# ON THE SUB-SUPERSOLUTION APPROACH FOR DIRICHLET PROBLEM DRIVEN BY $(p(x), q(x))$-LAPLACIAN OPERATOR WITH CONVECTION TERM

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Abstract. The method of sub and super-solution is applied to obtain existence and location of solutions to a quasilinear elliptic problem with variable exponent and Dirichlet boundary conditions involving a nonlinear term $f$ depending on solution and on its gradient. Under a suitable growth condition on the convection term $f$, the existence of at least one solution satisfying a priori estimate is obtained.

## 1. Introduction

The aim of this paper is to prove the existence of solutions for following Dirichlet problem

$$(P) \qquad \begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) is a non–empty bounded open domain with a Lipschitz boundary $\partial\Omega$.

In problem $(P)$ we assume that $p,\, q \in C(\overline{\Omega})$ and $1 < q(x) < p(x) < \infty$, for all $x \in \overline{\Omega}$. The so called $(p(x), q(x))$-Laplacian is the differential operator on the left-hand side of the equation and it is identified by operators $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ and $\Delta_{q(x)}u := \operatorname{div}(|\nabla u|^{q(x)-2}\nabla u)$ known as $p(x)-$Laplacian operator and $q(x)-$Laplacian operator, respectively. When $p(x) = p$ and $q(x) = q$ (constants) it becomes the usual $(p, q)$-Laplacian differential operator. The nonlinearity $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ of problem $(P)$ is a Carathéodory function , i.e. $f(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$. Problems with variable exponent find interesting applications in different physical problems, including the study of smart fluids, as electrorheological fluids ([1], [2], [6], [27], [28]), and in power electronic about the study of the thermistors ([4], [16]). There is a relevant literature about these problems, whose solutions are generally obtained by improving variational and topological methods (see for instance [5]-[25]). A detailed presentation of these problems can be found in the survey of Radulescu ([26]). The convection term $f$ depends on the solution $u$ and on its gradient $\nabla u$. In this case, problem $(P)$ can not be treated with classical variational methods. For this reason, we use another approach, based on the method of sub-supersolution, to obtain the existence of at least one solution. Recent results related to this method, applied to Dirichlet problems with $p(x)$-Laplacian operator without convection term can be found in [8], [9], [12], [15], [18], [19], [20].

*Key words and phrases.* $(p(x), q(x))$ −Laplacian, Dirichlet problem, gradient dependence, sub-supersolution, positive solution.

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For results with $p$-Laplacian or $(p,q)$-Laplacian operator ($p$, $q$= constant) and convection term we cite [14], [24] and therein references.

We want to emphasize that, although sub- supersolution method has been used to study problems in classical Sobolev spaces, for use it in Sobolev spaces with variable exponents new ideas and nontrivial techniques are needed. In order to apply sub-supersolution method the first objective is to find a subsolution and a supersolution of problem $(P)$. We point out that in [23] the authors studied a nonhomogeneous Neumann problems with gradient depedence showing explicitly how one can effectively determine sub-supersolutions.

In the case of $p$-Laplacian or $(p,q)$-Laplacian operator ($p$, $q$= constant) a subsolution can be construct using its first eigenvalue (see for example the recent paper of Motreanu [22]) but the $p(x)$-Laplacian operator is inhomogeneous and, usually the infimum of its eigenvalue is 0 (see [13]) so the first eingenvalue can not be used to costruct a subsolution. Then to obtain one subsolution of problem $(P)$, we consider an auxialiary Dirichlet problem, whose nonlinear term satisfies an appropriate growth condition respect on convection term. According to our knowlegde, it is the first time when the method of sup-supersolution is implemented for Dirichlet problems with convection term in variable exponent spaces. In particular, in Theorem 3.2 we prove that if $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution (respectively) of problem $(P)$ satisfying the condition $\underline{u} \leq \overline{u}$ a.e. in $\Omega$, then problem $(P)$ admits at least one solution $u$ satifying a priori estimate $\underline{u} \leq u \leq \overline{u}$ a.e. in $\Omega$.

In the final part of the paper we show, under verifiable conditions, how our abstract result, Theorem 3.2, can be effectively applied for obtaining the existence of at least one nonnegative solution for problem $(P)$.

The paper is arranged as follows. Section 2 contains some preliminary properties and basic notations on variable exponent Lebesgue and Sobolev spaces. Section 3 focuses on the solvability of an auxiliary problem, depending on a real parameter and which represents a coercive perturbation of problem $(P)$. Section 4 presents our main result and finally, Section 5 illustrates how the abstract result can be applied to obtain the existence of at least one nonnegative solution to problem $(P)$.

## 2. Preliminaries and basic notations

In this section, we recall definitions and tools used in the paper.

Let $p \in C(\overline{\Omega})$ and denote by

$$p^- = \min_{\overline{\Omega}} p(x), \quad p^+ = \max_{\overline{\Omega}} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, we introduce the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ defined by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \quad \text{such that } u \text{ is measurable and } \int_\Omega |u|^{p(x)}\, dx < +\infty \right\},$$

and equipped with the Luxemburg norm defined by

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \delta > 0 : \int_\Omega \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\}.$$

These spaces extend classical Banach spaces. So they are reflexive, separable and uniformly convex Banach spaces. The dual space of $L^{p(x)}(\Omega)$ is the space $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ for all $x\in\Omega$.

In the variable Lebesgue spaces the Hölder inequality holds and we recall it (for more details [12]).

$$\left|\int_\Omega uvdx\right|\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}\le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)},$$

for all $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$.

Now, we consider the modular given by

$$\rho_{p(x)}(u)=\int_\Omega |u(x)|^{p(x)}dx\quad\forall u\in L^{p(x)}(\Omega).$$

We have the following proposition

**Proposition 2.1.** *([12])*

(i) $\|u\|_{L^{p(x)}(\Omega)}<1\,(=1,\ >1)\ \Leftrightarrow\ \rho_{p(x)}(u)<1\ (=1,\ >1)$;

(ii) $\|u\|_{L^{p(x)}(\Omega)}>1\ \Rightarrow\ \|u\|^{p^-}_{L^{p(x)}(\Omega)}\le\rho_{p(x)}(u)\le\|u\|^{p^+}_{L^{p(x)}(\Omega)}$;

(iii) $\|u\|_{L^{p(x)}(\Omega)}<1\ \Rightarrow\ \|u\|^{p^+}_{L^{p(x)}(\Omega)}\le\rho_{p(x)}(u)\le\|u\|^{p^-}_{L^{p(x)}(\Omega)}$.

We want to observe that $\rho_{p(x)}(u)$ and $\|u\|_{L^{p(x)}(\Omega)}$ are linked by the following relation too.

$$\|u\|^{p^-}_{L^{p(x)}(\Omega)}-1\le\rho_{p(x)}(u)\le\|u\|^{p^+}_{L^{p(x)}(\Omega)}+1\quad\forall u\in L^{p(x)}(\Omega). \tag{2.1}$$

Moreover, see [17], if $p_1,\,p_2\in C(\overline{\Omega})$ , $p_1(x)>1$, $p_2(x)>1$ and $p_1(x)\le p_2(x)$ for all $x\in\overline{\Omega}$, then the embedding $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$ is continuous and it results

$$\|u\|_{L^{p_1(x)}(\Omega)}\le(1+|\Omega|)\|u\|_{L^{p_2(x)}(\Omega)}, \tag{2.2}$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$ in $\mathbb{R}^N$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega):=\left\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\right\},$$

and it is equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}:=\|u\|_{L^{p(x)}(\Omega)}+\||\nabla u|\|_{L^{p(x)}(\Omega)}.$$

By $W^{1,p(x)}_0(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\|:=\||\nabla u|\|_{L^{p(x)}(\Omega)}.$$

It is well known that $W^{1,p(x)}(\Omega)$, is separable, reflexive and uniformly convex Banach space.

Denoted by

$$p^*(x)=\begin{cases}\dfrac{p(x)N}{N-p(x)} & \text{if }\ p(x)<N,\\ +\infty & \text{if }\ p(x)\ge N,\end{cases}$$

the critical exponent of $p(x)$, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is continuous and compact for each $r \in C(\overline{\Omega})$ with $r(x) > 1$ and $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$. In the sequel we denote by $k_r$ the best constant for which one has

$$\|u\|_{L^{r(x)}(\Omega)} \leq k_r \|u\|_{W_0^{1,p(x)}(\Omega)}. \tag{2.3}$$

Finally, for every $u \in W_0^{1,p(x)}(\Omega)$ we introduce $u_+$, $u_- \in W_0^{1,p(x)}(\Omega)$ defined as $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$.

We refer to [11], [17] for more details on the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$.
Study of problem $(P)$ is based on the sub-supersolution method. For more details about this topic we cite [3] and [21].

A *solution* of problem $(P)$ is any function $u \in W_0^{1,p(x)}(\Omega)$ such that $f(x, u, \nabla u) \in L^{p'(x)}(\Omega)$ and

(2.4)
$$\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega |\nabla u(x)|^{q(x)-2}\nabla u(x) \cdot \nabla v(x)dx = \int_\Omega f(x, u, \nabla u)v\, dx,$$

$\forall v \in W_0^{1,p(x)}(\Omega)$.

A function $\overline{u} \in W^{1,p(x)}(\Omega)$ is a *supersolution* for problem $(P)$ if $\overline{u} \geq 0$ on $\partial\Omega$ such that $f(x, \overline{u}, \nabla\overline{u}) \in L^{p'(x)}(\Omega)$ and

(2.5)
$$\int_\Omega |\nabla \overline{u}(x)|^{p(x)-2}\nabla \overline{u}(x) \cdot \nabla v(x)dx + \int_\Omega |\nabla \overline{u}(x)|^{q(x)-2}\nabla \overline{u}(x) \cdot \nabla v(x)dx \geq \int_\Omega f(x, \overline{u}, \nabla \overline{u})v\, dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$, $v \geq 0$ a.e. in $\Omega$.

A function $\underline{u} \in W^{1,p(x)}(\Omega)$ is a *subsolution* for problem $(P)$ if $\underline{u} \leq 0$ on $\partial\Omega$ such that $f(x, \underline{u}, \nabla\underline{u}) \in L^{p'(x)}(\Omega)$ and

(2.6)
$$\int_\Omega |\nabla \underline{u}(x)|^{p(x)-2}\nabla \underline{u}(x) \cdot \nabla v(x)dx + \int_\Omega |\nabla \underline{u}(x)|^{q(x)-2}\nabla \underline{u}(x) \cdot \nabla v(x)dx \leq \int_\Omega f(x, \underline{u}, \nabla \underline{u})v\, dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$, $v \geq 0$ a.e. in $\Omega$.

Now, we recall the useful definitions and the main theorem on peudomonotone operators. Let $X$ be a real reflexive Banach space with norm $\|\cdot\|$, $X^*$ its dual space and $\langle .,. \rangle$ the duality paring between them. A mapping $A : X \to X^*$ is called coercive if $\lim_{\|u\|\to\infty} \frac{\langle Au,u\rangle}{\|u\|} = +\infty$; The map $A$ is called pseudomonotone if for all sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0$ imply that $Au_n \rightharpoonup Au$ and $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$.

**Theorem 2.1.** ([3, Theorem 2.99]) *Let $X$ be a real reflexive Banach space and let $A : X \to X^*$ be a bounded, coercive and pseudomonotone operator. Then, for every $b \in X^*$ the equation $Ax = b$ has at least one solution $x \in X$.*

In the sequel we will use the pair $\left(W_0^{1,p(x)}(\Omega),\, W^{-1,p'(x)}(\Omega)\right)$ where $W^{-1,p'(x)}(\Omega) := \left(W_0^{1,p(x)}(\Omega)\right)^*$. Important properties of the negative $p(x)$-Laplacian operator are listed in the next proposition and for the proof we refer to [12].

**Proposition 2.2.** *The negative $p(x)$-Laplacian operator $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined as*

$$\langle -\Delta_{p(x)}u,\, v\rangle = \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)\,dx$$

*for all $u, v \in W_0^{1,p(x)}(\Omega)$ is continuous, bounded, strictly monotone (hence maximal monotone and pseudomonotone too) and it has $S_+-$property i.e. every sequence $\{u_n\}$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\limsup_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle \le 0$ implies that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.*

We introduce the map $\Gamma : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$\langle \Gamma u, v\rangle = \langle -\Delta_{p(x)}u - \Delta_{q(x)}u,\, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx + \int_\Omega |\nabla u|^{q(x)-2}\nabla u\cdot\nabla v dx \tag{2.7}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$, corresponding to $(p(x), q(x))$-Laplacian, which is a bounded and a maximal monotone operator (see Proposition 2.2). By applying [21, Proposition 2.70] it has the $S_+-$property in turn. Moreover, since

$$\langle \Gamma u, u\rangle \ge \|u\|^{p^-} \quad \text{if } \|u\| > 1,$$

$\Gamma$ is coercive too.

Finally, with standard arguments it is possible to prove that the following comparison principle holds.

**Lemma 2.1.** *Let $u, v \in W^{1,p(x)}(\Omega)$ with $u \le v$ on $\partial\Omega$ and such that*

$$\int_{\{u>v\}} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\cdot\nabla(u-v)dx = 0.$$

*Then $u \le v$ a.e. in $\Omega$.*

## 3. Main result

Let us admit that a subsolution $\underline{u} \in W^{1,p(x)}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p(x)}(\Omega)$ for problem $(P)$ with $\underline{u}(x) \le \overline{u}(x)$ a.e. in $\Omega$ are given. Our main goal is to obtain a solution $u \in W_0^{1,p(x)}(\Omega)$ of problem $(P)$ which satisfies the property $\underline{u} \le u \le \overline{u}$ a.e. in $\Omega$. We are going to use comparison and truncation techniques. To this aim we define the truncation operator $T : W_0^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ associated with the ordered pair of sub-supersolution $\underline{u}$, $\overline{u}$ of problem $(P)$, given by

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) \ge \overline{u}(x), \\ u(x) & \text{if} \quad \underline{u}(x) < u(x) < \overline{u}(x), \\ \underline{u}(x) & \text{if} \quad u(x) \le \underline{u}(x), \end{cases} \tag{3.1}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. On the basis of (3.1) it is easy to verify that $T : W_0^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is continuous and bounded (in the sense that it maps bounded sets into bounded sets). The ordered interval $[\underline{u}, \overline{u}]$ associated to the ordered pair $\underline{u} \le \overline{u}$ is introduced as

$$[\underline{u}, \overline{u}] = \{u \in W_0^{1,p(x)}(\Omega) \;:\; \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ for a.e. } x \in \Omega\}.$$

Taking into account (3.1), we have $T(u) \in [\underline{u}, \overline{u}]$ for all $u \in W^{1,p(x)}(\Omega)$.

We assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ in $(P)$ satisfies the following growth condition

$(H)$ there exist a positive constant $a$ and a function $\sigma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^{p(x)-1} \quad \text{for a.e.}\, x \in \Omega, \text{ for all}\, s \in [\underline{u}(x), \overline{u}(x)], \text{ for all}\, \xi \in \mathbb{R}^N.$$

We point out that using hypothesis $(H)$ it is easy to prove that $f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ for all $u \in [\underline{u}, \overline{u}]$. Then $(H)$ guarantees the existence of integrals in (2.4), (2.5) and (2.6).

Consider the operator $N : [\underline{u}, \overline{u}] \to W^{-1,p'(x)}(\Omega)$, called Nemytskij operator and defined by

$$\langle N(u), v \rangle = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx,$$

for all $u \in [\underline{u}, \overline{u}]$, $v \in W_0^{1,p(x)}(\Omega)$ and where $f$ is the function which appears in $(P)$. It is well defined by virtue of hypothesis $(H)$, moreover the compact embedding theorem implies that the Nemytskij operator $N : [\underline{u}, \overline{u}] \to W^{-1,p'(x)}(\Omega)$ is completely continuous, since the operator $N$ is the composition of the mapping $u \to f(x,u,\nabla u)$ taking value in $L^{p'(x)}(\Omega)$ (which is continuous and bounded by hypothesis (H)) and the linear embedding $L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) = \left(W_0^{1,p(x)}(\Omega)\right)^*$ (which is compact, because it is the adjoint of the compact embedding $W_0^{1,p(x)}(\Omega) \to L^{p(x)}$ ).

We introduce the cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\pi(x,s) = \begin{cases} (s - \overline{u}(x))^{p(x)-1} & \text{if} \quad s > \overline{u}(x), \\ 0 & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x), \\ -(\underline{u}(x) - s)^{p(x)-1} & \text{if} \quad s < \underline{u}(x). \end{cases} \tag{3.2}$$

From (3.2) we obtain the following relation

$$|\pi(x,s)| \leq c|s|^{p(x)-1} + \varrho(x) \text{ for a.e.}\, x \in \Omega, \text{ all}\, s \in \mathbb{R} \tag{3.3}$$

with $c = \max\{1,\, 2^{p^+-1}\}$ and $\varrho \in L^{p'(x)}(\Omega)$ defined by

$$\varrho(x) = \max\{|\underline{u}(x)|^{p(x)-1},\, |\overline{u}(x)|^{p(x)-1}\}.$$

It is useful to point out some estimates related to function $\pi$.

**Proposition 3.1.** *The cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$, given by (3.2), satisfies the following estimates:*

$$\int_\Omega \pi(x,u(x))u(x)\,dx \geq r_1 \int_\Omega |u|^{p(x)}\,dx - r_2, \tag{3.4}$$

$$\int_\Omega |\pi(x,u(x))||v(x)|\,dx \leq \left( r_3 \|u\|_{L^{p(x)}(\Omega)}^{\frac{p^+}{p'^-}} + r_4 \right) \|v\|_{L^{p(x)}(\Omega)}\,, \tag{3.5}$$

*with $r_1, r_2, r_3, r_4$ positive constants.*

*Proof.* For the proof of the estimate (3.4) we observe that since $p \in C(\overline{\Omega})$ and $p(x) > 1$ for all $x \in \overline{\Omega}$, then there exist two positive constants $c_1$ and $c_2$ such that for each $\xi, \eta \in \mathbb{R}$ and for all $x \in \overline{\Omega}$ it results

$$(3.6)\qquad \begin{aligned} (\xi-\eta)^{p(x)-1}\xi &\geq c_1|\xi|^{p(x)} - c_2|\eta|^{p(x)-1}|\xi| \quad \textit{if}\ \xi \geq \eta \\ (\eta-\xi)^{p(x)-1}\xi &\leq -c_1|\xi|^{p(x)} + c_2|\eta|^{p(x)-1}|\xi| \quad \textit{if}\ \xi < \eta. \end{aligned}$$

From (3.6), for each $u \in W_0^{1,p(x)}(\Omega)$ one has

$$\int_\Omega \pi(x,u(x))u(x)\,dx = \int_{\{u>\bar{u}\}} (u(x)-\bar{u}(x))^{p(x)-1}\,u(x)\,dx - \int_{\{u<\underline{u}\}} (\underline{u}(x)-u(x))^{p(x)-1}\,u(x)\,dx \geq$$

$$\int_{\{u>\overline{u}\}} \left(c_1|u(x)|^{p(x)} - c_2|\bar{u}(x)|^{p(x)-1}|u(x)|\right)\,dx + \int_{\{u<\underline{u}\}} \left(c_1|u(x)|^{p(x)} - c_2|\underline{u}(x)|^{p(x)-1}|u(x)|\right)\,dx \geq$$

$$c_1\left[\int_\Omega |u(x)|^{p(x)}\,dx - \int_{\{\underline{u}\leq u\leq \underline{u}\}} |u(x)|^{p(x)}\,dx\right] - c_2\int_\Omega \left(|\bar{u}(x)|^{p(x)-1} + |\underline{u}(x)|^{p(x)-1}\right)|u(x)|\,dx.$$

Then Young inequality applied with $\bar{\epsilon} = \frac{c_1}{2c_2}$ and the relation $|u(x)| \leq |\bar{u}(x)| + |\underline{u}(x)|$ verified for each $x \in \Omega$ such that $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ show that

$$\begin{aligned} \int_\Omega \pi(x,u(x))u(x)\,dx &\geq c_1\left[\int_\Omega |u(x)|^{p(x)}\,dx - \int_\Omega (|\bar{u}(x)| + |\underline{u}(x)|)^{p(x)}\,dx\right] - \\ & c_2\left[\frac{c_1}{2c_2}\int_\Omega |u(x)|^{p(x)}\,dx + c_{\bar{\epsilon}}\int_\Omega \left(|\bar{u}(x)|^{p(x)-1} + |\underline{u}(x)|^{p(x)-1}\right)^{p'(x)}\,dx\right] = \\ &= r_1\int_\Omega |u(x)|^{p(x)}\,dx - r_2 \end{aligned}$$

with positive constants $c_{\bar{\epsilon}}$, $r_1 := \frac{c_1}{2}$ and

$r_2 := c_1\int_\Omega(|\underline{u}(x)| + |\bar{u}(x)|)^{p(x)}\,dx + c_2c_{\bar{\epsilon}}\int_\Omega \left(|\bar{u}(x)|^{p(x)-1} + |\underline{u}(x)|^{p(x)-1}\right)^{p'(x)}\,dx.$

The proof of estimate (3.5), follows by (3.2), using Hölder inequality and the modular. For each $u, v \in W_0^{1,p(x)}(\Omega)$ one has

(3.7)

$$\int_\Omega |\pi(x,u(x))||v(x)|\,dx = \int_{\{u>\overline{u}\}} (u(x)-\overline{u}(x))^{p(x)-1}\,|v(x)|\,dx + \int_{\{u<\underline{u}\}} (\underline{u}(x)-u(x))^{p(x)-1}\,|v(x)|\,dx \leq$$

$$2\left(\|(u-\overline{u})^{p(x)-1}\|_{L^{p'(x)}(\Omega)} + \|(\underline{u}-u)^{p(x)-1}\|_{L^{p'(x)}(\Omega)}\right)\|v\|_{L^{p(x)}(\Omega)}.$$

From (2.1), we have

$$\begin{aligned} &\|(u-\overline{u})^{p(x)-1}\|^{p'^-}_{L^{p'(x)}(\Omega)} \leq 1 + \rho_{p'(x)}\left((u-\overline{u})^{p(x)-1}\right) = 1 + \rho_{p(x)}(u-\overline{u}) = \\ &1 + \int_\Omega |u-\overline{u}|^{p(x)}dx \leq 1 + 2^{p^+-1}\left(\rho_{p(x)}(u) + \rho_{p(x)}(\overline{u})\right) \leq \\ &1 + 2^{p^+-1}\left(\|\overline{u}\|^{p^+}_{L^{p(x)}(\Omega)} + 2\right) + 2^{p^+-1}\|u\|^{p^+}_{L^{p(x)}(\Omega)}. \end{aligned}$$

In the same way, we obtain

$$\|(\underline{u}-u)^{p(x)-1}\|^{p'^-}_{L^{p'(x)}(\Omega)} \leq 1 + 2^{p^+-1}\left(\|\underline{u}\|^{p^+}_{L^{p(x)}(\Omega)} + 2\right) + 2^{p^+-1}\|u\|^{p^+}_{L^{p(x)}(\Omega)}.$$

Substituing previous relations in (3.7) and taking into account that $p'^- > 1$, we obtain

$$\int_\Omega |\pi(x,u(x))||v(x)|\,dx \le \left( r_3 \|u\|_{L^{p(x)}(\Omega)}^{\frac{p^+}{p'^-}} + r_4 \right) \|v\|_{L^{p(x)}(\Omega)}$$

where $r_3 := 2^{\frac{p^+-1}{p'^-}+2}$, $r_4 := 2\left[ \left(1 + 2^{p^+-1}\left(\|\overline{u}\|_{L^{p(x)}(\Omega)}^{p^+} + 2\right)\right)^{\frac{1}{p'^-}} + \left(1 + 2^{p^+-1}\left(\|\underline{u}\|_{L^{p(x)}(\Omega)}^{p^+} + 2\right)\right)^{\frac{1}{p'^-}} \right]$.

□

Now, we perturb problem $(P)$ using the Nemytskij operator $\Pi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by $\Pi(u) = \pi(\cdot, u(\cdot))$ where $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined as in (3.2), the Nemytskij operator $N : [\underline{u}, \overline{u}] \to W^{-1,p'(x)}(\Omega)$ composed with the truncation operator $T : W_0^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ given by (3.1) and a parameter $\lambda > 0$. In this way we obtain the auxiliary truncated problem

$$(T_\lambda) \qquad \begin{cases} -\Delta_{p(x)} u - \Delta_{q(x)} u + \lambda \Pi(u) = N \circ T(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

from which we have following the result.

**Theorem 3.1.** *Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem $(P)$, respectively, with $\underline{u} \le \overline{u}$ a.e. in $\Omega$ such that hypothesis $(H)$ is fulfilled. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$ there is at least one solution $u \in W_0^{1,p(x)}(\Omega)$ of the auxiliary problem $(T_\lambda)$.*

*Proof.* For every $\lambda > 0$ let $A_\lambda : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the operator defined by

$$\langle A_\lambda(u), v\rangle = \langle \Gamma(u) + \lambda\Pi(u) - N \circ T(u), v\rangle \tag{3.8}$$

for each $u$, $v \in W_0^{1,p(x)}(\Omega)$. The composed operator $N \circ T$ is bounded because $T$ is bounded and $N$ is completely continuous. Since $\Gamma$ and $\Pi$ are bounded, it follows from (3.8) that $A_\lambda$ is bounded.

The operator $A_\lambda$ is pseudomonotone. Indeed, let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a sequence such that

$$u_n \rightharpoonup u \ \text{ in } W_0^{1,p(x)}(\Omega), \tag{3.9}$$

and

$$\limsup_{n\to\infty} \langle A_\lambda u_n, u_n - u\rangle \le 0. \tag{3.10}$$

Consider

$$\begin{aligned} \langle A_\lambda u_n, u_n - u\rangle &= \langle \Gamma(u_n),\, u_n - u\rangle \\ &+ \lambda \int_\Omega \pi(x, u_n)(u_n - u)\,dx - \int_\Omega f(x, Tu_n, \nabla Tu_n)(u_n - u)\,dx. \end{aligned} \tag{3.11}$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$, using a subsequence if necessary, we have $u_n \to u$ in $L^{p(x)}(\Omega)$. Using (3.2), (2.1) and the Hölder inequality we have

$$\begin{aligned}\left|\int_\Omega \pi(x,u_n)(u_n-u)\,dx\right| &\le \int_{\{u_n<\underline{u}\}} (\underline{u}-u_n)^{p(x)-1}|u_n-u|\,dx \\ &+ \int_{\{u_n>\overline{u}\}} (u_n-\overline{u})^{p(x)-1}|u_n-u|\,dx \\ &\le 2\Big[\big(2+\|\underline{u}-u_n\|^{p^+}_{L^{p(x)}(\Omega)}\big)^{\frac{1}{(p')^-}} \\ &+ \big(2+\|u_n-\overline{u}\|^{p^+}_{L^{p(x)}(\Omega)}\big)^{\frac{1}{(p')^-}}\Big]\|u_n-u\|_{L^{p(x)}(\Omega)} \to 0 \qquad (3.12)\end{aligned}$$

as $n\to+\infty$. Using hypothesis $(H)$ we have

$$\begin{aligned}\left|\int_\Omega f(x,Tu_n,\nabla Tu_n)(u_n-u)\,dx\right| &\le \int_\Omega |\sigma||u_n-u|dx \\ &+ a\int_\Omega |\nabla Tu_n|^{p(x)-1}\,|u_n-u|\,dx. \qquad (3.13)\end{aligned}$$

Because of (H), $\sigma\in L^{p'(x)}(\Omega)$ and using Hölder inequality, we get

$$\int_\Omega |\sigma||u_n-u|dx \le 2\,\|\sigma\|_{L^{p'(x)}(\Omega)}\,\|u_n-u\|_{L^{p(x)}(\Omega)} \to 0 \,\text{as}\, n\to\infty. \qquad (3.14)$$

We want to prove that

$$\int_\Omega |\nabla Tu_n|^{p(x)-1}\,|u_n-u|\,dx \to 0 \text{ as } n\to+\infty. \qquad (3.15)$$

The definition (3.1) of the truncation operator $T$ yields

$$\int_\Omega |\nabla Tu_n|^{p(x)-1}\,|u_n-u|\,dx = \int_{\{u_n<\underline{u}\}} |\nabla\underline{u}|^{p(x)-1}\,|u_n-u|\,dx +$$

$$\int_{\{\underline{u}\le u_n\le\overline{u}\}} |\nabla u_n|^{p(x)-1}\,|u_n-u|\,dx + \int_{\{u_n>\overline{u}\}} |\nabla\overline{u}|^{p(x)-1}\,|u_n-u|\,dx.$$

From Hölder inequality and (2.1), we obtain

$$\int_{\{u_n<\underline{u}\}} |\nabla\underline{u}|^{p(x)-1}\,|u_n-u|\,dx \le \int_\Omega |\nabla\underline{u}|^{p(x)-1}\,|u_n-u|\,dx \le 2\big(2+\||\nabla\underline{u}|\|^{p^+}_{L^{p(x)}(\Omega)}\big)^{\frac{1}{(p')^-}}\,\|u_n-u\|_{L^{p(x)}(\Omega)},$$

$$\int_{\{\underline{u}\le u_n\le\overline{u}\}} |\nabla u_n|^{p(x)-1}\,|u_n-u|\,dx \le \int_\Omega |\nabla u_n|^{p(x)-1}\,|u_n-u|\,dx \le 2\big(2+\||\nabla u_n|\|^{p^+}_{L^{p(x)}(\Omega)}\big)^{\frac{1}{(p')^-}}\,\|u_n-u\|_{L^{p(x)}(\Omega)},$$

$$\int_{\{u_n>\overline{u}\}} |\nabla\overline{u}|^{p(x)-1}\,|u_n-u|\,dx \le \int_\Omega |\nabla\overline{u}|^{p(x)-1}\,|u_n-u|\,dx \le 2\big(2+\||\nabla\overline{u}|\|^{p^+}_{L^{p(x)}(\Omega)}\big)^{\frac{1}{(p')^-}}\,\|u_n-u\|_{L^{p(x)}(\Omega)}.$$

Therefore (3.15) is verified.

Owing to (3.14), (3.15) and (3.13) we obtain

$$\lim_{n\to\infty}\int_\Omega f(x,Tu_n,\nabla Tu_n)(u_n-u)\,dx = 0. \qquad (3.16)$$

Due to (3.12) and (3.16), inequality (3.10) becomes

$$\limsup_{n\to\infty}\langle \Gamma(u_n), u_n - u\rangle \leq 0.$$

From $(S)_+-$property of the operator $-\Delta_{p(x)} - \Delta_{q(x)}$ in conjuction with (3.9), we have the strong convergence $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Then

$$\Gamma(u_n) \to \Gamma(u). \tag{3.17}$$

In view of (3.17) and $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, one has $A_\lambda u_n \rightharpoonup A_\lambda u$, $\langle A_\lambda u_n, u_n\rangle \to \langle A_\lambda u, u\rangle$, then the operator $A_\lambda$ is pseudomonotone.

In order to prove that the operator $A_\lambda : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is coercive we observe that in view of (3.8) and (3.4) we obtain

$$\begin{aligned} \langle A_\lambda u, u\rangle &\geq \int_\Omega |\nabla u(x)|^{p(x)}\,dx + \lambda\int_\Omega \pi(x,u)u\,dx - \int_\Omega f(x,Tu,\nabla(Tu))u\,dx \\ &\geq \int_\Omega |\nabla u(x)|^{p(x)}\,dx + \lambda r_1\int_\Omega |u|^{p(x)} - \lambda r_2 - \int_\Omega f(x,Tu,\nabla(Tu))u\,dx. \end{aligned}\tag{3.18}$$

For every $u \in W^{1,p(x)}(\Omega)$, using hypothesis $(H)$, Young inequality, Hölder inequality and (2.3), for each $\varepsilon > 0$ we have

$$\begin{aligned} &\left|\int_\Omega f(x,Tu,\nabla(Tu))u\,dx\right| \leq \int_\Omega \Big(\sigma(x)|u(x)| + a|\nabla(Tu)|^{p(x)-1}|u(x)|\Big)\,dx \\ &\leq 2k_p\,\|\sigma\|_{L^{p'(x)}(\Omega)}\,\|u\| + 2ak_p\left(\||\nabla\underline{u}|^{p(x)-1}\|_{L^{p'(x)}(\Omega)} + \||\nabla\overline{u}|^{p(x)-1}\|_{L^{p'(x)}(\Omega)}\right)\|u\| + \\ &a\left[\varepsilon\int_\Omega |\nabla u|^{p(x)}\,dx + c_\varepsilon\int_\Omega |u|^{p(x)}\,dx\right]. \end{aligned}\tag{3.19}$$

Taking into account (3.19), (2.3), (3.4) and (3.5) then (3.18) becomes

$$\begin{aligned} \langle A_\lambda u, u\rangle &\geq (1-a\epsilon)\int_\Omega |\nabla u|^{p(x)}\,dx - d\,\|u\| + (\lambda r_1 - ac_\epsilon)\int_\Omega |u(x)|^{p(x)}\,dx - \lambda r_2 \\ &\geq (1-a\epsilon)\,\|u\|^{p^-} - d\,\|u\| + (\lambda r_1 - ac_\epsilon)\int_\Omega |u(x)|^{p(x)}\,dx - \lambda r_2 \text{ for all } \|u\| \geq 1. \end{aligned}\tag{3.20}$$

Because of $p^- > 1$, choosing $\epsilon \in\, ]0,1[$, from (3.20) we get coercivity of operator $A_\lambda$ for all $\lambda > \frac{ac_\epsilon}{r_1}$.

Since the operator $A_\lambda : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is bounded, pseudomonotone and coercive, Theorem 2.1 ensures that exists $u \in W_0^{1,p(x)}(\Omega)$ which solves $(T_\lambda)$ and the proof is complete. □

**Remark 3.1.** We observe that it is possible to improve growth condition $(H)$ replacing it with the more general condition

$(H_\beta)$ there exist a positive constant $a$, a function $\sigma \in L^{p'(x)}(\Omega)$ and $\beta \in C(\bar{\Omega})$ with $0 \leq \beta^- \leq \beta^+ < \left(\frac{p}{p^*}\right)^-$ such that

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^{\frac{\beta(x)}{p(x)-\beta(x)}} \quad \text{for a.e. } x \in \Omega, \text{ for all } s \in [\underline{u}(x), \overline{u}(x)], \text{ for all } \xi \in \mathbb{R}^N.$$

This condition extends to variable case the growth condition present, for example, in [22] and [24]. In particular, condition $\beta^+ < \left(\frac{p}{p^*}\right)^-$ ensures the well posedness of space $L^{\frac{p}{p-\beta}(x)}(\Omega)$ and, in the light of Theorem 2.3 of [11], provides the compact embedding of $W_0^{1,p(x)}(\Omega)$ in $L^{\frac{p}{p-\beta}(x)}(\Omega)$, that we need in proof of Theorem 3.1. We should be observed that condition $(H_\beta)$ entailes other estimates, similar to (3.4) and (3.5).

Our main result on problem $(P)$ is the following.

**Theorem 3.2.** *Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem $(P)$, respectively, with $\underline{u} \leq \overline{u}$ a.e. in $\Omega$ such that hypothesis $(H)$ is fulfilled. Then problem $(P)$ possesses at least one solution $u \in W_0^{1,p(x)}(\Omega)$ satisfying the location property $\underline{u} \leq u \leq \overline{u}$ a.e. in $\Omega$.*

*Proof.* Fixed $\lambda > 0$ sufficiently large, Theorem 3.1 ensures the existence of $u \in W_0^{1,p(x)}(\Omega)$ that is a solution of the truncated auxiliary problem $(T_\lambda)$. We want to prove that $\underline{u} \leq u \leq \overline{u}$ a.e. in $\Omega$.

Choosing $(u-\overline{u})_+ \in W_0^{1,p(x)}(\Omega)$ as a test function in (2.5) and in $(T_\lambda)$, we obtain that

$$\langle \Gamma(\overline{u}), (u-\overline{u})_+ \rangle \geq \int_\Omega f(x,\overline{u},\nabla\overline{u})(u-\overline{u})_+\, dx, \tag{3.21}$$

and

$$\begin{aligned} \langle \Gamma(u), (u-\overline{u})_+ \rangle + \lambda \int_\Omega \pi(x,u)(u-\overline{u})_+ dx = \\ = \int_\Omega f(x,Tu,\nabla(Tu))(u-\overline{u})_+ dx. \end{aligned} \tag{3.22}$$

Subtract (3.21) from (3.22) and use (3.1) we get

$$\begin{aligned} \langle \Gamma(u), (u-\overline{u})_+ \rangle \quad &- \quad \langle \Gamma(\overline{u}), (u-\overline{u})_+ \rangle + \lambda \int_\Omega \pi(x,u)(u-\overline{u})_+\, dx \\ &\leq \quad \int_\Omega \big(f(x,Tu,\nabla(Tu)) - f(x,\overline{u},\nabla\overline{u})\big)(u-\overline{u})_+\, dx \\ &= \quad \int_{\{u>\overline{u}\}} \big(f(x,Tu,\nabla(Tu)) - f(x,\overline{u},\nabla\overline{u})\big)(u-\overline{u})\, dx = 0. \end{aligned} \tag{3.23}$$

We observe that the classical inequality

$$(|\xi|^{h-2}\xi - |\eta|^{h-2}\eta) \cdot (\eta - \xi) \geq 0 \quad \text{for } \xi, \eta \in \mathbb{R}^N \text{ and for each } h > 1 \tag{3.24}$$

ensures

$$\langle\Gamma(u),(u-\overline{u})_+\rangle-\langle\Gamma(\overline{u}),(u-\overline{u})_+\rangle$$
$$=\int_{\{u>\overline{u}\}}\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla\overline{u}|^{p(x)-2}\nabla\overline{u}\right)\cdot\nabla(u-\overline{u})dx$$
$$\int_{\{u>\overline{u}\}}\left(|\nabla u|^{q(x)-2}\nabla u-|\nabla\overline{u}|^{q(x)-2}\nabla\overline{u}\right)\cdot\nabla(u-\overline{u})dx\geq 0. \tag{3.25}$$

From (3.2), (3.23) and (3.25) we obtain that

$$0\leq\lambda\int_{\{u>\overline{u}\}}(u-\overline{u})^{p(x)}\,dx=\lambda\int_\Omega\pi(x,u)(u-\overline{u})_+\,dx\leq$$

$$\langle\Gamma(u),(u-\overline{u})_+\rangle-\langle\Gamma(\overline{u}),(u-\overline{u})_+\rangle+\lambda\int_\Omega\pi(x,u)(u-\overline{u})_+\,dx\leq 0.$$

It follows that $u\leq\overline{u}$ a.e in $\Omega$.

With similar calculations we can show that $\underline{u}\leq u$ a.e in $\Omega$. Consequently, it results $Tu=u$, $\Pi(u)=0$ and so the solution $u$ of the auxiliary truncated problem $(T_\lambda)$ is a solution of the original problem $(P)$. □

## 4. Application and example

Goal of this section is to costruct a subsolution $\underline{u}\in W^{1,p(x)}(\Omega)$ and a supersolution $\overline{u}\in W^{1,p(x)}(\Omega)$ of problem $(P)$ with $0\leq\underline{u}<\overline{u}$ a.e. in $\Omega$ in order to apply Theorem 4.1. We introduce the following assumptions:

(H1) there exists a function $g\in C(\overline{\Omega}\times\mathbb{R})$ with $g(x,0)\geq 0$, $g(x,0)\not\equiv 0$ $\forall x\in\overline{\Omega}$, such that $s\to g(x,s)$ is nonincreasing in $s\in[0,\infty[$, for all $x\in\Omega$, and

$$f(x,s,\xi)\geq g(x,s)\quad a.e.\,x\in\Omega,\ \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N;$$

(H2) there esists a positive constant $\alpha_0\in\mathbb{R}$ such that

$$f(x,\alpha_0,0)<0\ a.e.\ x\in\Omega.$$

The following result allows to get existence of nonnegative solutions to problem $(P)$.

**Theorem 4.1.** *Assume that conditions* $(H1)$ *and* $(H2)$ *hold, and*

$$|f(x,s,\xi)|\leq\sigma(x)+a|\xi|^{p(x)-1}\quad a.e.\ in\,\Omega,\ for\ all\ s\in]0,\alpha_0],\ for\ all\,\xi\ \in\mathbb{R}^N,$$

*for* $\sigma\in L^{p'(x)}(\Omega)$ *and a positive constant* $a$*. Then, problem* $(P)$ *admits at least one non-negative solution* $u\in W_0^{1,p(x)}(\Omega)$ *satifying* $u(x)<\alpha_0$ *a.e. in* $\Omega$*.*

*Proof.* In virtue of hypothesis (H1) there exists a positive constant $M\in\mathbb{R}$ such that

$$M=\max_{x\in\overline{\Omega}}g(x,0).$$

Consider the problem

$$\begin{cases}-\Delta_{p(x)}u-\Delta_{q(x)}u=M & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{cases} \tag{4.1}$$

We claim that problem (4.1) admits one unique solution $w \in W_0^{1,p(x)}(\Omega)$. Consider the operator $\Gamma$ given by (2.7) and further comments. Using Theorem 2.1 there esists $w \in W_0^{1,p(x)}(\Omega)$ such that

$$\langle \Gamma(w), v\rangle = \langle M, v\rangle = \int_\Omega M v(x)\, dx$$

for all $v \in W_0^{1,p(x)}(\Omega)$. Then $w \in W_0^{1,p(x)}(\Omega)$ ia a weak solution to problem (4.1) and this solution is unique due to (3.24) and to the strict monotonicity of the operator $\Gamma$.

Choosing $-w_- \in W_0^{1,p(x)}(\Omega)$ in (4.1) as test function one has

$$\int_\Omega |\nabla w|^{p(x)-2}\nabla w \cdot \nabla(-w_-)dx + \int_\Omega |\nabla w|^{q(x)-2}\nabla(w)\cdot\nabla(-w_-)dx = \int_\Omega M(-w_-)dx,$$

then

$$0 \le \int_{\{w<0\}} |\nabla w|^{p(x)}dx \le \int_{\{w<0\}} |\nabla w|^{p(x)}dx + \int_{\{w<0\}} |\nabla w|^{q(x)}dx = \int_{\{w<0\}} Mwdx \le 0.$$

From Lemma 2.1 we have $w \ge 0$ a.e. in $\Omega$. On the other hand, condition $M > 0$ implies that $w \not\equiv 0$ Moreover, from Theorem 4.1 of [10] we have that $w \in L^\infty(\Omega)$.

Now, we claim that there exists a unique solution of following problem

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u = g(x, u(x)) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Consider the function $g^* : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ defined by

$$g^*(x,s) = \begin{cases} g(x,0) & s < 0,\\ g(x,s) & 0 \le s \le w(x),\\ g(x,w(x)) & s > w(x), \end{cases} \quad \forall x \in \overline{\Omega},\ \forall s \in \mathbb{R}$$

and put

$$G^*(x,s) = \int_0^s g^*(x,z)dz \quad \forall x \in \overline{\Omega}, \quad \forall s \in \mathbb{R}.$$

We observe that $g^*$ is a Carathéodory function and moreover we have

$$\min_{x\in\overline{\Omega}} g^*(x, \|w\|_\infty) \le g^*(x,s) \le M \quad \forall x \in \overline{\Omega},\ \forall s \in \mathbb{R}. \tag{4.3}$$

Consider the following problem

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u = g^*(x, u(x)) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

and the functional $I : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx - \int_\Omega G^*(x,u(x))dx.$$

Conditions (4.3) and $p^- > 1$ ensure that $I$ is coercive and it belongs to $C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$ with

$$\langle I'(u),v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx + \int_\Omega |\nabla u|^{q(x)-2}\nabla u\cdot\nabla v dx - \int_\Omega g^*(x,u(x))v(x)dx$$

for each $u,v\in W_0^{1,p(x)}(\Omega)$. Also, using the embedding theorem, we see that $I$ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass-Tonelli theorem, we can find $\underline{u}\in W_0^{1,p(x)}(\Omega)$ such that

$$I(\underline{u}) = \inf_{W_0^{1,p(x)}(\Omega)} I(u). \tag{4.5}$$

From (4.5) we have $I'(\underline{u})=0$ so we obtain

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\cdot\nabla h dx + \int_\Omega |\nabla \underline{u}|^{q(x)-2}\nabla \underline{u}\cdot\nabla h dx = \int_\Omega g^*(x,\underline{u})h dx, \tag{4.6}$$

for all $h\in W_0^{1,p(x)}(\Omega)$, i.e. $\underline{u}$ is a weak solution for problem (4.4). Choosing $h=-u_-\in W_0^{1,p(x)}(\Omega)$ as test function in (4.6) and taking into account $g(x,0)\geq 0$, we have

$$0\leq \int_{\{\underline{u}<0\}} |\nabla \underline{u}|^{p(x)}dx + \int_{\{\underline{u}<0\}} |\nabla \underline{u}|^{q(x)}dx =$$
$$= \int_{\{\underline{u}<0\}} g(x,0)(-u_-)dx \leq 0.$$

Hence, from Lemma 2.1, we have $\underline{u}\geq 0$ a.e. in $\Omega$. Because of $g^*(x,0)=g(x,0)\not\equiv 0$, we observe that $\underline{u}\not\equiv 0$.

By choosing $(\underline{u}-w)_+\in W_0^{1,p(x)}(\Omega)$ as test function in (4.1) and (4.4), we have

$$0\leq \int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\cdot\nabla(\underline{u}-w)_+ dx + \int_\Omega |\nabla \underline{u}|^{q(x)-2}\nabla \underline{u}\cdot\nabla(\underline{u}-w)_+ dx = \int_\Omega g^*(x,\underline{u})(\underline{u}-w)_+ dx$$

and

$$\int_\Omega |\nabla w|^{p(x)-2}\nabla w\cdot\nabla(\underline{u}-w)_+ dx + \int_\Omega |\nabla w|^{q(x)-2}|\nabla w\cdot\nabla(\underline{u}-w)_+ dx = \int_\Omega M(\underline{u}-w)_+ dx$$

and so we obtain

$$\int_\Omega \big(|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} - |\nabla w|^{p(x)-2}\nabla w\big)\cdot\nabla(\underline{u}-w)_+ dx$$
$$+\int_\Omega \big(|\nabla \underline{u}|^{q(x)-2}\nabla \underline{u} - |\nabla w|^{q(x)-2}\nabla w\big)\cdot\nabla(\underline{u}-w)_+ dx$$
$$= \int_\Omega \big(g^*(x,\underline{u}) - M\big)(\underline{u}-w)_+ dx \quad .$$

Taking into account hypothesis (H1) and classical inequality (3.24) we have

$$0\leq \int_{\{\underline{u}>w\}} \big(|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} - |\nabla w|^{p(x)-2}\nabla w\big)\cdot\nabla\big(\underline{u}-w\big)dx$$

$$\leq \int_{\{\underline{u}>w\}} \big(|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} - |\nabla w|^{p(x)-2}\nabla w\big) \cdot \nabla\big(\underline{u} - w\big)dx$$

$$+\int_{\{\underline{u}>w\}} \big(|\nabla \underline{u}|^{q(x)-2}\nabla \underline{u} - |\nabla w|^{q(x)-2}\nabla w\big) \cdot \nabla\big(\underline{u} - w\big)dx =$$

$$= \int_{\{\underline{u}>w\}} \big(g(x, w(x)) - M\big)(\underline{u} - w)dx \leq 0 \ ,$$

hence, from Lemma 2.1, we have $\underline{u} \leq w$ a.e. in $\Omega$. Therefore $g^*(x, \underline{u}) = g(x, \underline{u})$ and $\underline{u}$ is a nonnegative solution of (4.2). Moreover, $\underline{u}$ is unique (for the proof see for example [9], [19]).

Now we claim that function $\underline{u}$ is a subsolution of problem $(P)$ .
In fact, taking into account hypothesis (H1) and taking in the mind that $\underline{u}$ is solution of (4.2) we obtain

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} \cdot \nabla v dx + \int_\Omega |\nabla \underline{u}|^{q(x)-2}\nabla \underline{u} \cdot \nabla v dx = \int_\Omega g(x, \underline{u})v dx \leq \int_\Omega f(x, \underline{u}, \nabla \underline{u})v dx$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

From hypothesis (H2) we have that the constant function $\overline{u} = \alpha_0$ is a supersolution of problem $(P)$; in fact for all $v \in W_0^{1,p(x)}(\Omega)$, $v \geq 0$ it results

$$\int_\Omega |\nabla \overline{u}|^{p(x)-2}\nabla \overline{u} \cdot \nabla v dx + \int_\Omega |\nabla \overline{u}|^{q(x)-2}\nabla \overline{u} \cdot \nabla v dx = 0 \geq \int_\Omega f(x, \alpha_0, \nabla \alpha_0)v dx = \int_\Omega f(x, \alpha_0, 0)v dx.$$

Now, we claim that $\underline{u} \leq \alpha_0 = \overline{u}$ a.e $x \in \Omega$.

We observe that from hypotheses (H1) and (H2) with $\xi = 0$, $s = \alpha_0$ we have

$$g(x, \alpha_0) \leq f(x, \alpha_0, 0) < 0 \quad \forall x \in \overline{\Omega}.$$

Using $(\underline{u}-\alpha_0)_+ \in W_0^{1,p(x)}(\Omega)$ as test function in (4.2), and taking into account hypothesis (H1), we have

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} \cdot \nabla(\underline{u} - \alpha_0)_+ dx + \int_\Omega |\nabla \underline{u}|^{q(x)-2}\nabla \underline{u} \cdot \nabla(\underline{u} - \alpha_0)_+ dx$$

$$= \int_\Omega g(x, \underline{u})(\underline{u} - \alpha_0)_+ dx \leq \int_\Omega g(x, \alpha_0)(\underline{u} - \alpha_0)_+ dx,$$

then

$$0 \leq \int_{\{\underline{u}>\alpha_0\}} |\nabla \underline{u}|^{p(x)}dx + \int_{\{\underline{u}>\alpha_0\}} |\nabla \underline{u}|^{q(x)} \leq \int_{\{\underline{u}>\alpha_0\}} g(x, \alpha_o)(\underline{u} - \alpha_0)dx \leq 0.$$

From Lemma 2.1 we have $\underline{u} \leq \alpha_0$ a.e. $x \in \Omega$. We observe that hypothesis $(H)$ holds for the constructed subsolution $\underline{u}$ and supersolution $\overline{u}$ of our problem $(P)$. Therefore, Theorem 4.1 ensures the existence of a solution $u \in W_0^{1,p(x)}$ to problem $(P)$, which satisfies the property $\underline{u} \leq u \leq \overline{u}$ a.e. in $\Omega$. Taking into account that $\underline{u} \geq 0$, we conclude that $u$ is nonnegative. This completes the proof. □

Finally, we present an example of problem, which admits al least one nonnegative solution. Our aim is to apply Theorem 4.1.

**Example 4.1.** *Let $\Omega \subset \mathbb{R}^N$ be, with $N \geq 3$ and consider $p$, $q \in C(\overline{\Omega})$ with $1 < q(x) < p(x) < \infty$, for all $x \in \overline{\Omega}$. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be defined by*

$$f(x, s, \xi) = g(x, s) + h(\xi) \text{ for all } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

*with*

$$g(x, s) = \begin{cases} |x| + 1 & \text{if } x \in \Omega,\ s \leq 0, \\ (|x| + 1)(1 - s) & \text{if } x \in \Omega,\ 0 < s \leq 2, \\ -(|x| + 1) & \text{if } x \in \Omega,\ s > 2, \end{cases}$$

*and*

$$h(\xi) = \min\left\{|\xi|^{p^- -1}, |\xi|^{p^+ -1}\right\} \text{ for all } \xi \in \mathbb{R}^N.$$

*We observe that*

$$|f(x, s, \xi)| = |g(x, s) + h(\xi)| \leq |g(x, s)| + |h(\xi)| \leq (|x| + 1) + \min\left\{|\xi|^{p^- -1}, |\xi|^{p^+ -1}\right\} \leq (|x| + 1) + |\xi|^{p(x)-1},$$

*a.e. in $\Omega$, for all $s \in [0, 2]$, for all $\xi \in \mathbb{R}^N$ and so condition requested in Theorem 4.1 is verified by choosing $a = 1$, $\alpha_0 = 2$ and $\sigma(x) = |x| + 1$. On the other hand, since $h(\xi) \geq 0$ for all $\xi \in \mathbb{R}^N$, we have*

$$f(x, s, \xi) = g(x, s) + h(\xi) \geq g(x, s) \quad a.e.\ x \in \Omega,\ \forall s \in \mathbb{R},\ \xi \in \mathbb{R}^N,$$

*and*

$$f(x, 2, 0) = g(x, 2) = -(|x| + 1) < 0 \ a.e.\ x \in \Omega$$

*which return hypotheses $(H1)$ and $(H2)$ respectively. This allows the Theorem 4.1 to be applied to the problem $(P)$ with $f$ given above.*

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