Periodic solutions for systems with p-relativistic operator and unbounded discontinuous nonlinearities

Antonia CHINNI[*], Beatrice Di BELLA[†]

Department of Engineering, University of Messina
Contrada di Dio, S. Agata, 98166 Messina, Italy

Petru JEBELEAN[‡] and Călin ŞERBAN[§]

Department of Mathematics, West University of Timişoara
Blvd. V. Pârvan, no. 4, 300223 Timişoara, Romania

Abstract

We are concerned with the existence of periodic solutions for potential differential inclusions involving the p-relativistic operator

$$\mathcal{R}_p u := \left(\frac{|u'|^{p-2}u'}{(1-|u'|^p)^{1-1/p}} \right)'$$

and an (possible) unbounded discontinuous gradient. The approach relies on critical point theory for locally Lipschitz perturbations of convex, lower semicontinuous functions. The solutions we obtain appear as either minimizers or saddle points of the corresponding energy functional. Some examples of applications illustrating the general results are provided.

MSC 2010 Classification: 34A60, 47H14, 49J52, 49K21

Keywords: differential inclusions, p-relativistic operator, periodic solutions, nonsmooth critical point theory.

[*]achinni@unime.it
[†]bdibella@unime.it
[‡]petru.jebelean@e-uvt.ro
[§]calin.serban@e-uvt.ro

1 Introduction

In this paper we are concerned with differential inclusions systems of type

$$-\mathcal{R}_p u \in \partial F(t,u), \quad u(0)=u(T),\ u'(0)=u'(T), \tag{1.1}$$

where $T>0$, $p\in(1,\infty)$ and $\mathcal{R}_p u := (\varphi_p(u'))'$ with $\varphi_p : B_1 \to \mathbb{R}^N$ given by

$$\varphi_p(y) = \frac{|y|^{p-2}y}{(1-|y|^p)^{1-1/p}}, \quad \forall\, y\in B_1 \tag{1.2}$$

is the p-relativistic operator; here and below B_σ denotes the open ball of center 0 and radius σ in the Euclidean space $\mathbb{R}^N$. Notice that, as emphasized in [14], [15], the p-relativistic operator $\mathcal{R}_p$ extends the relativistic acceleration operator $\mathcal{R}_2$ which occurs in the dynamics of special relativity, in a manner similar to the one in which the vector p-Laplacian extends the classical acceleration operator $u \mapsto u''$. Setting

$$\sigma_{T,p} := \left(\frac{T(p-1)}{2p-1}\right)^{p-1} \quad \text{and} \quad \mu_q := \begin{cases} 1, & \text{if } 0<q<1; \\ 2^{q-1}, & \text{if } q\geq 1, \end{cases}$$

the mapping $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is assumed to satisfy hypothesis:

(H_F) *(i)* $F(\cdot,x):[0,T]\to\mathbb{R}$ *is measurable for every* $x\in\mathbb{R}^N$ *and* $F(\cdot,0)=0$;

(ii) $F(t,\cdot):\mathbb{R}^N\to\mathbb{R}$ *is locally Lipschitz in* $\mathbb{R}^N$ *for all* $t\in[0,T]$;

(iii) there exist $\alpha,\beta\in L^1([0,T];\mathbb{R})$ *with*

$$\|\alpha\|_{L^1} < \frac{2^p}{p\mu_{p-1}\sigma_{T,p}} \tag{1.3}$$

such that for all $t\in[0,T]$, $x\in\mathbb{R}^N$ *and* $\xi\in\partial F(t,x)$, *it holds*

$$|\xi| \leq \alpha(t)|x|^{p-1}+\beta(t),$$

where $\partial F(t,x)$ stands for the generalized Clarke gradient of $F(t,\cdot)$ at $x\in\mathbb{R}^N$.

A function $u\in C^1 := C^1([0,T];\mathbb{R}^N)$ is said to be a *solution* of problem (1.1) if $\|u'\|_C<1$, $\varphi_p(u')$ is absolutely continuous, u satisfies

$$-(\varphi_p(u'(t)))' \in \partial F(t,u(t)), \quad \text{for a.e. } t\in[0,T]$$

and the periodic boundary conditions. Here, $\|\cdot\|_C$ stands for the usual sup-norm on $C := C([0,T];\mathbb{R}^N)$.

In recent years the study of boundary value problems with singular operator has captured a special attention. Mainly, the obtained results are concerned with the existence and multiplicity of solutions for problems involving continuous perturbations of the relativistic operator $\mathcal{R}_2$ and less of them deal with discontinuous ones. In this last direction, we refer to the papers [6] and [19], where the existence of solutions of differential inclusions systems is derived by means of fixed-point and topological techniques, respectively to [13] - [15], [21] where variational approaches are employed.

If the potential F is assumed to verify (H_F) (i) together with

$(\widetilde{H}_F)$ *there is some* $\gamma \in L^1([0,T];\mathbb{R})$ *such that, for all* $t \in [0,T]$ *and* $x, y \in \mathbb{R}^N$, *it holds*

$$|F(t,x) - F(t,y)| \leq \gamma(t)|x-y| \tag{1.4}$$

and one of the Ahmad-Lazer-Paul [1] type conditions

$$\lim_{|x|\to\infty} \int_0^T F(t,x)dt = +\infty \tag{1.5}$$

or

$$\lim_{|x|\to\infty} \int_0^T F(t,x)dt = -\infty \tag{1.6}$$

holds true, then it was shown in [15] that (1.1) has at least one solution. Notice that condition $(\widetilde{H}_F)$ is stronger than (H_F), in the sense that $(\widetilde{H}_F)$ asks that $\partial F(t,\cdot)$ to be uniformly bounded by (the same) $\gamma \in L^1([0,T];\mathbb{R})$. In this view, here the novelty is that (H_F) (iii) allows to consider problems with unbounded generalized gradient. Specifically, the (generalized) gradient can have a $(p-1)$–polynomial growth. So, in the particular classical case (when $p = 2$), the gradient is allowed to have a linear growth and hence, the results obtained here are more general than the ones in [13]. Also, we note that conditions like (H_F) (iii) occur, among others, in studies concerning differential inclusions involving the p-Laplacian operator (see e.g. [4], [8], [10]). Thus, it is the aim of this paper to provide weaker conditions than the ones in [15], ensuring the solvability of system (1.1). It is worth to point out that the results we obtain here are new even in the more "classical" cases $p = 2$ and/or if $F(t,\cdot)$ is of class C^1 (meaning that we have equations instead of inclusions – see Section 5).

The first result is the following

Theorem 1.1 *If* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (H_F) *and there exists a positive constant* η *such that*

$$\max_{x\in\mathbb{R}^N, |x|=\eta} \int_0^T F(t,x)dt + \frac{(\sigma_{T,p})^{\frac{1}{p-1}}}{2} \cdot \frac{\left(\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)^{\frac{p}{p-1}}}{\left(\dfrac{2}{p} - \dfrac{\mu_{p-1}\sigma_{T,p}}{2^{p-1}}\|\alpha\|_{L^1}\right)^{\frac{1}{p-1}}} < 0, \tag{1.7}$$

then problem (1.1) *has at least one solution.*

Obviously, when $\alpha = 0$, (1.6) implies condition (1.7). To prove Theorem 1.1, we consider a modified system involving a non-singular operator, for which we obtain the existence of a minimizer of the associated action functional and then, we show that this actually solves problem (1.1). Here we use some ideas from [12].

Next, if instead of (H_F) (iii), the mapping F verifies

($\mathcal{H}_F$) *there exist* $\alpha_1, \beta_1 \in L^1([0,T];\mathbb{R})$ *and* $r \geq 0$ *such that for all* $t \in [0,T]$, $x \in \mathbb{R}^N$ *and* $\xi \in \partial F(t,x)$, *it holds*

$$|\xi| \leq \alpha_1(t)|x|^r + \beta_1(t),$$

we obtain the following generalized Ahmad-Lazer-Paul type result.

Theorem 1.2 *Assume that* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (H_F) *(i), (ii) and* $(\mathcal{H}_F)$. *If either*

$$\lim_{|x|\to\infty} \frac{1}{|x|^r} \int_0^T F(t,x)dt = +\infty \tag{1.8}$$

or

$$\lim_{|x|\to\infty} \frac{1}{|x|^r} \int_0^T F(t,x)dt = -\infty \tag{1.9}$$

holds true, then problem (1.1) *has at least one solution.*

Conditions like (1.8) and (1.9) are often encountered in studies concerning second order differential systems with unbounded nonlinearities – we refer to e.g. [5], [9], [22], [23]. Also, we note that there are functions F satisfying (1.7) and not satisfying (1.9) as it is highlighted in Example 5.1. To prove Theorem 1.2 we use the direct method in the calculus of variations and the saddle-point theorem in the frame of the critical point theory for locally Lipschitz perturbations of convex, lower semicontinuous functionals, developed by Motreanu and Panagiotopoulos in [17]. So, the solutions which we obtain appear either as saddle points – if (1.8) holds true, or as minimizers – when (1.9) is fulfilled.

Before concluding this introductory part, for the convenience of the reader, we recall some basic facts in the nonsmooth critical point theory.

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. If $\mathcal{G} : X \to \mathbb{R}$ is locally Lipschitz, then the *generalized directional derivative* at $x \in X$ in the direction of $v \in X$ is defined by

$$\mathcal{G}^0(x;v) = \limsup_{y\to x,\ s\searrow 0} \frac{\mathcal{G}(y+sv) - \mathcal{G}(y)}{s}$$

and the *generalized gradient* (in the sense of Clarke [3]) of $\mathcal{G}$ at $x \in X$ is the set

$$\partial\mathcal{G}(x) = \left\{x^* \in X^* :\ \mathcal{G}^0(x;v) \geq \langle x^*, v\rangle,\ \ \forall\, v \in X\right\},$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X.

Now, if $J : X \to (-\infty, +\infty]$ is a functional having the structure

$$J = \Phi + \mathcal{G}, \tag{1.10}$$

with $\mathcal{G} : X \to \mathbb{R}$ locally Lipschitz and $\Phi : X \to (-\infty, +\infty]$ proper, convex and lower semicontinuous, then an element $x \in X$ is said to be *a critical point* of J if the inequality

$$\mathcal{G}^0(x; v - x) + \Phi(v) - \Phi(x) \geq 0$$

holds true for all $v \in X$. The number $c = J(x)$ is called *a critical value* of J corresponding to the critical point x. One has that $x \in X$ is a critical point of J iff $0 \in \partial \mathcal{G}(x) + \overline{\partial}\Phi(x)$, where the set $\overline{\partial}\Phi(x)$ is the subdifferential of Φ at x in the sense of convex analysis [20] (see e.g. [18, Remark 2.2]). The reader will emphasize that the above definition of a critical point of J coincides with the one for a locally Lipschitz function in [2] provided that additionally Φ in (1.10) is of class C^1. This means that we could employ the theory in [2] in this situation, and this is the case when dealing with the proof of Theorem 1.1. However, the whole generality of the theory developed in [17] is exploited to prove Theorem 1.2. So, the reason for choosing this framework is motivated by its unifying feature.

We say that $(x_n) \subset X$ is a (PS) *sequence* for J in (1.10) provided that $(J(x_n))$ is bounded and

$$\mathcal{G}^0(x_n; v - x_n) + \Phi(v) - \Phi(x_n) \geq -\varepsilon_n \|v - x_n\|, \quad \forall\, v \in X,$$

for a sequence $(\varepsilon_n) \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$. The functional J is said *to satisfy the Palais-Smale condition* (in short, (PS) *condition*) if every (PS) sequence possesses a convergent subsequence.

Theorem 1.3 (see [17, Corollary 3.3]) *Suppose that* $X = X_1 \oplus X_2$, *with* $\dim X_1 < \infty$ *and there exists* $\rho > 0$ *such that*

$$\inf_{X_2} J > \sup_{\partial B_\rho \cap X_1} J. \tag{1.11}$$

If J satisfies the (PS) condition, then J possesses a critical value $c \geq \inf_{X_2} J$ given by

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in D} J(\gamma(u)),$$

where $D = \overline{B}_\rho \cap X_1$ *and* $\Gamma = \left\{\gamma \in C(D, X) : \ \gamma|_{\partial B_\rho \cap X_1} = id\right\}$.

2 A non-singular system

Having in view (1.1), we construct a system with a non-singular operator, using an idea introduced in [12] for $p = 2$ (also see [13, 14]), which consists in a suitable cutting of the singular homeomorphism φ_p in (1.2). With this aim, let $R \in (0, 1)$ be arbitrarily chosen such that

$$c_{R,p} := \frac{R^{p-1}}{(1 - R^p)^{1-1/p}} > \sqrt{N}\|\beta\|_{L^1} \tag{2.1}$$

and let the homeomorphism $\psi_p : \mathbb{R}^N \to \mathbb{R}^N$ be given by

$$\psi_p(y) = \frac{|y|^{p-2} y}{\left(1 - \min\{|y|^p, R^p\}\right)^{1-1/p}} \quad \left(y \in \mathbb{R}^N\right).$$

We define $\Psi_p : \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi_p(y) = 1 - \frac{(p-1)(1-\min\{|y|^p, R^p\}) + 1 - |y|^p}{p(1-\min\{|y|^p, R^p\})^{1-1/p}} \quad \left(y \in \mathbb{R}^N\right).$$

Straightforward computations show that $\psi_p = \nabla \Psi_p$ on $\mathbb{R}^N$ and

$$\Psi_p(y) \geq \frac{1}{p}|y|^p \quad \left(y \in \mathbb{R}^N\right). \tag{2.2}$$

Now, we consider the problem

$$-(\psi_p(u'))' \in \partial F(t,u), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{2.3}$$

where F fulfils condition (H_F).

By a *solution* of problem (2.3) we understand a function $u \in C^1$ with $\psi_p(u')$ absolutely continuous, which satisfies

$$-(\psi_p(u'(t)))' \in \partial F(t, u(t)), \quad \text{for a.e. } t \in [0,T]$$

and the periodic boundary conditions.

Proposition 2.1 *Assume* (H_F). *If* $u \in C^1$ *is a solution of problem* (2.3) *and*

$$\|u\|_C \leq \left(\frac{p\mu_{p-1}\sigma_{T,p}}{2^p\sqrt{N}}\left(c_{R,p} - \sqrt{N}\|\beta\|_{L^1}\right)\right)^{\frac{1}{p-1}}, \tag{2.4}$$

then u *solves* (1.1).

Proof. Let $u = (u_1, ..., u_N) \in C^1$ be a solution of (2.3) which satisfies (2.4). One has

$$-(\psi_p(u'))' = v, \tag{2.5}$$

with $v(t) \in \partial F(t, u(t))$ for a.e. $t \in [0,T]$. If $|u'| \leq R$ on $[0,T]$, clearly $\varphi_p(u') = \psi_p(u')$ and the proof is complete. If we suppose that there exists $t_0 \in [0,T]$ with $|u'(t_0)| > R$, then we get a contradiction. Indeed, because u is such that $u(0) = u(T)$, there exist $\xi_i \in [0,T]$ with $u_i'(\xi_i) = 0$ for all $i = 1, \ldots, N$. It follows

$$\begin{aligned}
\int_{\xi_i}^{t_0} (\psi_{p_i}(u'(\tau))' d\tau &= \psi_{p_i}(u'(t_0)) \\
&- \psi_{p_i}\left(u_1'(\xi_i), \ldots, u_{i-1}'(\xi_i), 0, u_{i+1}'(\xi_i), \ldots, u_N'(\xi_i)\right) \\
&= \psi_{p_i}(u'(t_0)) = \frac{|u'(t_0)|^{p-2}u_i'(t_0)}{(1-R^p)^{1-1/p}}.
\end{aligned} \tag{2.6}$$

Then, integrating (2.5) componentwise, between ξ_i and t_0, taking the absolute value and using (2.6) and (H_F) (iii), we obtain

$$\begin{aligned}\frac{|u'(t_0)|^{p-2}|u_i'(t_0)|}{(1-R^p)^{1-1/p}} &= \left|\int_{\xi_i}^{t_0} v_i(\tau)d\tau\right| \le \int_0^T |v(\tau)|d\tau \\ &\le \int_0^T [\alpha(\tau)|u(\tau)|^{p-1}+\beta(\tau)]d\tau \\ &\le \|\alpha\|_{L^1}\|u\|_C^{p-1}+\|\beta\|_{L^1} \le \frac{2^p\|u\|_C^{p-1}}{p\mu_{p-1}\sigma_{T,p}}+\|\beta\|_{L^1}\end{aligned}$$

for $i=\overline{1,N}$. This gives

$$c_{R,p} < \frac{|u'(t_0)|^{p-1}}{(1-R^p)^{1-1/p}} \le \sqrt{N}\left(\frac{2^p\|u\|_C^{p-1}}{p\mu_{p-1}\sigma_{T,p}}+\|\beta\|_{L^1}\right),$$

which contradicts (2.4). ■

Next, assuming hypothesis (H_F), a variational approach is introduced for problem (2.3). In this view, the space

$$W_T^{1,p} := \left\{u \in W^{1,p}([0,T];\mathbb{R}^N) : \ u(0)=u(T)\right\},$$

will be considered with the norm

$$\|u\|_{W_T^{1,p}} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p}.$$

Hereafter, if $u \in L^1([0,T];\mathbb{R}^N)$ we write $u=\overline{u}+\widetilde{u}$, with

$$\overline{u} := \frac{1}{T}\int_0^T u(t)\,dt$$

and we note the inequality [16, p. 385] (also see [24]):

$$\|\widetilde{u}\|_C \le \frac{(\sigma_{T,p})^{1/p}}{2}\|u'\|_{L^p}, \quad \forall\ u \in W_T^{1,p}. \tag{2.7}$$

Next, we define the functional $\mathcal{I}_\Psi : W_T^{1,p} \to \mathbb{R}$ by

$$\mathcal{I}_\Psi(u) = \int_0^T \Psi_p(u')dt \quad (u \in W_T^{1,p}). \tag{2.8}$$

Standard arguments show that the convex function $\mathcal{I}_\Psi$ is of class C^1 on $W_T^{1,p}$ and

$$\langle \mathcal{I}_\Psi'(u), v\rangle = \int_0^T (\psi_p(u')|v')dt \quad (u,v \in W_T^{1,p}), \tag{2.9}$$

where $(\cdot|\cdot)$ is the usual scalar product in the Euclidean space $\mathbb{R}^N$. Then, on account of (H_F), we can define the locally Lipschitz function $\mathcal{F}: C \to \mathbb{R}$ by

$$\mathcal{F}(u) = -\int_0^T F(t,u)dt \quad (u \in C) \tag{2.10}$$

and taking into account the embedding $W_T^{1,p} \overset{i}{\hookrightarrow} C$, we introduce the functional

$$\mathcal{I}_F := \mathcal{F}|_{W_T^{1,p}} = \mathcal{F} \circ i \tag{2.11}$$

which is still locally Lipschitz on $W_T^{1,p}$.

Proposition 2.2 *Assume that $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H_F). If $u \in W_T^{1,p}$ and $\ell \in \partial\mathcal{I}_F(u)$ then there is some $u_\ell \in L^1([0,T];\mathbb{R}^N)$ such that $u_\ell(t) \in \partial F(t,u(t))$ for a.e. $t \in [0,T]$ and*

$$\langle \ell, v\rangle = -\int_0^T (u_\ell|v)dt, \quad \forall\ v \in W_T^{1,p}. \tag{2.12}$$

Proof. First, by the chain rule [3, Theorem 2.3.10] we have

$$\partial\mathcal{I}_F(u) \subset \partial\mathcal{F}(u), \quad \forall\ u \in W_T^{1,p}. \tag{2.13}$$

Also, for $\rho > 0$, $x, y \in \overline{B}_\rho$ and $t \in [0,T]$, from Lebourg's mean value theorem [3, Proposition 2.3.7], there exist $\tau \in (0,1)$ and $z^* \in \partial F(t, \tau x + (1-\tau)y)$ such that

$$|F(t,y) - F(t,x)| \le |z^*||y-x|.$$

Using hypothesis (H_F) (iii) and the elementary inequality

$$|x+y|^q \le \mu_q(|x|^q + |y|^q) \quad (x, y \in \mathbb{R}^N,\ q > 0), \tag{2.14}$$

one gets

$$\begin{aligned} |F(t,y) - F(t,x)| &\le \left(\alpha(t)|\tau x + (1-\tau)y|^{p-1} + \beta(t)\right)|y-x| \\ &\le \left(\mu_{p-1}\alpha(t)(|x|^{p-1} + |y|^{p-1}) + \beta(t)\right)|y-x| \\ &\le \left(2\mu_{p-1}\alpha(t)\rho^{p-1} + \beta(t)\right)|y-x| \\ &= \gamma_\rho(t)|y-x|, \end{aligned} \tag{2.15}$$

where we have denoted by γ_ρ the L^1-function $\gamma_\rho(t) := 2\mu_{p-1}\alpha(t)\rho^{p-1} + \beta(t)$.

Then, by virtue of (2.13) and (2.15), the proof follows exactly by the same arguments as in the proof of Proposition 5.3 in [11]. ■

Next, with $\mathcal{I}_\Psi$ in (2.8) and $\mathcal{I}_F$ given by (2.11), we introduce the action functional associated to (2.3) by

$$\mathcal{I}(u) := \mathcal{I}_\Psi(u) + \mathcal{I}_F(u) \quad (u \in W_T^{1,p}). \tag{2.16}$$

It is worth to point out that $\mathcal{I}$ depends on $R \in (0,1)$ satisfying (2.1). Clearly, from (2.9) we have that

$$\mathcal{I}^0(u;v) = \int_0^T (\psi_p(u')|v')dt + \mathcal{I}_F^0(u;v), \quad \forall\ u, v \in W_T^{1,p}. \tag{2.17}$$

Proposition 2.3 *If hypothesis* (H_F) *holds true and* $u \in W_T^{1,p}$ *is a critical point of* $\mathcal{I}$*, then* $u \in C^1$ *and is a solution of problem* (2.3).

Proof. This relies on (2.17) and Proposition 2.2 and follows exactly the outline of the proof of [13, Proposition 3.3]. ■

For $\eta > 0$, we set

$$K_\eta := \{u \in W_T^{1,p} : \ |\overline{u}| \leq \eta\}. \tag{2.18}$$

Lemma 2.1 *Assume that* (H_F) *holds true and let* $\eta > 0$*. If* R *satisfies* (2.1) *together with*

$$\eta + \frac{1}{2}\left(\frac{\sigma_{T,p}\left(\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)}{\dfrac{2}{p} - \dfrac{\mu_{p-1}\sigma_{T,p}}{2^{p-1}}\|\alpha\|_{L^1}}\right)^{\frac{1}{p-1}} \leq \left(\frac{p\mu_{p-1}\sigma_{T,p}}{2^p\sqrt{N}}\left(c_{R,p} - \sqrt{N}\|\beta\|_{L^1}\right)\right)^{\frac{1}{p-1}} \tag{2.19}$$

then each minimizer $u \in K_\eta$ *of* $\mathcal{I}$ *on* K_η *necessarily satisfies* (2.4).

Proof. Since $\mathcal{I}(u) \leq \mathcal{I}(\overline{u})$, from (2.2), one has

$$\frac{1}{p}\|u'\|_{L^p}^p \leq \int_0^T \Psi_p(u')dt \leq \int_0^T [F(t,u) - F(t,\overline{u})]dt. \tag{2.20}$$

From Lebourg's mean value theorem, there exist $\tau \in (0,1)$ and $z^*(t) \in \partial F(t, \overline{u} + \tau\widetilde{u}(t))$ such that

$$|F(t,u(t)) - F(t,\overline{u})| \leq |z^*(t)||\widetilde{u}(t)| \quad (t \in [0,T])$$

and by virtue of (H_F) (iii), (2.14) and (2.7) we infer

$$\begin{aligned}
\left|\int_0^T [F(t,u) - F(t,\overline{u})]dt\right| &\leq \int_0^T \left(\alpha(t)|\overline{u} + \tau\widetilde{u}(t)|^{p-1} + \beta(t)\right)|\widetilde{u}(t)|dt \\
&\leq \int_0^T \left(\mu_{p-1}\alpha(t)(|\widetilde{u}(t)|^p + |\overline{u}|^{p-1}|\widetilde{u}(t)|) + \beta(t)|\widetilde{u}(t)|\right)dt \\
&\leq \mu_{p-1}\|\alpha\|_{L^1}\|\widetilde{u}\|_C^p + \|\widetilde{u}\|_C\left(\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right) \\
&\leq \frac{(\sigma_{T,p})^{1/p}}{2}\|u'\|_{L^p}\left(\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right) \\
&\quad + \frac{\mu_{p-1}\sigma_{T,p}}{2^p}\|\alpha\|_{L^1}\|u'\|_{L^p}^p.
\end{aligned} \tag{2.21}$$

This and (2.20) yield

$$\|u'\|_{L^p} \leq (\sigma_{T,p})^{\frac{1}{p(p-1)}}\left(\frac{\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}}{\dfrac{2}{p} - \dfrac{\mu_{p-1}\sigma_{T,p}}{2^{p-1}}\|\alpha\|_{L^1}}\right)^{\frac{1}{p-1}}. \tag{2.22}$$

On the other hand, using again (2.7), we have

$$|u(t)| \leq |\overline{u}| + |\widetilde{u}(t)| \leq \eta + \|\widetilde{u}\|_C \leq \eta + \frac{(\sigma_{T,p})^{1/p}}{2}\|u'\|_{L^p} \tag{2.23}$$

and hence, on account of (2.22) and (2.19), we obtain that u satisfies (2.4). ■

3 Proof of Theorem 1.1

Let $\eta > 0$ be so that (1.7) holds true. We pick $R \in (0,1)$ such that (2.1) and (2.19) are satisfied and consider K_η as in (2.18). We will show that $\mathcal{I}$ in (2.16) has a minimizer in K_η which is not a boundary point. Then, the conclusion follows from Lemma 2.1, Propositions 2.3 and 2.1.

Similarly to (2.21), using (2.2), Lebourg's theorem, (H_F) (iii), inequalities (2.14) and (2.7), we estimate $\mathcal{I}$ as follows:

$$\begin{aligned}
\mathcal{I}(v) &= \int_0^T \Psi_p(v')dt - \int_0^T F(t,v)dt \\
&\geq \frac{1}{p}\|v'\|_{L^p}^p - \int_0^T [F(t,v) - F(t,\overline{v})]dt - \int_0^T F(t,\overline{v})dt \\
&\geq \frac{1}{p}\|v'\|_{L^p}^p - \int_0^T \left(\mu_{p-1}\alpha(t)(|\widetilde{v}(t)|^p + |\overline{v}|^{p-1}|\widetilde{v}(t)|) + \beta(t)|\widetilde{v}(t)|\right) dt \\
&\quad - \int_0^T F(t,\overline{v})dt \geq \frac{1}{p}\|v'\|_{L^p}^p - \int_0^T F(t,\overline{v})dt \\
&\quad -\mu_{p-1}\|\alpha\|_{L^1}\|\widetilde{v}\|_C^p - \|\widetilde{v}\|_C \left(\mu_{p-1}|\overline{v}|^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right) \\
&\geq \frac{1}{2}\left(\frac{2}{p} - \frac{\mu_{p-1}\sigma_{T,p}}{2^{p-1}}\|\alpha\|_{L^1}\right)\|v'\|_{L^p}^p - \int_0^T F(t,\overline{v})dt \\
&\quad - \frac{(\sigma_{T,p})^{1/p}}{2}\left(\mu_{p-1}|\overline{v}|^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)\|v'\|_{L^p} \quad (v \in W_T^{1,p}).
\end{aligned} \tag{3.1}$$

Hence, since $\int_0^T F(t,\overline{v})dt$ is bounded on K_η, one has that $\mathcal{I}$ is bounded from below on K_η. We set

$$\zeta := \inf_{v \in K_\eta} \mathcal{I}(v) \tag{3.2}$$

and let (v_k) be a minimizing sequence in K_η with

$$\mathcal{I}(v_k) \to \zeta \quad \text{as } k \to \infty.$$

By virtue of (3.1), we get that $(\widetilde{v}_k')$ is bounded in $L^p([0,T];\mathbb{R}^N)$ and hence we infer that (v_k) is bounded in $W_T^{1,p}$, which implies that there exists a subsequence of (v_k), still denoted by (v_k), which weakly converges to some $u \in W_T^{1,p}$. Since K_η is a convex closed subset of $W_T^{1,p}$, one has that $u \in K_\eta$.

On account of the compactness of the embedding $W^{1,p}_T \subset C$ and of the continuity of $\mathcal{F}$ on C, the functional $\mathcal{I}_F$ is sequentially weakly continuous on $W^{1,p}_T$. Then, since the C^1 functional $\mathcal{I}_\Psi$ is convex, it is weakly lower semicontinuous. Hence, we deduce that $\mathcal{I}$ is sequentially weakly lower semicontinuous and so, one gets

$$\zeta = \lim_{k\to\infty} \mathcal{I}(v_k) \geq \mathcal{I}(u) \geq \zeta,$$

showing that $\zeta = \mathcal{I}(u)$. So, u is a minimum point of $\mathcal{I}$ in K_η.

Next, if we suppose by contradiction that $u \in \partial K_\eta$, then $|\overline{u}| = \eta$ and from (2.22) and (1.7), we get (see (3.1)):

$$\begin{aligned} \mathcal{I}(u) &\geq -\frac{(\sigma_{T,p})^{1/p}}{2}\left(\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)\|u'\|_{L^p} - \int_0^T F(t,\overline{u})dt \\ &\geq -\frac{(\sigma_{T,p})^{\frac{1}{p-1}}}{2}\cdot\frac{\left(\mu_{p-1}\eta^{p-1}\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)^{\frac{p}{p-1}}}{\left(\dfrac{2}{p} - \dfrac{\mu_{p-1}\sigma_{T,p}}{2^{p-1}}\|\alpha\|_{L^1}\right)^{\frac{1}{p-1}}} \\ &\quad - \max_{x\in\mathbb{R}^N, |x|=\eta}\int_0^T F(t,x)dt > 0. \end{aligned}$$

Thus, we obtain $\mathcal{I}(u) > 0 = \mathcal{I}(0) \geq \mathcal{I}(u)$, a contradiction. Hence, $u \notin \partial K_\eta$ and the proof is complete. ■

4 Proof of Theorem 1.2

First, assuming hypotheses (H_F) (i), (ii) and $(\mathcal{H}_F)$, a variational approach is directly introduced for problem (1.1). With this aim, let $L^\infty := L^\infty([0,T];\mathbb{R}^N)$ be considered with the usual norm $\|\cdot\|_\infty$ and $W^{1,\infty} := W^{1,\infty}([0,T];\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{W^{1,\infty}} = \|u\|_\infty + \|u'\|_\infty \quad (u \in W^{1,\infty}).$$

Setting

$$\mathcal{K} := \left\{u \in W^{1,\infty} : \ \|u'\|_\infty \leq 1, \ u(0) = u(T)\right\},$$

let $\Phi : C \to (-\infty, +\infty]$ be defined by

$$\Phi(u) = \begin{cases} \displaystyle\int_0^T [1 - (1 - |u'|^p)^{1/p}], & \text{if } u \in \mathcal{K}, \\ +\infty, & \text{otherwise.} \end{cases}$$

One has that Φ is proper, convex and lower semicontinuous (see [15]). Next, with the locally Lipschitz function $\mathcal{F}$ given in (2.10), the energy functional associated to (1.1) will be

$$\mathcal{J} := \Phi + \mathcal{F},$$

which has the structure from (1.10). Also, for $\rho > 0$, $x, y \in \overline{B}_\rho$ and $t \in [0, T]$, one obtains (see (2.15))

$$|F(t,y) - F(t,x)| \leq \widetilde{\gamma}_\rho(t)|y - x|, \tag{4.1}$$

where we have denoted by $\widetilde{\gamma}_\rho$ the L^1-function $\widetilde{\gamma}_\rho(t) := 2\mu_r \alpha_1(t)\rho^r + \beta_1(t)$ if $r > 0$, respectively $\widetilde{\gamma}_\rho(t) := \alpha_1(t) + \beta_1(t)$ for $r = 0$. Then, on account of (4.1), we have the following

Proposition 4.1 (see [15, Proposition 4.1]) *If u is a critical point of $\mathcal{J}$, then u is a solution of problem* (1.1).

If $u \in W^{1,\infty}$, then each component $\widetilde{u}_i$ vanishes at some $\xi_i \in [0, T]$ $(i = \overline{1, N})$ and hence

$$|\widetilde{u}_i(t)| = |\widetilde{u}_i(t) - \widetilde{u}_i(\xi_i)| \leq \int_0^T |u_i'(\tau)|\, d\tau \leq T\|u'\|_\infty \quad (i = \overline{1, N}),$$

so, one has that

$$\|\widetilde{u}\|_\infty \leq T\sqrt{N}\ \|u'\|_\infty.$$

Thus,

$$\|\widetilde{u}\|_\infty \leq T\sqrt{N}, \tag{4.2}$$

for all $u \in \mathcal{K}$. Also, for $\theta > 0$, we put

$$\mathcal{K}_\theta := \{u \in \mathcal{K} : \ |\overline{u}| \leq \theta\}.$$

Lemma 4.1 (see [15, Lemma 4.1]) *Assume that there is some $\theta > 0$ such that*

$$\inf_{\mathcal{K}_\theta} \mathcal{J} = \inf_{\mathcal{K}} \mathcal{J}. \tag{4.3}$$

Then $\mathcal{J}$ is bounded from below on C and attains its infimum at some $u \in \mathcal{K}_\theta$, which solves problem (1.1).

Proof of Theorem 1.2. Since if $r = 0$ condition (1.4) in hypothesis $(\widetilde{H}_F)$ is fulfilled with $\gamma = \alpha_1 + \beta_1$, the result is obtained in [15], we have to treat the remaining case when $r > 0$.

Assume first that (1.9) holds true. For arbitrary $u \in \mathcal{K}$, using $(\mathcal{H}_F)$ with $r > 0$, Lebourg's theorem and (4.2), together with (2.14), one has

$$\begin{aligned}\left|\int_0^T [F(t,u) - F(t,\overline{u})]dt\right| &\leq \int_0^T \left(\mu_r \alpha_1(t)(|\widetilde{u}(t)|^r + |\overline{u}|^r) + \beta_1(t)\right)|\widetilde{u}(t)|dt \\ &\leq \mu_r\|\alpha_1\|_{L^1}\|\widetilde{u}\|_\infty^{r+1} + \|\widetilde{u}\|_\infty \left(\mu_r|\overline{u}|^r\|\alpha_1\|_{L^1} + \|\beta_1\|_{L^1}\right) \\ &\leq \mu_r(T\sqrt{N})^{r+1}\|\alpha_1\|_{L^1} + T\sqrt{N}\|\beta_1\|_{L^1} + \mu_r T\sqrt{N}\|\alpha_1\|_{L^1}|\overline{u}|^r. \end{aligned} \tag{4.4}$$

Then, by the definition of Φ, we estimate $\mathcal{J}$ as follows

$$\mathcal{J}(u) \geq -\int_0^T F(t,\overline{u})dt - \int_0^T [F(t,u) - F(t,\overline{u})]dt \geq -\int_0^T F(t,\overline{u})dt \\ - \mu_r T\sqrt{N}\|\alpha_1\|_{L^1}|\overline{u}|^r - \mu_r(T\sqrt{N})^{r+1}\|\alpha_1\|_{L^1} - T\sqrt{N}\|\beta_1\|_{L^1}.$$

By virtue of (1.9) we can find $\theta > 0$ such that $\mathcal{J}(u) > 0$ provided that $|\overline{u}| > \theta$. As $\mathcal{J}(0) = 0$ (see (H_F) (i)), we have that (4.3) is fulfilled and the conclusion follows from Lemma 4.1.

In the second case – when (1.8) holds true, we apply Theorem 1.3. First, notice that if $v = c \in \mathbb{R}^N$ is a constant function, from (1.8), one has

$$\mathcal{J}(c) = -\int_0^T F(t,c)dt \to -\infty, \quad \text{as } |c| \to \infty. \tag{4.5}$$

Splitting $C = \mathbb{R}^N \oplus \widetilde{C}$, with $\widetilde{C} := \{u \in C : \ \overline{u} = 0\}$, using again the definition of Φ, hypothesis $(\mathcal{H}_F)$, Lebourg's theorem and (4.2), one obtains

$$\mathcal{J}(u) \geq -\int_0^T F(t,\widetilde{u})dt \geq -(T\sqrt{N})^{r+1}\|\alpha_1\|_{L^1} - T\sqrt{N}\|\beta_1\|_{L^1}, \tag{4.6}$$

for all $u \in \mathcal{K} \cap \widetilde{C}$. So, from (4.5) and (4.6), condition (1.11) in Theorem 1.3 is fulfilled. It remains to show that $\mathcal{J}$ satisfies the (PS) condition. Let $(u_n) \subset \mathcal{K}$ be a (PS) sequence. Since $(\mathcal{J}(u_n))$ is bounded and Φ is bounded in $\mathcal{K}$, using (4.4) with u_n instead of u, from

$$\mathcal{J}(u_n) = \Phi(u_n) - \int_0^T F(t,\overline{u}_n)dt - \int_0^T [F(t,u_n) - F(t,\overline{u}_n)]dt$$

it follows that there exists a constant $\overline{k} \in \mathbb{R}$ such that

$$\int_0^T F(t,\overline{u}_n)dt - \mu_r T\sqrt{N}\|\alpha_1\|_{L^1}|\overline{u}_n|^r \leq \overline{k}.$$

Then, by (1.8), the sequence $(\overline{u}_n)$ is bounded. Using now (4.2) we obtain

$$\|u_n\|_{W^{1,\infty}} \leq |\overline{u}_n| + T\sqrt{N} + 1,$$

showing that (u_n) is bounded in $W^{1,\infty}$ and on account of the compactness of the embedding $W^{1,\infty} \subset C$ we have that (u_n) has a convergent subsequence in C. Consequently, $\mathcal{J}$ satisfies the (PS) condition and the proof is then accomplished by virtue of Theorem 1.3 and Proposition 4.1. ■

5 Some corollaries and examples

First, notice that if instead of hypothesis (H_F) (iii) in Theorem 1.1 the mapping F is asked to satisfy $(\mathcal{H}_F)$ with $r \in [0, p-1)$ and α_1 verifying (1.3), then this still remain applicable, with $\alpha(t) = \alpha_1(t)$ and $\beta(t) = \alpha_1(t) + \beta_1(t)$. Indeed, for all $t \in [0, T]$, we get

$$\alpha_1(t)|x|^r \leq \alpha_1(t)|x|^{p-1}, \quad \forall\, x \in \mathbb{R}^N,\ |x| > 1.$$

On the other hand, one has

$$\alpha_1(t)|x|^r \leq \alpha_1(t), \quad \forall\, x \in \mathbb{R}^N,\ |x| \leq 1$$

and hence

$$\alpha_1(t)|x|^r \leq \alpha_1(t)|x|^{p-1} + \alpha_1(t), \quad \forall\, x \in \mathbb{R}^N.$$

Therefore, (H_F) (iii) holds with $\alpha(t) = \alpha_1(t)$ and $\beta(t) = \alpha_1(t) + \beta_1(t)$.

Corollary 5.1 *Assume that* (H_F) (i) *is satisfied and there exists a positive constant* η *such that* (1.7) *holds true. If, in addition,* F *verifies*

$(\widehat{H}_F)$ *(i)* $F(t, \cdot) : \mathbb{R}^N \to \mathbb{R}$ *is of class* C^1 *on* $\mathbb{R}^N$ *for a.e.* $t \in [0, T]$*;*

(ii) there exist $\alpha, \beta \in L^1([0, T]; \mathbb{R})$ *with* α *satisfying* (1.3) *and such that for all* $t \in [0, T]$ *and* $x \in \mathbb{R}^N$*, it holds* $|\nabla_x F(t, x)| \leq \alpha(t)|x|^{p-1} + \beta(t)$*,*

then the differential system

$$-\mathcal{R}_p u = \nabla_u F(t, u), \quad u(0) = u(T),\ u'(0) = u'(T) \tag{5.1}$$

has as least one solution.

Proof. On account of $(\widehat{H}_F)$ (i), one has that $\partial F(t, x) = \{\nabla_x F(t, x)\}$ for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^N$. Then, the conclusion follows from Theorem 1.1. ■

Corollary 5.2 *Assume* (H_F) (i), $(\widehat{H}_F)$ (i) *and*

$(\widehat{\mathcal{H}}_F)$ *there exist* $\alpha_1, \beta_1 \in L^1([0, T]; \mathbb{R})$ *and* $r \geq 0$ *such that for all* $t \in [0, T]$ *and* $x \in \mathbb{R}^N$*, it holds* $|\nabla_x F(t, x)| \leq \alpha_1(t)|x|^r + \beta_1(t)$*.*

If either (1.8) *or* (1.9) *holds true, then* (5.1) *has as least one solution.*

For the reader convenience we highlight below the results in Theorem 1.1 and Corollary 5.1 in the particular case when $p = 2$.

Corollary 5.3 *Assume that* $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (H_F) (i), (ii) *and*

there exist $\alpha, \beta \in L^1([0, T]; \mathbb{R})$ *with* $\|\alpha\|_{L^1} < 6/T$ *so that for all* $t \in [0, T]$*,* $x \in \mathbb{R}^N$ *and* $\xi \in \partial F(t, x)$*, it holds* $|\xi| \leq \alpha(t)|x| + \beta(t)$*.*

If there exists $\eta > 0$ *such that*

$$\max_{x \in \mathbb{R}^N, |x| = \eta} \int_0^T F(t, x)dt + \frac{T\left(\eta\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)^2}{6 - T\|\alpha\|_{L^1}} < 0, \tag{5.2}$$

then

$$-\left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)' \in \partial F(t, u), \quad u(0) = u(T),\ u'(0) = u'(T),$$

has at least one solution.

Corollary 5.4 *Assume* (H_F) *(i) and* $(\widehat{H}_F)$ *(i). If there exists* η *a positive constant such that* (5.2) *holds true and* F *verifies*

> *there exist* $\alpha, \beta \in L^1([0,T];\mathbb{R})$ *with* $\|\alpha\|_{L^1} < 6/T$ *such that for all* $t \in [0,T]$ *and* $x \in \mathbb{R}^N$*, it holds* $|\nabla_x F(t,x)| \le \alpha(t)|x| + \beta(t)$,

then system

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' = \nabla_u F(t,u), \quad u(0) = u(T),\ u'(0) = u'(T)$$

has as least one solution.

Next, we give some examples of applications of the above results to the existence of Filippov type periodic solutions [7] for differential inclusions. For simplicity we restrict ourselves to the following one-dimensional ($N = 1$) discontinuous boundary value problem:

$$-\mathcal{R}_p u \in [\underline{f}(u), \overline{f}(u)], \qquad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{5.3}$$

where $f \in L^\infty_{loc}(\mathbb{R};\mathbb{R})$ and for $x \in \mathbb{R}$, as usual, we have denoted

$$\underline{f}(x) = \lim_{\delta \searrow 0} \operatorname{essinf}\{f(y):\ |x-y| < \delta\}$$

and

$$\overline{f}(x) = \lim_{\delta \searrow 0} \operatorname{esssup}\{f(y):\ |x-y| < \delta\}.$$

Setting

$$F_1(x) = \int_0^x f(\tau)d\tau, \quad (x \in \mathbb{R}),$$

one has (see e.g., [17, Proposition 1.7]):

$$\partial F_1(x) = [\underline{f}(x), \overline{f}(x)], \quad \forall\ x \in \mathbb{R}. \tag{5.4}$$

Example 5.1 Problem (5.3) with $T = 1$, $p = 2$ and

$$f(x) = \begin{cases} \dfrac{1}{4}\left(-\dfrac{3}{2} + x\cos\left(\dfrac{1}{x}\right) + \dfrac{1}{2}\sin\left(\dfrac{1}{x}\right)\right), & x > 0, \\[2ex] \dfrac{1}{4}\left(-\dfrac{1}{2} - \sin x - x\cos x\right), & x < 0 \end{cases}$$

has at least one solution. Indeed, one has

$$F_1(x) = \begin{cases} \dfrac{1}{4}\left(-\dfrac{3x}{2} + \dfrac{x^2}{2}\cos\left(\dfrac{1}{x}\right)\right), & x > 0, \\[2ex] \dfrac{1}{4}\left(-\dfrac{x}{2} - x\sin x\right), & x \le 0 \end{cases}$$

and, on account of (5.4) (also see [2, Example 2])

$$[\underline{f}(x), \overline{f}(x)] = \partial F_1(x) = \begin{cases} \{f(x)\}, & x \neq 0, \\ \left[-\frac{1}{2}, -\frac{1}{8}\right], & x = 0. \end{cases}$$

Then, $\alpha \equiv 1/4$ $(< 6 = 6/T)$, $\beta \equiv 1/2$ and taking $\eta = \pi/2$, one gets

- $\xi := \dfrac{T\left(\eta\|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)^2}{6 - T\|\alpha\|_{L^1}} = \dfrac{4}{23}\left(\dfrac{\pi}{8} + \dfrac{1}{2}\right)^2 \cong 0,13859;$
- $F_1(\eta) = -\dfrac{3\pi}{16} + \dfrac{\pi^2}{32}\cos\left(\dfrac{2}{\pi}\right) \cong -0,34104;$
- $F_1(-\eta) = -\dfrac{\pi}{16} \cong -0,19634.$

Thus

$$\max_{|x|=\eta} F_1(x) + \xi \cong -0,05775 < 0$$

and (5.2) holds true. Hence, the result follows from Corollary 5.3 with $F(t,x) = F_1(x)$. We point out that F here does not satisfy (1.9) $(r = 1)$ and so we cannot apply Theorem 1.2.

For the next example we consider problem (5.3) depending on a positive parameter λ, i.e.,

$$-\mathcal{R}_p u \in \lambda[\underline{f}(u), \overline{f}(u)], \qquad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{5.5}$$

Example 5.2 For all $\lambda > 0$, problem (5.5) with $T = 1$, $p = 4$ and

$$f(x) = \begin{cases} -1 - x^3, & x > 0, \\ -\frac{1}{16} - x^3, & x < 0 \end{cases}$$

is solvable. Here, one has

$$F_1(x) = \begin{cases} -x - \frac{x^4}{4}, & x \geq 0, \\ -\frac{x}{16} - \frac{x^4}{4}, & x < 0 \end{cases}$$

and by virtue of (5.4),

$$[\underline{f}(x), \overline{f}(x)] = \partial F_1(x) = \begin{cases} \{f(x)\}, & x \neq 0, \\ \left[-1, -\frac{1}{16}\right], & x = 0. \end{cases}$$

Then, for all positive λ, condition (1.9) is fulfilled ($r = 3$) and so we can apply Theorem 1.2 with $F(t,x) = \lambda F_1(x)$. Also, as $\alpha = \beta \equiv \lambda$, taking $\eta = 1$ in (5.2), one gets

$$\max_{x\in\mathbb{R}^N, |x|=1} \lambda F_1(x) + \frac{(\sigma_{1,4})^{\frac{1}{3}} \left(\mu_3 \|\alpha\|_{L^1} + \|\beta\|_{L^1}\right)^{\frac{4}{3}}}{2\left(\dfrac{1}{2} - \dfrac{\mu_3\sigma_{1,4}}{8}\|\alpha\|_{L^1}\right)^{\frac{1}{3}}} = \lambda \left(\frac{5\cdot 343^{\frac{2}{3}}(135\lambda)^{\frac{1}{3}}}{686\left(\frac{1}{2} - \frac{27\lambda}{686}\right)^{\frac{1}{3}}} - \frac{3}{16} \right).$$

Hence, for sufficiently small λ, conditions (1.3) and (5.2) hold true and in this case we can use also Theorem 1.1 to obtain the solvability of problem (5.5). We point out that if we take a large λ (for example $\geq 343/27 = 2/(\mu_3\sigma_{1,4})$), then we cannot apply Theorem 1.1 because α does not verify (1.3).

We conclude by an example when condition (1.8) is fulfilled.

Example 5.3 Let $T > 0$, $p > 1$, $q > 2$ and $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(t,x) = \left(\frac{2T}{3} - t\right)\frac{|x|^q}{q} + |x_1| + (e(t)|x) \quad (t \in [0,T],\ x = (x_1,x_2) \in \mathbb{R}^2),$$

where $e = (e_1, e_2) \in L^1([0,T];\mathbb{R}^2)$. Denoting

$$h_i(t,x) := \left(\frac{2T}{3} - t\right)|x|^{q-2}x_i + e_i(t) \quad (i = 1,2)$$

one has

$$\partial F(t,x) = \begin{cases} \{(h_1(t,x)+1, h_2(t,x))\}, & x_1 > 0, \\ (h_1(t,x) + [-1,1], h_2(t,x)), & x_1 = 0, \\ \{(h_1(t,x)-1, h_2(t,x))\}, & x_1 < 0. \end{cases}$$

From Theorem 1.2, the two dimensional system ($u = (u_1,u_2)$)

$$\begin{cases} -\left(\dfrac{|u'|^{p-2}u_1'}{(1-|u'|^p)^{1-1/p}}\right)' \in \begin{cases} \{h_1(t,u)+1\}, & u_1(t) > 0, \\ h_1(t,u) + [-1,1], & u_1(t) = 0, \\ \{h_1(t,u)-1\}, & u_1(t) < 0, \end{cases} \\ -\left(\dfrac{|u'|^{p-2}u_2'}{(1-|u'|^p)^{1-1/p}}\right)' = h_2(t,u), \\ u(0) = u(T),\ u'(0) = u'(T) \end{cases}$$

has at least one solution because (1.8) holds true with $r = q - 1$.

Acknowledgements. The work of Călin Şerban was supported by the grant of Ministery of Research and Innovation, CNCS - UEFISCDI, project number **PN-III-P1-1.1-PD-2016-0040**, title "*Multiple solutions for systems with singular ϕ-Laplacian operator*", within PNCDI III.
This study was partly funded by Research project of MIUR (Italian Ministry of Education, University and Research) Prin 2017 "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods"
(Grant Number:2017AYM8XW).
The first two authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

The authors thank the anonymous referee for his (her) useful comments, leading to the improvement of the presentation of the paper.

References

[1] S. Ahmad, A.C. Lazer and J.L. Paul, Elementary critical point theory and perturbations of elliptic boundary value problems at resonance, *Indiana Math. J.* **25** (1976), 933–944.

[2] K.-C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981), 102–129.

[3] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

[4] F. Colasuonno, A. Iannizzotto and D. Mugnai, Three solutions for a Neumann partial differential inclusion via nonsmooth Morse theory, *Set-Valued Var. Anal.* **25** (2017), 405–425.

[5] J.X. Feng, Z.Q. Han, Periodic solutions to differential systems with unbounded or periodic nonlinearities, *J. Math. Anal. Appl.* **323** (2006), 1264–1278.

[6] L. Ferracuti and F. Papalini, Boundary-value problems for strongly nonlinear multivalued equations involving different Φ-Laplacians, *Adv. Differ. Equ.* **14** (2009), 541–566.

[7] A.F. Filippov, Right hand side discontinuous differential equations, *Trans. Amer. Math. Soc.* **42** (1964), 199–227.

[8] L. Gasiński, Multiplicity theorems for periodic systems with a p-Laplacian-like operator, *Nonlinear Anal.* **67** (2007), 2632–2641.

[9] Z.Q. Han, 2π-periodic solutions to n-dimensional systems of Duffing's type, *J. Qingdao Univ.* **7** (1994), 19–26 (in Chinese).

[10] A. Iannizzotto, S.A. Marano and D. Motreanu, Positive, negative, and nodal solutions to elliptic differential inclusions depending on a parameter, *Adv. Nonlinear Stud.* **13** (2013), 431–445.

[11] P. Jebelean, Variational methods for ordinary p-Laplacian systems with potential boundary conditions, *Adv. Differ. Equ.* **13** (2008), 273–322.

[12] P. Jebelean, J. Mawhin and C. Şerban, Multiple periodic solutions for perturbed relativistic pendulum systems, *Proc. Amer. Math. Soc.* **143** (2015), 3029–3039.

[13] P. Jebelean, J. Mawhin and C. Şerban, Periodic solutions for discontinuous perturbations of the relativistic operator, *Bull. Sci. Math.* (2015), **140** (2016), 99–117.

[14] P. Jebelean, J. Mawhin and C. Şerban, A vector p-Laplacian type approach to multiple periodic solutions for the p-relativistic operator, *Commun. Contemp. Math.* **19** (2017), 16 pages.

[15] P. Jebelean and C. Şerban, Boundary value problems for discontinuous perturbations of singular ϕ-Laplacian operator, *J. Math. Anal. Appl.* **431** (2015), 662–681.

[16] V.E. Levin and S.B. Stechkin, *Supplement to the Russian edition of Hardy, Littlewood and Pólya*, in: G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities (Russian edition), Foreign Literature, Moscow, 1948, 361–441.

[17] D. Motreanu and P.D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Springer, Nonconvex Optimization and Its Applications, Vol. 29, Dordrecht, 1999.

[18] D. Motreanu and V. Rădulescu, *Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems*, Springer, Nonconvex Optimization and Its Applications, Vol 67, Dordrecht, 2003.

[19] F. Papalini, Strongly nonlinear multivalued systems involving singular Φ-Laplacian operators, *Comm. Pure Appl. Analysis* **9 (4)** (2010), 1025–1040.

[20] R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, 1972.

[21] S. Shang, Z. Bai, Y. Tian and Y. Yue, Periodic solution for second-order impulsive differential inclusions with relativistic operator, *Bound. Value Probl.* **173** (2018), 19 pp.

[22] C.L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, *Proc. Amer. Math. Soc.* **126** (1998), 3263–3270.

[23] X.P. Wu and C.L. Tang, Periodic solutions of a class of non-autonomous second order systems, *J. Math. Anal. Appl.* **236** (1999), 227–235.

[24] E.A. Zernyshkina, The Wirtinger-Steklov inequality between the norm of a periodic function and the norm of the positive cutoff of its derivative, *Proc. Steklov Inst. Math.* **264** (2009), Supplement 1, 199–213.