phi-\lambda\Psi$ associated to problem $(N_{\lambda,f^{+}})$. In other words, we have that $u\in X$ satisfies the following condition

$$\sum_{k=1}^{N+1}\phi_p(\Delta u(k-1))\Delta v(k-1)+\sum_{k=1}^{N}q(k)\phi_p(u(k))v(k)=\sum_{k=1}^{N}f^{+}(k,u(k))v(k),\ \forall u,v\in X. \tag{11}$$

From this, taking as test function $v=-u^{-}$, it is a simple computation to prove that $\|u^{-}\|=0$, that is u is nonnegative. Moreover, arguing by contradiction, we show that u is also a positive solution of problem $(N_{\lambda,\underline{f}})$. Suppose that $u(k)=0$ for some $k\in[1,N]$. Being u a solution of problem $(N_{\lambda,\underline{f}})$ we have

$$\phi_p(\Delta u(k-1))-\phi_p(\Delta u(k))=f(k,0)\geq 0,$$

which implies that

$$0\geq -|u(k-1)|^{p-2}u(k-1)-|u(k+1)|^{p-2}u(k+1)\geq 0.$$

So, we have that $u(k-1)=u(k+1)=0$. Hence, iterating this process, we get that $u(k)=0$ for every $k\in[1,N]$, which contradicts that u is nontrivial and this completes the proof.

Now, we present a particular case of Theorem 2.

Corollary 1. *Assume that f is a continuous function such that $f(k,0)>0$ for all $k\in[0,N]$ and*

$$\limsup_{t\to 0^{+}}\frac{F(k,t)}{t^{p}}=+\infty, \tag{12}$$

and

$$\lim_{t\to+\infty}\frac{F(k,t)}{t^{p}}=+\infty,$$

for all $k\in[0,N]$, and put $\lambda^{*}=\dfrac{q}{p}\sup_{c>0}\dfrac{c^{p}}{\sum_{k=1}^{N}\max_{|\xi|\leq c}F(k,\xi)}$.

Then, for each $\lambda\in\,]0,\lambda^{*}[$, the problem $(N_{\lambda,\underline{f}})$ admits at least two positive solutions.

Proof. First, note that $L_\infty = +\infty$. Then, fix $\lambda \in]0, \lambda^*[$ and $c > 0$ such that

$$\lambda < \frac{q}{p} \frac{c^p}{\sum_{k=1}^{N} \max_{|\xi| \le c} F(k, \xi)}.$$

From (12) we have

$$\limsup_{t \to 0^+} \frac{\sum_{k=1}^{N} F(k,t)}{t^p} = +\infty,$$

then there is $d > 0$ with $d < c$ such that $\frac{p}{Q} \frac{\sum_{k=1}^{N} F(k,d)}{d^p} > \frac{1}{\lambda}$. Hence, Theorem 2 ensures the conclusion.

Acknowledgements

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017- Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW).

References

1. R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York, Basel, (2000).
2. R. P. Agarwal, K. Perera, D. O'Regan, *Multiple positive solutions of singular and nonsingular discrete problems via variational methods*, Nonlinear Anal. **58** (2004), 69–73.
3. R. P. Agarwal, K. Perera, D. O'Regan, *Multiple positive solutions of singular discrete p-Laplacian problems via variational methods*, Adv. Diff. Equ. **2** (2005), 93–99.
4. R. P. Agarwal, *On multipoint boundary value problems for discrete equations*, J. Math. Anal. Appl. **96** (2) (1983), 520-534.
5. A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973) 349–381.
6. D. R. Anderson, I. Rachuånková, C. C. Tisdell, *Solvability of discrete Neumann boundary value probles*, Adv. Diff. Equ. **2** (2005), 93–99.
7. G. Bonanno, G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, *Z. Anal. Anwend.* **35** (2016), No. 4, 449–464.
8. C. Bereanu, J. Mawhin, *Boundary value problems for second-order nonlinear difference equations with discrete ϕ-Laplacian and singular ϕ*, J. Difference Equ. Appl. **14** (2008), 1099–1118.

9. C. Bereanu, P. Jebelean, C. Şerban, *Ground state and mountain pass solutions for discrete* $p(\cdot)-$*Laplacian*, Bound. Value Probl. 2012 (104) (2012).
10. G. Bonanno, P. Candito, *Nonlinear difference equations investigated via critical point methods*, Nonlinear Anal. **70** (2009), 3180–3186.
11. G. Bonanno, P. Candito, *Infinitely many solutions for a class of discrete nonlinear boundary value problems*, Appl. Anal. **88** (2009), 605–616.
12. G. Bonanno, P. Candito, *Nonlinear difference equations through variational methods, Handbook on Nonconvex Analysis*, Int. Press, Somerville, MA, 2010, pp.1–44.
13. G. Bonanno, P. Candito, G. D'Aguì, *Variational methods on finite dimensional Banach spaces and discrete problems*, Advanced Nonlinear Studies **14** (2014), 915–939.
14. G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012) 2992–3007.
15. G. Bonanno, P. Jebelean, C. Şerban, *Superlinear discrete problems*, Appl. Math. Lett. **52** (2016) 162–168.
16. P. Candito, G. D'Aguì, *Three solutions for a discrete nonlinear Neumann problem involving the p-Laplacian*, Adv. Difference Equ. 2010, Art. ID 862016, 11 pp.
17. P. Candito, G. D'Aguì, *Three solutions to a perturbed nonlinear discrete Dirichlet problem*, J. Math. Anal. Appl. **375** (2011), 594–601.
18. P. Candito, G. D'Aguì, *Constant-sign solutions for a nonlinear neumann problem involving the discrete p-laplacian*, Opuscula Math. **34**, no. 4 (2014), 683–690.
19. P. Candito, N. Giovannelli, *Multiple solutions for a discrete boundary value problem involving the p-Laplacian*, Comput. Math. Appl. **56** (2008), 959–964.
20. G. D'Aguì, J. Mawhin, A. Sciammetta *Positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian*, J. Math. Anal.Appl. **447**, (2017),383–397.
21. M. Galewski, S. Głąb, *On the discrete boundary value problem for anisotropic equation*, J. Math. Anal. Appl. **386** (2012), 956–965
22. A. Guiro, I. Nyanquini, S. Ouaro, *On the solvability of discrete nonlinear Neumann problems involving the p(x)-Laplacian*, Adv. Difference Equ. 2011 (32) (2011).
23. L. Jiang, Z. Zhou, *Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations*, Adv. Difference Equ. (2008), Article ID 345916, 10 pg.
24. W. G. Kelly, A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, San Diego, New York, Basel, (1991).
25. I. Rachuảnková, C. C.Tisdell, *Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions*, Nonlinear Anal. **67** (2007), 1236–1245.
26. C. Şerban, *Existence of solutions for discrete p-Laplacian with potential boundary conditions*, J. Difference Equ. Appl. **19** (2013), 527–537.
27. M. Struwe, Variational Methods, Springer, Berlin, 1996.
28. Y. Tian , W. Ge, *The existence of solutions for a second-order discrete Neumann problem with p-Laplacian*, J. Appl. Math. Comput. **26** (2008), 333-340.